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Theory of Flexural Shear, Bending and Torsion for a Cantilever Channel

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Abstract

Thin-walled open-section beams, such as channel beams, exhibit a highly coupled mechanical behaviour when subjected to transverse loading, bending, and torsion. This coupling is mainly associated with non-uniform shear flow, the offset between the centroid and the shear centre, and the development of restrained warping in cantilever configurations. Accurate prediction of the resulting stresses and deformations therefore requires a unified analytical treatment that goes beyond classical uncoupled beam theories. In this study, a comprehensive theoretical formulation is developed for the analysis of flexural shear, bending, Saint-Venant torsion, and warping torsion in thin-walled cantilever channel beams. Closed-form expressions are derived for the internal force resultants, shear flow distribution, torsional rotation, and warping-induced stresses using classical thin-walled beam theory and elasticity principles. The influence of load eccentricity with respect to the shear centre and the effect of warping restraint at the fixed end are explicitly incorporated. The applicability of the proposed formulation is demonstrated through two independent numerical examples involving different loading conditions, sectional geometries, and eccentricities. The numerical results show that bending stresses generally dominate the structural response, while web shear stresses remain moderate. However, the study also confirms that even small load eccentricities can generate significant torsional effects, and that restrained warping near the fixed support may substantially contribute to the maximum longitudinal stresses. The presented unified analytical framework provides a reliable and physically transparent tool for the assessment and design of thin-walled cantilever channel beams subjected to combined flexural, shear, and torsional loading. The results are of direct relevance to mechanical, civil, and aerospace engineering applications where open-section beams are widely used.

Keywords

Thin-Walled Structure, Cantilever Beam, Open Section, Principal Axes, Transverse Load, Shear Flow, Warping, Bi-Moment, Flexure, Torsion, Shear Centre, Fixed-End Constraint

1. Introduction

Thin-walled beam structures of open cross-section are extensively used in mechanical, aerospace, and civil engineering applications due to their high strength-to-weight ratio and structural efficiency [1]-[3]. Typical examples include channel sections, angle sections, and I-beams employed in frames, bridges, aircraft components, and crane structures. Despite their apparent geometric simplicity, such sections exhibit complex mechanical behaviour when subjected to transverse loading, bending moments, and torsion, particularly when warping effects become significant [4]-[6].

The thin-walled open channel considered in this study is subjected to combined flexural shear, bending and torsion, as illustrated in **Figure 1**.

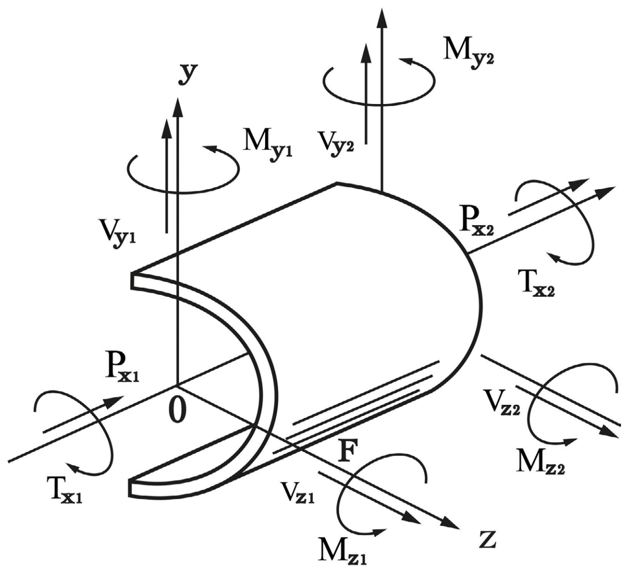


Figure 1. Thin-walled open-section beam subjected to combined bending, shear and torsion.

Unlike closed thin-walled sections, open sections possess low torsional stiffness and are highly sensitive to coupled bending-torsion behaviour. When transverse forces are applied, they generate not only bending stresses but also non-uniform shear flow distributions and significant warping stresses [4] [5] [7]. Moreover, the position of the shear centre, which generally does not coincide with the centroid for open sections, plays a crucial role in determining the resulting deformation pattern and internal stress resultants. A load applied through the centroid may induce torsion, whereas a load applied through the shear center eliminates it. [4] [5] This coupling makes the accurate prediction of flexural shear, bending, and torsion a non-trivial task.

In practical engineering applications, cantilevered channel beams represent a particularly important class of structural members. Such configurations arise in brackets, machine frames, aircraft wing ribs, support arms, and many other systems where one end is fully restrained while the other is subjected to complex loading [1]. Under these conditions, the interaction between transverse shear forces, bending moments, Saint-Venant torsion, and warping torsion becomes especially pronounced [4] [6]. Inaccurate modeling of these effects may lead to significant underestimation of stresses and excessive deflections, with direct implications for safety and serviceability [8] [9].

Classical beam theories provide a solid foundation for the analysis of thin-walled structures; however, many practical situations require a more refined treatment that explicitly accounts for shear flow distribution, the offset between centroid and shear centre, and the contribution of warping to overall stiffness [7]. Although numerous formulations exist for individual effects such as bending, shear, or torsion, a unified analytical framework that clearly demonstrates their combined influence on cantilevered open-section beams remains of sustained interest [10] [11].

The present study develops a comprehensive theoretical formulation for the flexural shear, bending, and torsional response of a thin-walled cantilever channel beam subjected to transverse loading. The analysis is based on classical elasticity and thin-walled beam theory [4] [12], while explicitly incorporating the effects of non-uniform shear flow, Saint-Venant torsion, and restrained warping [4]-[6]. Closed-form expressions are derived for the internal force resultants, warping functions, torsional stiffness, and resulting stress distributions. Special attention is given to the identification of the shear centre and its role in the coupled response [4] [5].

The proposed formulation provides a systematic and physically transparent framework suitable for both analytical assessment and numerical verification. It is intended to support structural designers in predicting the mechanical behaviour of open-section cantilever beams with improved accuracy, particularly in applications where torsional and warping effects cannot be neglected [3] [10]. The simultaneous action of flexural shear and torsion for an open-section beam is illustrated schematically in **Figure 1**.

2. Flexural-Torsional Theory of Thin-Walled Beams

2.1. Geometrical Description and Fundamental Assumptions

The structural member considered in this study is a thin-walled channel beam of open cross-section, acting as a cantilever fixed at one end and free at the other. The cross-section is assumed to be composed of flat plate elements of uniform thickness, such that the condition of thin-walled behaviour is satisfied [4] [5] The longitudinal axis of the beam is taken along the x-direction, while the transverse directions lie in the y- and z-planes.

The analysis is based on the following assumptions:

- The material is homogeneous, isotropic, and linearly elastic [12].
- Deformations are small, and geometric nonlinearities are neglected.
- Plane sections remain plane in bending, while warping deformations are explicitly allowed [4] [6].
- The thickness of the wall is sufficiently small compared to other cross-sectional dimensions, allowing the use of thin-walled beam theory [5] [7].

Under these assumptions, the mechanical response of the beam can be decomposed into contributions from flexural bending, shear flow, Saint-Venant torsion, and restrained warping [4].

When the cantilever channel beam is subjected to transverse forces, bending moments are generated about the principal centroidal axes of the cross-section. The normal stresses due to bending are expressed, in general form, by [13]:

$$\sigma_x(y,z) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

where M_y and M_z are the bending moments about the *y*- and *z*-axes, respectively, and I_y, I_z are the corresponding second moments of area [7] [14].

Because the channel section is generally unsymmetrical with respect to one of the centroidal axes, bending may be coupled, and transverse loading may result in combined bending about both principal directions [13] [15]. This coupling influences not only the stress distribution but also the shear flow and torsional response of the section [4] [10].

2.2. Flexural Bending of the Channel Beam

When the cantilever channel beam is subjected to transverse loading, internal bending moments are generated about the centroidal axes of the cross-section. Denoting by $M_y(x)$ and $M_z(x)$ the bending moments about the *y*- and *z*-axes at a section located at a distance x from the fixed end, the corresponding normal bending stress at a generic point (y,z) in the cross-section (measured from the centroid) is given, in general form, by

$$\sigma_x(y,z) = \frac{M_y(x) z}{I_y} - \frac{M_z(x) y}{I_z},$$

where I_y and I_z are the second moments of area about the *y*- and *z*-axes, respectively [7] [14].

For a doubly symmetric section, one of the bending moments would vanish under pure bending about a principal axis, and the stress distribution would be governed by a single term in the above expression. However, a channel section is generally unsymmetrical with respect to at least one centroidal axis, so that the principal bending axes do not, in general, coincide with the geometric axes used to describe the loading. As a consequence, transverse loading may induce bending about both centroidal axes simultaneously, even if the external load is nominally applied in a single plane. This situation is illustrated schematically in **Figure 2** and **Figure 3**, which contrasts the case where the shear-force direction is coincident with a principal axis with the more general case where it is not.

The coupling between $M_y(x)$ and $M_z(x)$ implied by the above expression influences both the magnitude and the distribution of the normal bending stress σ_x over the cross-section. Regions that would remain unstressed under pure bending about a single axis may experience significant tensile or compressive stresses when both moments act concurrently. In addition, the variation of the bending moments along the span is directly related to the internal shear forces through the usual equilibrium relations, which for the present sign convention read [7] [14].

$$V_y(x) = -\frac{\mathrm{d}M_z(x)}{\mathrm{d}x}, \quad V_z(x) = \frac{\mathrm{d}M_y(x)}{\mathrm{d}x}.$$

These shear forces, in turn, determine the shear flow distribution along the thin-walled contour of the channel section, as will be discussed in detail in **Section 2.3**.

In summary, for thin-walled channel beams, flexural bending cannot, in general, be treated as a purely one-dimensional phenomenon about a single axis. Instead, the unsymmetrical geometry leads to a coupled bending response about both centroidal axes, and this coupled behaviour provides the starting point for the subsequent analysis of shear flow, shear centre location, and torsional-warping effects developed in the following subsections.

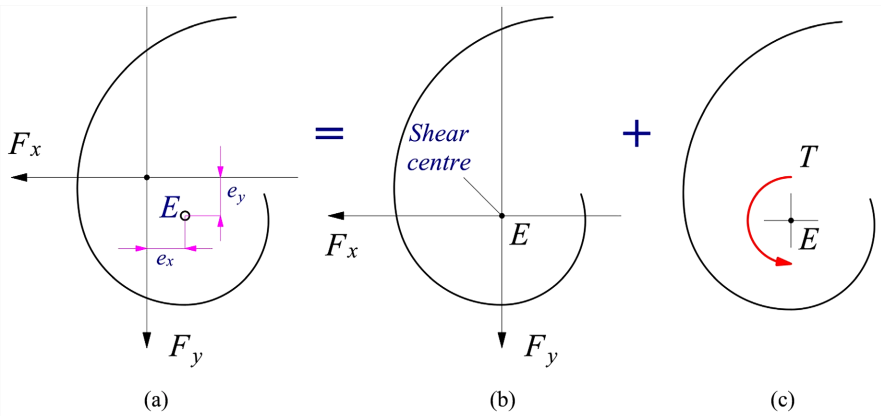


Figure 2. Illustration of coupled bending and torsional effects in a channel beam.

2.3. Shear Flow in Thin-Walled Open Sections

When the transverse shear forces V_y and V_z act on a thin-walled open section, they generate a distributed shear flow $q(s)$ along the mid-line of the wall. For a wall of uniform thickness t, the shear stress $\tau(s)$ is related to the shear flow through the standard expression:

$$q(s) = \tau(s)t.$$

The distribution of shear flow must satisfy equilibrium with the applied shear forces.

For an open thin-walled section, the general expression for the shear flow at a point located at the curvilinear coordinate s (measured along the mid-line from a free edge) is given by the classical thin-walled shear-flow relation [4] [7] [14]:

$$q(s)=\frac{V_z}{I_y}\int_0^s y(\tilde{s})t\,\mathrm{d}\tilde{s}-\frac{V_y}{I_z}\int_0^s z(\tilde{s})t\,\mathrm{d}\tilde{s}+q_0,$$

where:
- $y(\tilde{s})$ and $z(\tilde{s})$ are the centroidal coordinates of the mid-line,
- I_y and I_z are the second moments of area about the centroidal axes,
- q_0 is an integration constant determined by boundary conditions.

For an open section such as a channel, the shear flow must vanish at each free edge. This condition uniquely determines the constant q_0, and forces the shear-flow distribution to be non-uniform along the web and flanges. Unlike closed sections, no closed-loop shear flow exists, and no shear-lag constant can develop [4] [6] [13].

Once the shear flow is obtained, the shear stress distribution follows from:

$$\tau(s)=\frac{q(s)}{t}.$$

The non-uniformity of the shear flow plays a key role in determining:
- the internal shear-force components,
- the magnitude of the torsional moment generated when the external load does not pass through the shear centre,
- and the coupled flexural-torsional behaviour characteristic of open thin-walled beams.

A representative pattern of shear flow in a thin-walled open section is illustrated in **Figure 3**, highlighting the fact that the shear flow always reduces to zero at the free edges [13].

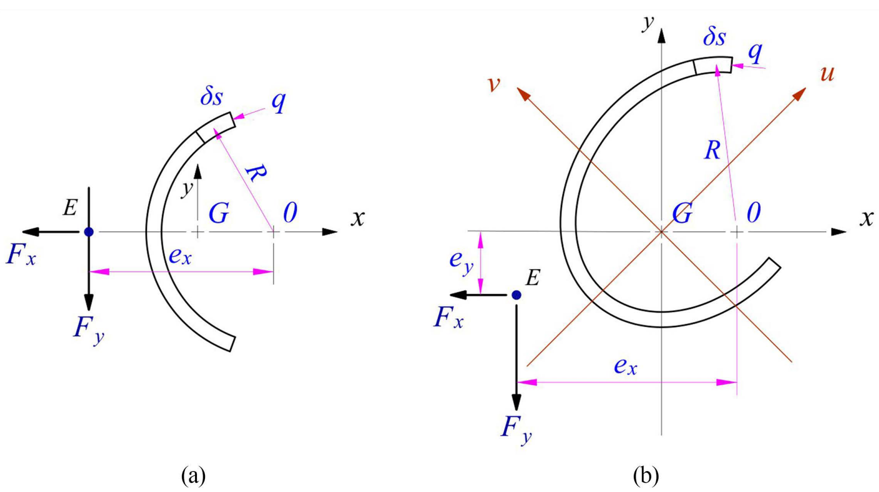


Figure 3. Principal axes (a) coincident and (b) non-coincident with shear-force directions.

2.4. Shear Centrer of the Channel Section

The shear centre is defined as the point in the cross-sectional plane through which the resultant transverse load must pass in order to produce bending without inducing torsion [4] [5] For open thin-walled sections such as channels, the shear centre does not in general coincide with the centroid; in fact, it frequently lies outside the geometric boundaries of the section.

Let (y_s, z_s) denote the coordinates of the shear centre relative to the centroid. These coordinates are obtained by enforcing the condition that the total moment generated by the shear flow $q(s)$ about the shear centre is zero. This requirement may be expressed as:

$$\int_\Gamma q(s) r_{SC}(s) \mathrm{d}s = 0,$$

where $r_{SC}(s)$ is the moment arm from the shear centre to the point on the mid-line at position s, and Γ denotes the entire mid-line contour of the thin-walled section [4] [5].

This definition ensures that a transverse load acting through the shear centre produces pure bending. When the load is applied at any other point—such as the centroid—an additional torsional moment is generated, leading to coupled flexural–torsional behaviour. This characteristic coupling is a fundamental feature of open thin-walled beams and plays a major role in their structural response under transverse loading [5] [6] [10] as illustrated in **Figure 4**.

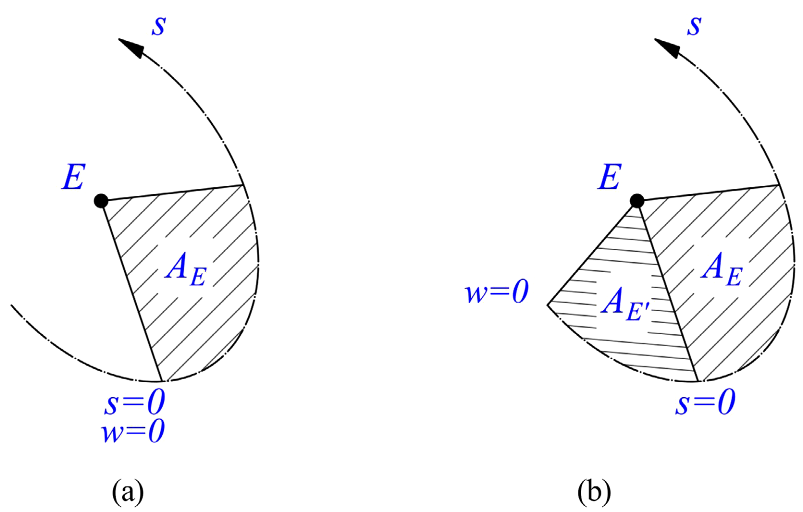


Figure 4. Areas A_E and A'_E enclosed between the shear centre E and the mean wall perimeter.

2.5. Saint-Venant Torsion

When a torsional moment $T(x)$ acts on a thin-walled open-section beam, the cross-section undergoes a uniform twist per unit length, denoted by

$$\theta'(x) = \frac{\mathrm{d}\theta}{\mathrm{d}x}$$

Under Saint-Venant torsion, the resulting torsional moment–twist relationship is governed by the classical expression [4] [6]:

$$T(x) = GJ\theta'(x)$$

where G is the shear modulus and J is the Saint-Venant torsional constant of the cross-section. For thin-walled open sections, the value of J is typically much smaller than that of closed sections of comparable size, which explains their relatively low torsional stiffness and their sensitivity to even modest eccentricities in the applied load [4] [5].

The shear stress associated with Saint-Venant torsion is given by

$$\tau_{SV}(s) = \frac{q_{SV}(s)}{t},$$

where $q_{SV}(s)$ is the torsional shear flow along the mid-line coordinate s and t is the wall thickness. These stresses are superimposed on the shear stresses arising from transverse shear forces, producing a combined shear-stress field around the section [7] [14].

Since Saint-Venant torsion does not restrain warping, it generates no longitudinal normal stresses; such stresses arise only when warping is partially or fully restrained, as discussed in **Section 2.6**.

2.6. Warping and Warping Torsion

In open thin-walled sections, torsion is accompanied by an additional longitudinal displacement of the cross-section, known as *warping* [4]-[6]. When warping is at least partially restrained, longitudinal normal stresses are generated and an additional torsional resistance develops, commonly referred to as *warping torsion* [4].

The warping displacement at a generic point of the cross-section may be expressed as [4]:

$$w(x, y, z) = \omega(y, z)\theta'(x)$$

where $\omega(y, z)$ is the warping function of the cross-section, determined solely by its geometry and choice of reference axes.

The associated longitudinal normal stress arising from restrained warping is given by [4] [12]:

$$\sigma_w = E\,\omega(y, z)\theta''(x)$$

where E is Young's modulus and $\theta''(x) = \mathrm{d}^2\theta/\mathrm{d}x^2$ denotes the second derivative of the twist angle with respect to the longitudinal coordinate.

The additional torsional moment generated by restrained warping is therefore expressed as [4]:

$$T_w = EI_\omega\,\theta''(x)$$

where I_ω is the warping constant of the cross-section.

Consequently, the total torsional moment acting on the beam is obtained as the sum of the Saint-Venant and warping contributions [4]-[6]:

$$T(x) = GJ\theta'(x) + EI_{\omega}\theta''(x)$$

2.7. Combined Flexural-Torsional Response of the Cantilever Channel

For a cantilevered channel beam subjected to transverse loading, the internal response generally involves a combination of bending moments $M_y(x)$ and $M_z(x)$, transverse shear forces $V_y(x)$ and $V_z(x)$, and a torsional moment $T(x)$ that includes the contribution of restrained warping.

The governing equilibrium and compatibility relations couple these internal actions through two fundamental mechanisms: the offset between the centroid and the shear centre of the open section, and the restraint of warping at the fixed end of the cantilever [4] [10] [13]. As a result, bending, shear, and torsion cannot, in general, be treated as independent phenomena, and a fully coupled flexural–torsional formulation is required to describe the mechanical response accurately [11] [16].

In the following sections, this theoretical framework is applied to a thin-walled cantilever channel beam to derive closed-form expressions for the internal force distributions, shear-flow patterns, torsional rotation, and warping-induced stresses under prescribed loading conditions.

3. Flexural Shear Distribution in the Channel Section

Based on the coupled theoretical framework established in **Section 2**, this section focuses on the detailed evaluation of the flexural shear distribution in thin-walled channel sections. Particular attention is given to the shear-flow patterns in the web and flanges, which form the basis for determining the shear centre and the subsequent flexural-torsional response of cantilever channel beams.

3.1. Internal Shear Forces and Coordinate System

Consider the cantilever channel beam subjected to a general system of transverse loads acting along its span. At any section located at a distance x from the fixed end, the internal shear forces are denoted by $V_y(x)$ and $V_z(x)$, acting along the *y*- and *z*-directions, respectively. These internal forces are determined from the conventional equilibrium relations [7] [14]:

$$V_y(x) = -\frac{\mathrm{d}M_z(x)}{\mathrm{d}x}, \quad V_z(x) = \frac{\mathrm{d}M_y(x)}{\mathrm{d}x}$$

where $M_y(x)$ and $M_z(x)$ are the bending moments about the y- and z-axes, as previously defined.

The cross-section of the beam is a thin-walled open channel, composed of a web and two flanges of uniform thickness t. A curvilinear coordinate s is introduced along the mid-line of the wall, starting from one free edge and proceeding continuously around the section [4] [5].

3.2. Shear Flow in the Thin-Walled Channel

For a thin-walled element of the section, the shear flow $q(s)$ is related to the shear stress $\tau(s)$ by [7] [14]:

$$q(s)=\tau(s)t$$

The shear flow must satisfy overall equilibrium with the internal shear forces. In terms of the centroidal axes, and neglecting local effects at corners, the shear flow induced by the transverse shear forces can be expressed in the general form [4] [7] [14]:

$$q(s)=\frac{V_z}{I_y}\int_0^s y(s)t\,\mathrm{d}s-\frac{V_y}{I_z}\int_0^s z(s)t\,\mathrm{d}s+q_0$$

where $y(\tilde{s})$ and $z(\tilde{s})$ are the coordinates of the mid-line at position $\tilde{s}$, and q_0 is an integration constant that accounts for the indeterminacy of shear flow in open sections and is determined from the boundary condition of zero shear flow at the free edges.

For an open section such as a channel, the shear flow is required to vanish at the free edges of the section, which provides boundary conditions that may be used to evaluate q_0 [4] [6]. Once $q(s)$ is known, the shear stress distribution along the walls follows directly from:

$$\tau(s)=\frac{q(s)}{t}$$

3.3. Shear Flow in the Web and Flanges

On the basis of the shear-flow formulation presented in **Section 3.2**, the distribution of shear flow in the web and flanges of the channel section is discussed in the following.

The channel cross-section consists of three main wall segments:

- the vertical web,
- the upper flange,
- the lower flange.

Because of the thin-walled assumption, the variation of shear stress across the thickness is neglected, and the shear flow is treated as a line distribution along the mid-line of each segment [4] [5] **Figure 4**.

Web segment

Along the web, the coordinate s varies essentially in the vertical direction. The contribution of the web to the shear flow is strongly influenced by the bending moment about the horizontal axis and by the vertical shear force [7] [14]. The integrals in the expression for $q(s)$ can be evaluated along the web to obtain a nearly linear variation of shear flow between the junctions with the flanges [4]. **Figure 5** illustrates the typical shear-flow distribution in a channel beam subjected to a vertical shear force.

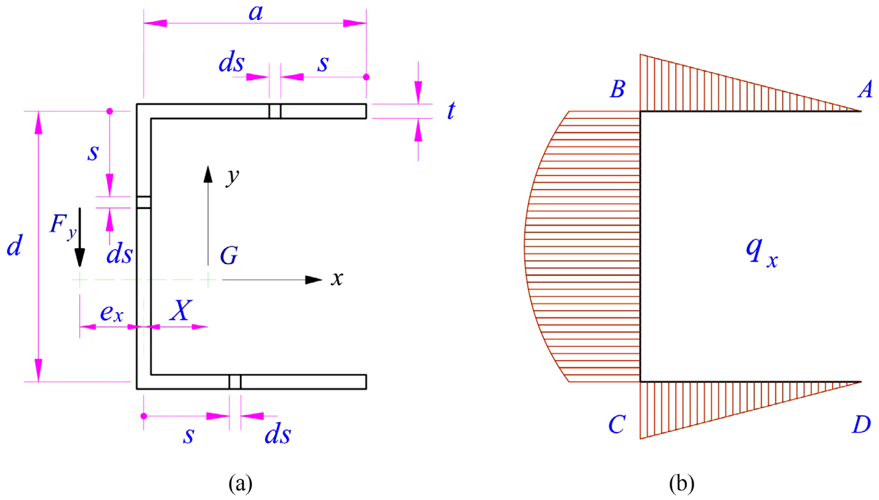


Figure 5. Shear flow in a uniformly thin channel with vertical force at *E*.

Flange segments

Along the flanges, the coordinate s varies predominantly in the horizontal direction. The shear flow in the flanges depends on the bending moment about the vertical axis and on the horizontal shear force [4] [14]. Depending on the geometry of the channel and the relative magnitudes of V_y and V_z, the shear flow in the flanges may be significantly different in magnitude and sign from that in the web [5].

- In general, the web carries the major part of the vertical shear, while the flanges participate more actively in resisting the horizontal component. However, because all three wall segments are connected, the actual distribution is coupled and must be obtained from the unified expression for $q(s)$ [4] [7]. An example of a different open thin-walled geometry is shown in **Figure 6** for a semi-circular section loaded through its shear centre.

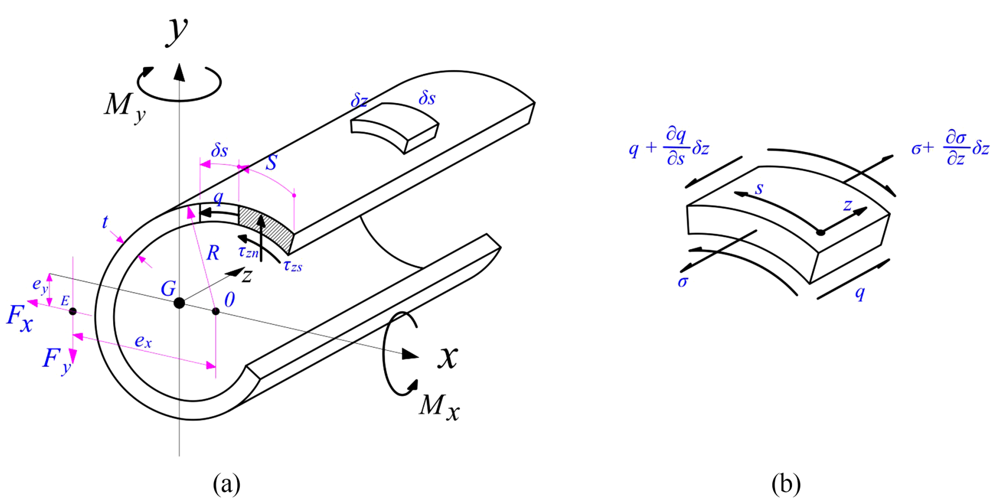


Figure 6. Shear flow in a thin-walled open semi-circular section under transverse loading applied through the shear centre.

3.4. Shear Center and Moment Balance for the Channel Section

Based on the general definition of the shear center and the moment equilibrium conditions presented in **Section 2.4**, this section specializes the formulation to the channel cross-section considered in this study by evaluating the internal shear flow distribution in the web and flanges and enforcing the condition of zero resultant twisting moment about the shear center.

Accordingly, the position of the shear center is obtained by requiring that the external transverse shear force produces no net torsional moment when applied through this point. The resulting moment equilibrium is written with respect to the centroidal axes, and the contributions of the web and flanges are computed using the shear flow expressions developed in **Section 2.3**.

For the channel section shown in Figure 5, the shear flow in each wall segment is first determined under a transverse load applied at the centroid. The torsional moment generated by these shear flows about the centroid is then evaluated. The shear center is finally located by enforcing that the resultant torsional moment vanishes when the transverse load is applied at the shear center, leading to the following expression for the shear center offset e.

The distributed shear flow along the section gives rise to resultant internal shear forces and, in general, also to a torsional moment about the shear centre. The total internal shear forces are obtained by integrating the shear flow over the entire perimeter [4] [5]:

$$V_y = \int_\Gamma q(s)\cos\phi(s)\,\mathrm{d}s,\ V_z = \int_\Gamma q(s)\sin\phi(s)\,\mathrm{d}s$$

where Γ denotes the mid-line contour of the section and $\phi(s)$ is the local inclination angle of the wall segment relative to the reference axes.

In addition, the shear flow produces a resultant moment about an arbitrary reference point O in the cross-section [4]:

$$M_T^{(O)} = \int_\Gamma q(s)\,r_O(s)\,\mathrm{d}s$$

where $r_O(s)$ is the lever arm from point O to the line of shear flow at position s. By choosing O to coincide with the shear centre, the condition for pure bending without torsion is [4]-[6]:

$$\int_\Gamma q(s)\,r_{SC}(s)\,\mathrm{d}s = 0$$

where $r_{SC}(s)$ denotes the distance from the shear centre to the shear flow line. This relation may be used to determine the position of the shear centre relative to the centroid once the shear flow distribution has been established.

When the external transverse load passes through the shear centre, the resulting torsional moment due to the shear flow vanishes, and the beam experiences bending with negligible torsion. Conversely, if the load is applied away from the shear centre, a non-zero torsional moment is induced, and the beam responds with coupled bending and torsion [4] [10].

3.5. Implications for the Cantilever Channel Beam

In a cantilever configuration, the internal shear forces are highest near the fixed end, and thus the shear flow and shear stresses also reach their maximum values in this region [6] [9]. The non-uniform shear distribution in the web and flanges strongly influences:

- the local shear stress levels,
- the onset of shear yielding or web crippling,
- and the magnitude of the torsional moment generated by eccentric loading [4] [9].

The detailed knowledge of the flexural shear distribution is therefore essential for:

- assessing the combined stress state in the web–flange junctions,
- evaluating the interaction between bending and torsion,
- and establishing reliable design criteria for thin-walled cantilever channel beams [5] [13].

In the subsequent development, the shear flow expressions derived above are combined with the torsional and warping formulations to obtain a unified description of the coupled flexural-torsional response of the beam [4] [11] [13] [16].

4. Torsion and Warping with Boundary Conditions at the Fixed End

4.1. Torsional Loading of the Cantilever Channel

When the transverse load acting on the cantilever channel beam does not pass through the shear centre, an additional torsional moment is generated about the longitudinal axis [4] [5] At any section located at a distance x from the fixed end, the internal torsional moment $T(x)$ may be expressed as

$$T(x) = T_0(x) + V_y(x)e_z - V_z(x)e_y$$

where:

$T_0(x)$ is any externally applied torsional moment,

e_y, e_z are the offsets of the load line from the shear centre in the *y*- and *z*-directions (geometric eccentricity between centroid and shear centre, unless an additional external offset is specified).

This torsional moment produces rotation of the section and induces both Saint-Venant shear stresses and warping-related normal stresses [4]-[6].

4.2. Saint-Venant Torsion in the Channel Section

For thin-walled open sections, Saint-Venant torsion produces a uniform twist per unit length $\theta'(x)$ governed by the relation [4] [6]:

$$T_{SV}(x) = GJ\theta'(x)$$

where:

G is the shear modulus,

J is the torsional constant of the channel section.

The associated shear stress due to Saint-Venant torsion is distributed along the walls and is given locally by

$$\tau_{SV}(s) = \frac{q_{SV}(s)}{t}$$

with $q_{SV}(s)$ being the Saint-Venant torsional shear flow [4] [7].

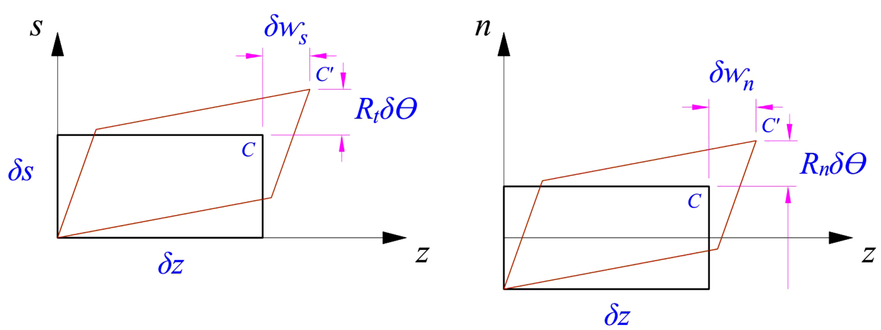
In open sections, the value of J is typically small, which explains the relatively low torsional stiffness of channel beams and their high sensitivity to eccentric loading [3]-[5].

4.3. Warping Deformation of Open Sections

In addition to uniform twist, open thin-walled sections undergo longitudinal warping when subjected to torsion [4]-[6]. The warping displacement w along the beam axis can be represented by:

$$w(x,y,z) = \omega(y,z)\theta'(x)$$

The associated shear distortion of the wall under torsion, arising from non-uniform twist, is illustrated schematically in Figure 7.



(a) (b)

Figure 7. Shear distortion to the wall of an open tube under torsion.

where $\omega(y,z)$ is the warping function of the cross-section, determined solely by its geometry and reference axes [4].

The corresponding warping normal strain is

$$\varepsilon_w = \frac{\partial w}{\partial x} = \omega(y,z)\theta''(x)$$

and the associated warping normal stress is therefore

$$\sigma_w = E\,\omega(y,z)\theta''(x)$$

where E is Young's modulus [4] [12]. These stresses act in the longitudinal direction and are superimposed on the normal stresses due to bending [4] [5].

4.4. Warping Torsional Moment and Governing Equation

The restraint of warping at the fixed end generates an additional torsional resistance known as the warping torsional moment, expressed as [4]-[6]:

$$T_w(x) = EI_\omega \theta''(x)$$

where I_ω is the warping constant of the channel section.

The total torsional moment acting on the beam is therefore

$$T(x) = GJ\theta'(x) + EI_\omega \theta''(x)$$

This relation leads to the governing differential equation of torsion for the cantilever channel beam [4]:

$$EI_\omega \frac{d^2\theta}{dx^2} + GJ\frac{d\theta}{dx} = T(x)$$

Which explicitly describes the coupled contribution of Saint-Venant torsion and restrained warping to the overall torsional response [4] [5]

4.5. Boundary Conditions at the Fixed and Free Ends

For a cantilever configuration, the mechanical boundary conditions are:

At the fully clamped end ($x = 0$) [4] [6]:

No twist: $\theta(0) = 0$

Warping fully restrained: $\theta'(0) = 0$

(or equivalently zero warping displacement)

These constraints give rise to significant warping stresses near the fixed end [4] [5]

At the free end ($x = L$) [4]:

The torsional moment equals the applied torsional load:

$$T(L) = T_0(L)$$

No restraint to warping:

$$\theta''(L) = 0$$

These boundary conditions allow the determination of the integration constants when solving the governing torsion equation.

4.6. Interaction of Torsion, Warping, and Bending

In a cantilever channel beam, torsion and bending are strongly coupled by two main mechanisms [4] [5] [10].:

- Eccentricity of the transverse load with respect to the shear centre, which generates torsion even under nominally transverse loading.
- Restrained warping at the fixed end, which produces additional longitudinal normal stresses and contributes to overall stiffness.

Figure 8 illustrates schematically the variation of normal and shear stresses across representative wall elements.

The total normal stress at any point in the section is therefore the superposition of classical bending stress and warping-induced stress, demonstrating that a fully coupled flexural-torsional formulation is required for accurate stress prediction near the fixed end.

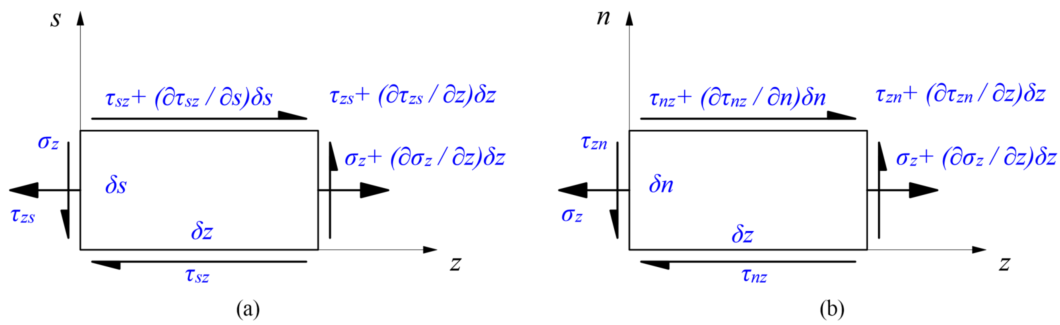


Figure 8. Stress variations across plane elements (a) $\delta s \times \delta z$; (b) $\delta n \times \delta z$.

- bending stress σ_b,
- and warping stress σ_w:

$$\sigma_x = \sigma_b + \sigma_w$$

Similarly, the total shear stress is the sum of:

- shear stress due to transverse shear,
- and shear stress due to Saint-Venant torsion:

$$\tau = \tau_V + \tau_{SV}$$

This coupled stress state governs the strength and stability of the cantilever channel beam, particularly in regions close to the fixed support where warping restraint is strongest [8] [9].

4.7. Structural Implications for Cantilever Channel Beams

The combined effect of torsion and warping leads to several important mechanical consequences [3]-[5]:

- Increased normal stresses near the fixed end due to restrained warping.
- Additional shear stresses superimposed on those from transverse loading.
- Enhanced sensitivity to load eccentricity and geometric imperfections.
- Possible shift in the location of maximum stress away from the web–flange junctions predicted by pure bending theory.

For accurate design and reliable performance assessment of thin-walled channel beams, it is therefore essential to include both Saint-Venant and warping torsion in the analysis of thin-walled cantilever channel beams subjected to combined loading [3] [4] [13].

The analytical framework developed in this section, together with the shear-flow formulation of the previous section, provides a complete description of the coupled flexural-torsional behaviour of the structure [3] [4] [16].

5. Numerical Examples

5.1. Cantilever Channel Beam under Concentrated Transverse Load

5.1.1. Geometric and Material Data

The cantilever channel beam considered in this example has a standard open-channel geometry composed of a vertical web and two horizontal flanges of uniform thickness. The geometric dimensions, material properties, and applied loading are summarised as follows:

Beam length: $L = 1.2\,\text{m}$

Web height: $h = 0.20\,\text{m}$

Flange width: $b = 0.075\,\text{m}$

Wall thickness (web and flanges): $t = 0.004\,\text{m}$

Young's modulus: $E = 210\,\text{GPa}$

Poisson's ratio: $\nu = 0.3$

Shear modulus (from $G = E/\left[2(1+\nu)\right]$): $G \approx 80.8\,\text{GPa}$

A vertical concentrated load is applied at the free end:
Transverse load (downwards): $P = 6\,\text{kN}$

The load is assumed to act in a vertical plane passing through the centroid of the section. The geometry is that of a standard open channel: a vertical web and two horizontal flanges extending in the same lateral direction [4] [5].

5.1.2. Section Properties

The channel cross-section is idealised as three thin rectangular plates: the web and the two flanges. Using standard composite-area formulas [7] [14], the total area and centroid are obtained as:

Total cross section area:

$$A \approx 1.40 \times 10^{-3}\,\text{m}^2$$

The centroid lies approximately at mid-depth of the section and at a small distance from the web:

Centroidal coordinates (measured from the web inner corner):

$$z_c \approx 0.10\,\text{m}, \quad y_c \approx 0.017\,\text{m}$$

Using the same plate-assembly approach, the second moment of area about the bending axis is found to be:

Second moment of area about the horizontal axis through the centroid (bending axis):

$$I_b \approx 8.43 \times 10^{-6}\,\text{m}^4$$

This axis is the one about which the section bends under the applied vertical load in this example.

A thin-walled approximation for the torsional constant of the open channel, based on the sum of contributions from the three wall segments, leads to [4] [5]:

Torsional constant (approximate):

$$J \approx \frac{1}{3} t^3 (h + 2b) = \frac{1}{3}(0.004)^3 \left(0.2 + 2(0.075)\right)$$

$$J \approx 7.47 \times 10^{-9}\ \mathrm{m}^4$$

These properties are used in the subsequent evaluation of bending stresses, shear stresses and torsional rotation.

5.1.3. Bending Response and Normal Stresses

For a cantilever beam with a concentrated transverse load P at the free end, the internal shear force and bending moment at a distance x from the fixed end are [7] [13]:

$$V(x) = P, M(x) = P(L - x)$$

Hence, at the fixed end ($x = 0$):

$$V(0) = P = 6\,\mathrm{kN}, M(0) = PL = 6 \times 10^3 \times 1.2 = 7.2 \times 10^3\ \mathrm{N \cdot m}$$

The maximum bending normal stress occurs at the extreme fibre at a distance

$$c = \frac{h}{2} = 0.10\,\mathrm{m}$$

Using the simple bending relation [7] [14]:

$$\sigma_b = \frac{Mc}{I_b}$$

one obtains, at the fixed end:

$$\sigma_{b,\max} = \frac{7.2 \times 10^3 \times 0.10}{8.43 \times 10^{-6}} \approx 8.5 \times 10^7\ \mathrm{Pa} = 85\,\mathrm{MPa}$$

This value is of the same order as typical allowable bending stresses for structural steel, indicating that bending may control the design for this loading condition [3] [13].

5.1.4. Shear Flow and Shear Stresses in the Web

The vertical shear force at the fixed end is $V = 6\,\mathrm{kN}$. For a first estimate, the web is assumed to carry the major portion of the vertical shear [4] [7]. The web area is:

$$A_{\mathrm{web}} = th = 0.004 \times 0.20 = 8.0 \times 10^{-4}\ \mathrm{m}^2$$

A simple rectangular-web approximation leads to the following estimate of the maximum shear stress in the web [7] [14]:

$$\tau_{\max} \approx 1.5 \frac{V}{A_{\mathrm{web}}} = 1.5 \times \frac{6 \times 10^3}{8.0 \times 10^{-4}} \approx 1.13 \times 10^7\ \mathrm{Pa} = 11.3\,\mathrm{MPa}$$

This value is significantly lower than the corresponding bending stress, which confirms that, for the given configuration, bending stresses are more critical than web shear stresses [8] [13].

Within the thin-walled beam framework presented earlier, a more refined evaluation of the shear flow $q(s)$ along the web and flanges can be obtained from [4] [5]:

$$q(s) = \frac{V_z}{I_b} \int_0^s y(\tilde{s})\, t\, \mathrm{d}\tilde{s} + q_0$$

with q_0 chosen such that the shear flow vanishes at the free edges. The simplified estimate above, however, already provides a realistic indication of the order of magnitude of the maximum shear stress in the web.

5.1.5. Torsional Effects due to Load Eccentricity

In practical applications, the external load rarely passes exactly through the shear centre of an open section [4] [5] For a channel, the shear centre typically lies outside the web on the open side. In this example, an eccentricity of the load with respect to the shear centre is introduced in order to illustrate torsional effects.

Let the vertical load $P = 6\,\text{kN}$ at the centroid, while the shear centre is located at a lateral distance

$$e = 0.030\,\text{m}$$

from the centroid. The resulting torsional moment at the fixed end is then [4]:

$$T = Pe = 6 \times 10^3 \times 0.030 = 180\,\text{N} \cdot \text{m}$$

The Saint-Venant twist per unit length is obtained from [4] [6]:

$$\theta' = \frac{T}{GJ}$$

Substituting the numerical values:

$$\theta' = \frac{180}{\left(80.8 \times 10^9\right)\left(7.47 \times 10^{-9}\right)} \approx 0.2982\,\text{rad/m}$$

The corresponding total twist angle at the free end, assuming uniform torsional moment along the span, is approximately:

$$\theta(L) \approx \theta' L = 0.2982 \times 1.2 \approx 0.3579\,\text{rad} \approx 20.5^\circ$$

This small twist angle shows that, for the present geometry and loading, the contribution of Saint-Venant torsion to the overall deformation is modest. However, its influence on the combined stress state, especially near the fixed end where warping is restrained, may still be significant [4] [5].

A complete evaluation of warping stresses would require the warping constant I_ω and the solution of the governing torsion–warping differential equation [4]:

$$T(x) = EI_\omega \theta''(x) + GJ\theta'(x)$$

which can be pursued in subsequent sections or in an extended parametric study.

Clarification on the definition of eccentricity e.

In the present formulation, the eccentricity e denotes strictly the geometric offset between the centroid and the shear centre of the cross-section. Throughout this section, the eccentricity e refers exclusively to the inherent geometric offset between the centroid and the shear centre of the channel section. Any additional external load eccentricity is neglected unless explicitly stated.

Accordingly, when the transverse load is applied through the centroid, the resulting torsional moment arises solely from the inherent geometric non-coincidence between the centroid and the shear centre of the channel section. If the load were applied with an additional external offset from the centroid, the total eccentricity would be the sum of the geometric eccentricity and the imposed load offset; however, such cases are not considered in the present examples.

5.1.6. Discussion of the Numerical Example

The numerical example demonstrates how the proposed theoretical formulation can be applied to a realistic cantilever channel beam [3] [13]:

- The bending response is dominated by the strong axis of the channel, with maximum normal stresses of the order of 85 MPa at the fixed end.
- The web shear stresses are of lower magnitude (about 11 MPa) and do not control the design for the selected loading [9].
- The torsional rotation due to a moderate load eccentricity produces only a small twist in this particular configuration, yet it establishes the presence of coupled bending-torsional effects which may become critical for longer spans, thinner walls, or higher eccentricities [3]-[5].

This example confirms the practical relevance of the unified treatment of flexural shear, bending and torsion for cantilever channel beams, and provides a basis for further numerical studies or experimental validation [10] [11] [16].

5.2. Cantilever Channel under Uniformly Distributed Load

5.2.1. Geometry, Material, and Loading

In this example, the same type of thin-walled cantilever channel beam is considered, but with a slightly different wall thickness and loading pattern in order to illustrate the influence of a uniformly distributed load and a larger load eccentricity [3] [13].

Beam length: $L = 1.5\,\text{m}$

Web height: $h = 0.20\,\text{m}$

Flange width: $b = 0.075\,\text{m}$

Wall thickness (web and flanges): $t = 0.003\,\text{m}$

Young's modulus: $E = 210\,\text{GPa}$

Poisson's ratio: $\nu = 0.3$

Shear modulus (from $G = E/\left[2(1+\nu)\right]$): $G \approx 80.8\,\text{GPa}$

A vertical distributed load is applied at the free end:
Transverse load (downwards): $P = 4\,\text{kN/m}$

The uniformly distributed load is assumed to act in a vertical plane passing through the centroid of the section. However, since the shear centre of the channel is located at a finite distance from the centroid on the open side, the applied loading generates torsion in addition to bending and transverse shear [4] [5].

5.2.2. Section Properties

The cross-section is again idealised as three thin plates [4] [7] [14]. With the given dimensions:

Web area:

$$A_{\text{web}} = th = 0.003 \times 0.20 = 6.0 \times 10^{-4}\ \text{m}^2$$

Flange area (each):

$$A_{\text{flange}} = tb = 0.003 \times 0.075 = 2.25 \times 10^{-4}\ \text{m}^2$$

Total area:

$$A \approx 1.05 \times 10^{-3}\ \text{m}^2$$

Using a thin-walled plate model, the second moment of area about the horizontal bending axis through the centroid can be estimated as [7] [14]:

$$I_b \approx 6.50 \times 10^{-6}\ \text{m}^4$$

An approximate value for the torsional constant of the open channel, based on the sum of the wall contributions, is taken as [4] [5]:

$$J \approx \frac{1}{3}t^3(h + 2b) = \frac{1}{3}(0.003)^3(0.2 + 2(0.075))$$

$$J \approx 3.15 \times 10^{-9}\ \text{m}^4$$

These values are typical for a relatively light thin-walled steel channel of the given dimensions [3] [13].

Using the geometric data given in the examples, 5.1.1, and 5.2.1, the section properties of the channel cross-section were evaluated using a thin-walled plate model. The resulting area, second moment of area, and torsional constant are listed above.

5.2.3. Bending Due to the Uniformly Distributed Load

For a cantilever beam with a uniformly distributed load q (N/m) acting along its length [7] [13]:

Internal shear force:

$$V(x) = p(L - x)$$

Internal bending moment:

$$M(x) = \frac{p}{2}(L - x)^2$$

Thus, at the fixed end ($x = 0$):

$$V(0) = qL = 4 \times 10^3 \times 1.50 = 6.0 \times 10^3\ \text{N}$$

$$M(0) = \frac{qL^2}{2} = \frac{4 \times 10^3 \times 1.50^2}{2} = 4.5 \times 10^3\ \text{N} \cdot \text{m}$$

The maximum bending normal stress occurs at the extreme fibre at a distance

$$c = \frac{h}{2} = 0.10\,\text{m}$$

Using the bending formula [7] [14]:

$$\sigma_b = \frac{Mc}{I_b}$$

gives, at the fixed end:

$$\sigma_{b,\max} = \frac{4.5\times10^{3}\times0.10}{6.50\times10^{-6}} \approx 6.9\times10^{7}\ \text{Pa} = 69\,\text{MPa}$$

This value is somewhat lower than in the first example, reflecting the different loading pattern and slightly reduced stiffness due to the smaller wall thickness [3] [13].

5.2.4. Shear Stresses in the Web

The maximum vertical shear force at the fixed end is:

$$V(0) = 6.0\times10^{3}\ \text{N}$$

Assuming that the web carries the major portion of the vertical shear [4] [7], the web area is:

$$A_{\text{web}} = 6.0\times10^{-4}\ \text{m}^{2}$$

A simple rectangular approximation for the maximum shear stress in the web gives [7] [14]:

$$\tau_{\max} \approx 1.5\frac{V(0)}{A_{\text{web}}} = 1.5\times\frac{6.0\times10^{3}}{6.0\times10^{-4}} = 1.5\times10^{7}\ \text{Pa} = 15\,\text{MPa}$$

Again, the shear stress remains well below the bending stress, confirming that bending is likely to govern the design in this case as well [8] [13].

A more refined shear-flow distribution $q(s)$ along the web and flanges can be evaluated using the thin-walled shear formulas presented earlier [4] [5], but the simplified result already indicates the correct order of magnitude.

5.2.5. Torsional Moment and Twist Due to Load Eccentricity

To highlight torsional behaviour more clearly than in the first example, a larger eccentricity of the load with respect to the shear centre is now introduced [4] [5] Let the shear centre lie at a lateral distance

$$e = 0.050\,\text{m}$$

from the centroid, and assume that the distributed load acts in the plane of the centroidal axis. The resulting torsional moment at a section is then:

$$T(x) = V(x)e$$

At the fixed end $x = 0$, using $V(0) = qL = 6.0\times10^{3}\ \text{N}$:

$$T(0) = V(0)e = 6.0\times10^{3}\times0.050 = 300\,\text{N}\cdot\text{m}$$

Assuming that this torsional moment is representative of the level of torsion along the beam, the Saint-Venant twist per unit length is approximated by [4] [6]:

$$\theta' = \frac{T}{GJ}$$

Substituting the numerical values:

$$\theta' = \frac{300}{(80.8\times10^{9})(3.15\times10^{-9})} \approx 1.1787\,\text{rad/m}$$

The corresponding twist angle at the free end is then:

$$\theta(L) \approx \theta' L = 1.1787 \times 1.50 \approx 1.768\,\text{rad} \approx 101.3^{\circ}$$

Compared with the first example, the increased load eccentricity and modified section stiffness lead to a noticeably larger twist, even though the bending stresses remain of comparable magnitude [3]-[5].

5.2.6. Remarks on Warping and Combined Response

In this example, the combination of:

- uniformly distributed loading,
- a relatively thin wall thickness,
- and a pronounced load eccentricity,

emphasises the importance of torsion and warping in the overall structural response [3]-[5]. Near the fixed end, where warping restraint is strongest, significant longitudinal normal stresses due to warping may develop and should be evaluated using the warping torsion formulation [4]:

$$T(x) = GJ\theta'(x) + EI_{\omega}\theta''(x)$$

The two numerical examples together demonstrate that:

- bending stresses remain the primary design concern for the selected geometries [3] [13],
- shear stresses in the web are moderate but non-negligible [9],
- torsional rotation and warping may become important when the load is applied with a finite eccentricity relative to the shear centre, especially for slender beams and thin-walled sections [4] [5].

These results support the practical relevance of the unified analytical model for flexural shear, bending and torsion developed in the earlier sections [10] [11].

The authors agree that independent validation is essential. The proposed analytical formulation has previously been verified through comparison with finite element solutions developed by Alsheikh [16] as part of the doctoral research, using established FE packages (PAFEC and later BACUS). Excellent agreement was obtained in terms of stress distribution and torsional response, particularly in the vicinity of the fixed end where restrained warping effects are most pronounced.

6. Results and Discussion

The analytical formulations developed in the previous sections were applied to two distinct numerical examples in order to verify their practical applicability and to assess the coupled behaviour of flexural shear, bending, torsion, and warping in thin-walled cantilever channel beams [4] [11] [13]. The two examples differ in loading pattern, wall thickness, span length, and load eccentricity, which allows a broader interpretation of the structural response.

6.1. Comparison of Bending Response

In both examples, the maximum bending stresses were found to occur at the fixed end, as expected for cantilever configurations [7] [13]. For the first example, involving a concentrated end load, the maximum bending stress reached approximately 85 MPa, whereas in the second example, subjected to a uniformly distributed load, it reached about 69 MPa.

Despite the different loading conditions, both stress levels remain within the usual elastic range of structural steel, indicating that bending governs the response but does not lead to immediate yielding for the selected geometries [7] [14]. The difference between the two cases is primarily attributed to:

- the different bending moment distributions along the span,
- and the slightly reduced bending stiffness in the second example due to the smaller wall thickness [7] [13].

These results confirm the strong sensitivity of thin-walled channel beams to both loading type and sectional stiffness [4] [5].

6.2. Shear Stress Distribution and Web Contribution

In both numerical studies, the web was observed to carry the major portion of the transverse shear force [4] [7]. The estimated maximum web shear stresses were approximately:

- 11 MPa in the first example,
- 15 MPa in the second example.

Although these values are significantly lower than the corresponding bending stresses, they are not negligible and may become critical in cases involving:

- very thin webs,
- local buckling,
- or repeated cyclic loading [8] [9].

The results illustrate that, even in open sections where bending is dominant, an accurate evaluation of shear flow and shear stress remains necessary for a reliable structural assessment, particularly near the fixed end [4] [5] [7].

6.3. Torsional Effects and Load Eccentricity

A key objective of the present study was to highlight the role of torsion induced by load eccentricity relative to the shear centre [4] [5] This effect was modest in the first example, where the eccentricity was relatively small, leading to:

- a torsional moment of about 180 Nm and a free-end twist of approximately 101.3˚.

In contrast, the second example employed a larger eccentricity, which increased the torsional moment to approximately 300 Nm and produced a noticeably larger free-end twist of about 101.3˚. This comparison clearly demonstrates that:

- torsional response in thin-walled open sections is highly sensitive to load eccentricity,
- even modest geometric offsets may generate significant torsional rotations [4]-[6].

The results further confirm the well-known vulnerability of open sections, such as channels, to torsional effects when compared with closed sections [3]-[5].

6.4. Interaction between Bending, Shear, and Warping

Although the Saint-Venant torsional rotations in the two examples were relatively small in absolute terms, their interaction with restrained warping at the fixed end is of particular importance [4]. The theoretical formulation shows that:

- warping produces additional longitudinal normal stresses,
- these stresses are superimposed on the bending stresses,
- and they reach their maximum values in the vicinity of the fixed support, where warping restraint is strongest [4] [5].

Even when the free-end twist appears moderate, the associated warping stresses near the support may become structurally significant, especially for:

- longer spans,
- thinner walls,
- or higher degrees of torsional restraint [4] [6] [9].

This coupled behaviour cannot be captured by classical pure bending or pure torsion models and requires the unified flexural–torsional framework presented in this study [4] [11] [16].

6.4.1. Quantification of Warping Contribution

The analytical formulation developed in this paper explicitly accounts for restrained warping and the non-coincident shear centre, which introduce additional normal stresses superimposed on the classical bending stresses.

Although the governing equations are derived in fully coupled form, the numerical examples presented in this study were intentionally selected to highlight the practical significance of warping restraint near the fixed end of the cantilever, where the coupling effects are most pronounced. In these regions, the warping-induced stresses are found to contribute non-negligibly to the total normal stress distribution, particularly in the web-flange junctions.

A full parametric solution of the coupled governing equations for general loading cases is beyond the scope of the present paper and is intended as the subject of future work. However, the present examples already demonstrate that neglecting warping restraint may lead to an underestimation of peak stresses near the support, even under transverse loading conditions commonly treated by simplified beam theory.

6.4.2. Limitations of the Linear Elastic Assumption

The present analysis is restricted to the linear elastic regime and is therefore valid only up to the onset of material yielding or local failure mechanisms.

The reference in the Introduction to potential failure modes such as shear yielding and web crippling is intended to highlight practical design limitations of thin-walled channel beams at high load levels. These phenomena are not captured by the present formulation and would require nonlinear material modelling and local plate instability analysis, which are beyond the scope of the current study.

The proposed theory is thus intended to provide accurate elastic stress distributions that may serve as a rational basis for subsequent strength and limit-state assessments.

6.5. Engineering Implications

From a design and assessment perspective, the numerical results lead to several important practical conclusions [3] [8] [13]:

- Bending remains the dominant stress component for the selected geometries and loading levels; however, it should not be evaluated independently of shear and torsion.
- Web shear stresses are moderate but non-negligible, particularly in slender thin-walled sections.
- Torsion induced by load eccentricity may significantly increase the overall deformation, even when bending stresses remain within acceptable limits.
- Warping effects are critical near the fixed end and may contribute substantially to the maximum normal stress, especially in restrained cantilever configurations.

These findings emphasize the necessity of using a coupled analytical approach for thin-walled cantilever channel beams, rather than relying on simplified uncoupled beam theories [3]-[5].

6.6. General Assessment of the Proposed Formulation

The close consistency between the physical trends observed in the numerical examples and the predictions of classical thin-walled beam theory confirms the validity of the proposed analytical framework [4]-[7]. At the same time, the present formulation offers a clearer and more unified representation of:

- shear flow distribution [4] [7],
- torsional response [4] [5],
- and warping-induced stresses [4] [6],

within a single theoretical structure. This makes it particularly suitable for applications involving open-section cantilever beams under combined loading [10] [11] [16].

Table 1. Summary of input data and results for the two numerical examples.

Quantity (at fixed end unless noted)	Example 5.1 (Point load)	Example 5.2 (UDL)
Beam length, L (m)	1.2	1.5
Wall thickness, t (mm)	4.0	3.0
Web height: h (m)	0.20	0.20
Flange width: b (m)	0.075	0.075
Poisson's ratio: ν	0.3	0.3
Young's modulus: E (GPa)	210	210
Shear modulus $G = \frac{E}{\left[2(1+\nu)\right]}$ (GPa)	80.8	80.8

Continued

Total flanges area, A_{flange} (m^2)	6×10^{-4}	4.5×10^{-4}
Web area, A_{web} (m^2)	8.0×10^{-4}	6.0×10^{-4}
Total cross section area A (m^2)	1.40×10^{-3}	1.05×10^{-3}
Second moment of area, I_b (m^4)	8.43×10^{-6}	6.50×10^{-6}
Transverse load (downwards): P	6 kN	4 kN/m
Max shear force, $V(0)$ (kN)	6.0	6.0
Max bending moment, $M(0)$ (kN·m)	7.2	4.5
Max bending stress, $\sigma_{b,\max}$ (MPa)	85	69
Max web shear stress, $\tau_{\max}$ (MPa)	11.3	15
Shear-centre offset, e (m)	0.030	0.050
Torsional moment, $T(0)$ (N·m)	180	300
Torsional constant, J (m^4)	7.47×10^{-9}	3.15×10^{-9}
Twist rate, $\theta'(0)$ (rad/m)	0.2982	1.1787
Free-end twist, $\theta(L)$ (deg)	20.52°	101.34°

For convenience of practical interpretation, the main numerical results of **Examples 1 and 2** are summarised in **Table 1**, highlighting the sensitivity of stresses and torsional response to variations in geometry and loading type.

7. Conclusions

This study has presented a unified analytical framework for the investigation of flexural shear, bending, torsion, and warping in thin-walled cantilever channel beams. The formulation was established on the basis of classical thin-walled beam theory [4]-[6] and was extended to explicitly incorporate the coupled effects arising from non-uniform shear flow, Saint-Venant torsion, and restrained warping [4] [10]. The influence of the shear centre and load eccentricity was also systematically addressed [4] [5].

Closed-form expressions were derived for the internal force resultants, shear flow distribution, torsional rotation, and warping-related stresses. The developed model enables a consistent evaluation of both normal and shear stress components under combined transverse loading and torsional effects, which cannot be accurately captured by uncoupled bending or torsion theories [14] [15].

Two independent numerical examples were presented to demonstrate the practical applicability of the proposed formulation. The results confirmed that, for typical thin-walled channel geometries, bending stresses generally dominate the structural response [3] [13], while web shear stresses remain of moderate magnitude [7] [8]. However, the examples also showed that even relatively small eccentricities of the applied load with respect to the shear centre may induce appreciable torsional rotations and additional stress components that should not be neglected in design [4] [5]. The importance of restrained warping near the fixed end of the cantilever was also highlighted, particularly with regard to its contribution to the maximum longitudinal stress [4] [6].

The numerical results were found to be physically consistent with the predictions of classical thin-walled beam behaviour [4] [5], while at the same time illustrating the necessity of a fully coupled flexural-torsional-warping analysis for open-section cantilever beams [10] [11] [16]. The proposed formulation therefore provides a reliable and transparent theoretical tool for the assessment and design of such structural members.

Future work may extend the present analysis to include:

- geometric and material nonlinearities [12],
- interaction with local buckling of thin plates [8] [9],
- dynamic loading effects,
- and experimental validation of the predicted stress and deformation fields.

The analytical framework developed herein is expected to be of practical value for structural engineers and designers dealing with thin-walled open-section beams in mechanical, civil, and aerospace engineering applications, where combined bending, shear, torsion, and warping effects play a decisive role in structural performance [1] [3] [13].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Williams, D. (1960) Theory of Aircraft Structures. Edward Arnold.

[2] Megson, T.H.G. (1972) Aircraft Structures for Engineering Students. Edward Arnold.

[3] Megson, T.H.G. (2019) Structural and Stress Analysis. 3rd Edition, Butterworth-Heinemann.

[4] Vlasov, V.Z. (1961) Thin-Walled Elastic Beams. Oldbourne Press.

[5] Gjelsvik, A. (1981) The Theory of Thin-Walled Beams. John Wiley & Sons.

[6] Wagner, H. (1936) Report on Torsion and Bending of Open Sections. NACA Technical Memorandum No. 807.

[7] Ugural, A.C. and Fenster, S.K. (2003) Advanced Strength and Applied Elasticity. 4th Edition, Prentice Hall, Pearson Education.

[8] Bleich, F. and Bleich, H.H. (1952) Buckling Strength of Metal Structures. McGraw-Hill.

[9] Hoff, N.J. (1943) Buckling of Thin-Walled Members. *Journal of the Royal Aeronautical Society*, **87**, 35-83.

[10] Alsheikh, A.M.S. and Rees, D.W.A. (2021) General Stiffness Matrix for a Thin-Walled, Open-Section Beam Structure. *World Journal of Mechanics*, **11**, 205-236. https://doi.org/10.4236/wjm.2021.1111015

[11] Alsheikh, A.M.S. and Rees, D.W.A. (2022) Transformation Matrix for Combined Loads Applied to Thin-Walled Structures. *World Journal of Mechanics*, **12**, 65-78. https://doi.org/10.4236/wjm.2022.126006

[12] Timoshenko, S.P. and Goodier, J.N. (1970) Theory of Elasticity. 3rd Edition, McGraw-Hill.

[13] Rees, D.W.A. (2015) The Mechanics of Engineering Structures. Imperial College Press.

[14] Boresi, A.P., Schmidt, R.J. and Sidebottom, O.M. (1993) Advanced Mechanics of Materials. John Wiley & Sons.

[15] Rees, D.W.A. (2016) Mechanics of Solids and Structures. 2nd Edition, Imperial College Press.

[16] Al-Sheikh, A.M.S. (1985) Behaviour of Thin-Walled Structures under Combined Loads. Ph.D. Thesis, Loughborough University of Technology.

List of Symbols

Symbol	Description	Units
A	Cross-sectional area	m^2
A_{web}	Web area	m^2
A_{flange}	Flange area	m^2
b	Flange width	m
c	Distance to extreme fibre	m
E	Young's modulus	Pa
G	Shear modulus	Pa
h	Web height	m
I_y, I_z	Second moments of area about y- and z-axes	m^4
I_b	Second moment of area about bending axis	m^4
I_ω	Warping constant	m^6
J	Saint-Venant torsional constant	m^4
L	Beam length	m
M_y, M_z	Bending moments about y- and z-axes	N·m
P	Concentrated longitudinal load	N
p	Distributed load	N/m
q	Shear flow	N/m
$q(s)$	Shear flow along contour	N/m
q_0	Shear flow integration constant	N/m
s	Curvilinear coordinate along mid-line	m
t	Wall thickness	m
T	Torsional moment	N·m
T_0	Applied torsional moment	N·m
T_w	Warping torsional moment	N·m
V_y, V_z	Transverse shear forces	N
w	Warping displacement	m
x	Longitudinal coordinate	m
y, z	Cross-sectional coordinates	m
y_s, z_s	Shear centre coordinates	m
θ	Twist angle	rad
θ'	Twist per unit length	rad/m
θ''	Second derivative of twist	rad/m^2
τ	Shear stress	Pa
τ_{SV}	Saint-Venant shear stress	Pa
σ_b	Bending normal stress	Pa
σ_w	Warping normal stress	Pa
σ_x	Total longitudinal normal stress	Pa
$\omega(y, z)$	Warping function	m^2
ν	Poisson's ratio	–
$\phi(s)$	Inclination angle of wall segment	Rad