

# Oscillation Analysis of a Bifilar Pendulum with *Mathematica*

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## Abstract

Utilizing a Computer Algebra System (CAS), namely *Mathematica*<sup>®</sup>, the characteristics of a bifilar disk-shaped pendulum have been studied. By applying the Lagrangian methodology, the disk's motion equation is formulated. This is conducive to an ODE, and its numeric solution co-insides with intuitive expectation. The period of the oscillations and tension in the strings are calculated and graphed.

## Keywords

Bifilar Pendulum, Oscillation Period, ODE, *Mathematica*

## 1. Introduction

The main differences between a typical bifilar vs. a torsional pendulum are: 1) a typical circular-shaped disk of the former is suspended with two massless unstretchable strings, while the latter is hung only one elastic soft twist-able rod “string”, and 2) twisting the disk of the former lifts the disk while the disk of the latter stays at the same initial height. The analysis of the latter, torsion pendulum, is well-known [1]. At the same time, literature and physics textbooks [2] [3] have overlooked the issue, especially with the Lagrangian methodology.

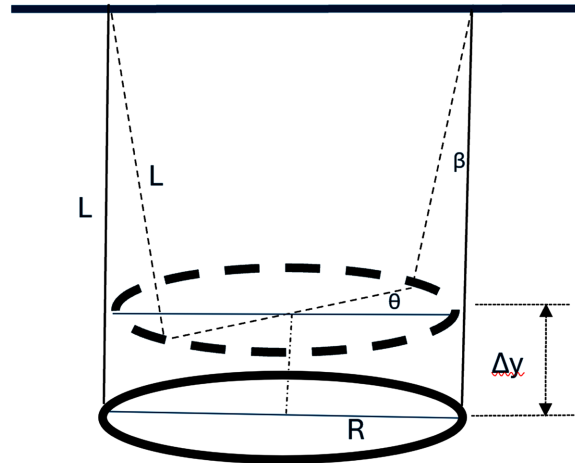
This short study presents a simple approach to analyzing the characteristics of a bifilar circular disk. Contributions of this report are: 1) Lagrangian methodology and 2) explicit utilization of a CAS, *Mathematica* [4] conducive to a practical situation.

This article is composed of three sections. In Section 2, the Lagrangian method [5] is introduced. The corresponding Equation of Motion is solved. By utilizing its solution, all the related kinematic quantities needed are determined, and some are plotted. The last section, Conclusions, addresses the achievements and sug-

gests further challenging explorations.

## 2. Formulation and Coding

The profile of a circular disk-shaped bifilar pendulum is shown in **Figure 1**.



**Figure 1.** A circular disk of radius  $R$  is suspended horizontally with *two* vertical strings attached to the ends of a diagonal.

A disk of mass  $m$  and radius  $R$  is suspended with *two* massless strings of identical lengths  $L$ . The dashed circle shows the “displaced/twisted” initial circle after being lifted  $\Delta y$  vertically due to twisted angle  $\theta$ . The  $\beta$  is the angle the string of the twisted disk makes concerning its initial vertical position. The strings are connected at opposite ends of a diagonal of the disk perpendicular to the surface of the circle. The objective is to turn/twist the disk from its natural hanging position, CW or CCW, by an angle  $\theta$  and then release it free. The applied torque and elasticity of the strings make the disk pass its initial natural position, repeating the process, in other words, making the disk oscillate. The significant difference between the situation explained and that of a torsion pendulum is the vertical height the disk gains through oscillations. Given the Lagrangian methodology, it corresponds to the gravitational potential energy, *i.e.*,  $V = mg\Delta y$ . Where  $\Delta y$ , shown in **Figure 1**, is the gained/lost height during oscillations. Specifically,

$$\Delta y = L(1 - \cos[\beta]), \tag{1}$$

$\beta$  is the angle the strings make it with vertical positions.

In other words, the potential energy  $V$  is,

$$V = mg\Delta y, \tag{2}$$

The disk begins rotating about COM. The rotational kinetic energy of the disk is,  $1/2 I_0 \omega^2$  [2] [3] where the moment of inertia of the disk about its center is  $I_0 = \frac{1}{2} mR^2$  [2] [3]. Putting this information together yields the Lagrangian  $L$ ;

$$L = T - V = 1/2 I_0 \dot{\theta}^2 - mg\Delta y, \tag{3}$$

where  $\omega = \dot{\theta}$  is the angular velocity of the disk.

Applying Euler-Lagrange [4] equation,  $\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$ , yields,

$$\ddot{\theta}(t) + 2g/R \sin[R/L\theta(t)] = 0, \quad (4)$$

$\ddot{\theta}(t)$  is the second-order time derivative. The argument of the Sin in (4) comes about from the geometry shown in **Figure 1**, namely, equating the circular arc-length  $L\beta = R\theta$ . By utilizing the latter, the argument of the Sin in (4) is established. Equation (4) is independent of mass,  $m$ . They are hinting at the independence of the oscillation period to mass.

Although formally, one might claim the goal of this article is reached, to put the results in action, a set of practical values for the quantities involved is suggested. These are stored in the **values** listing,

$$\mathbf{values} = \{g \rightarrow 9.8, R \rightarrow 1010^{-2}, \ell \rightarrow 0.5, m \rightarrow 0.5\};$$

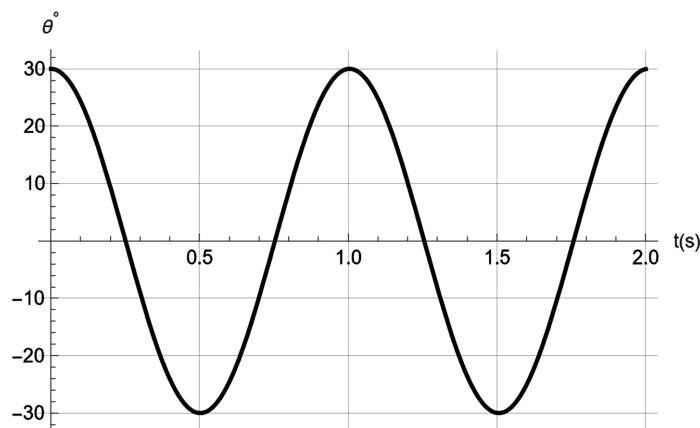
The symbols are self-explanatory, and the units are MKS. See **Figure 1** to correspond. *Mathematica* code forming (4) follows,

$$\mathbf{eq\theta1} = \theta''[t] + 2(g/R)\text{Sin}[(R/\ell)\theta[t]]/.values;$$

$$\mathbf{sol\theta1} = \text{NDSolve}[\{\mathbf{eq\theta1} == 0, \theta[0] == \pi/6, \theta'[0] == 0\}, \theta, \{t, 0, 10\}];$$

graph of the solution of the above code converted in degrees,

$$\mathbf{plot\theta1} = \text{Plot}[(180/\pi)\theta[t]/.\mathbf{sol\theta1}, \{t, 0, 2\}, \text{PlotStyle} \rightarrow \text{Black}, \text{AxesLabel} \rightarrow \{t(\text{s}), \theta^\circ\}, \text{GridLines} \rightarrow \text{Automatic}]$$



**Figure 2.** Oscillating angle vs. time. The initial twisted angle is set at  $30^\circ$ .

**Figure 2** shows the oscillations of the twisted angle of the disk vs. time on the horizontal plane.

Equation (4) is a nonlinear ODE. Its nonlinear term may be linearized, attempting to convert it into an SHM. And by Taylor [6] expansion to the 3<sup>rd</sup> order, the  $\mathbf{eq\theta1}$  yields,

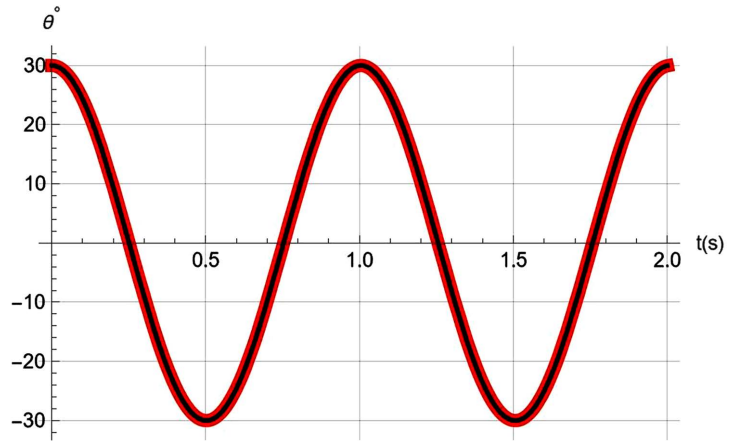
$$\mathbf{eq\theta2} = \theta2''[t] + 2g/R \text{Normal}[\text{Series}[\text{Sin}[R/\ell \theta2[t]], \{\theta2[t], 0, 3\}]]/.values;$$

$$\mathbf{sol\theta2} = \text{NDSolve}[\{\mathbf{eq\theta2} == 0, \theta2[0] == \pi/6, \theta2'[0] == 0\}, \theta2, \{t, 0, 10\}];$$

The solution is plotted in **Figure 3**.

```
plotθ2 = Plot[(180/π)θ2[t]/.solθ2, {t, 0, 2}, AxesLabel → {"t(s)", "θ"},
  PlotStyle → {Red, Thickness[0.02]}, GridLines → Automatic];
```

The last two solutions are compared,

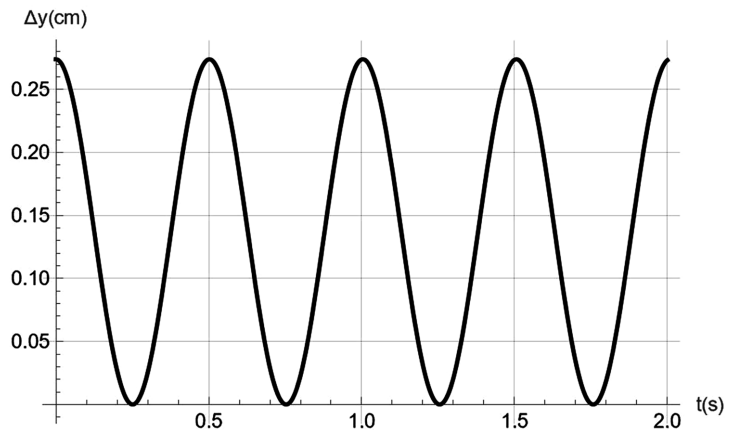


**Figure 3.** The original oscillations, eqθ1, in black is, compared to the 3<sup>rd</sup> order Taylor expanded, eqθ2, in red.

According to **Figure 3**, there is no need to replace the Sin function with the Taylor polynomials; the results are the same.

$$\Delta y = \ell(1 - \text{Cos}[R/\ell\theta[t]/.\text{sol}\theta1])/.values;$$

```
Plot[100Δy, {t, 0, 2}, AxesLabel → {t(s), Δy(cm)}, GridLines → Automatic,
  PlotStyle → Black]
```



**Figure 4.** The vertical oscillating graph of the height vs. time.

The vertical axis is in CGS units.

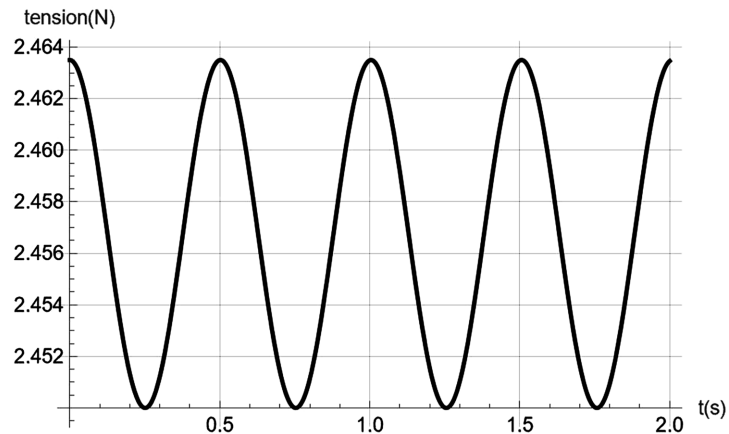
As shown in **Figure 4**, the disk vibrates vertically. Its oscillation period is different from the one shown in **Figure 2**.

Because of the setup’s symmetry, the strings’ tensions are the same. The tension in the string is  $T(\theta) = 1/2mg \cdot 1/\text{Cos}[\beta]$ , as depicted in **Figure 5**.

```

Tension = (1/2mg/.values) (1/Cos[R/lθ[t]/.solθ1]/.values);
Plot[Tension, {t, 0, 2}, AxesLabel → {"t(s)", "tension(N)"}, PlotStyle → Black,
GridLines → Automatic, PlotRange → All]

```



**Figure 5.** Tension in either one of the strings in Newtons vs. time.

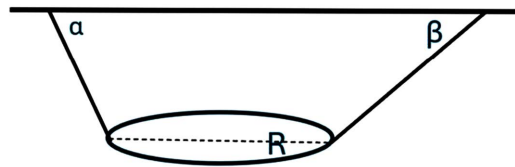
Because of the symmetry of the case at hand, tensions in the strings are the same.

Finally, based on the given information, the evaluation of the oscillation period is straightforward, as shown in **Figure 3**; it is 1 second. **Figure 3** shows no difference between the oscillation period corresponding to the exact equation of motion. This is the graph of eqθ1 in the black curve vs. the third-order Taylor expanded version, eqθ2, in the red curve. Hence, because there is no apparent difference between the black and red curve, one may develop the eqθ1 to a “less accurate” version, *i.e.*, to the first order linearizing the equation conducive to SHM with an associated standard period  $T = 2\pi\sqrt{\frac{\ell}{2g}}$ . Its value for the chosen numeric parameters listed in **values** yields the mentioned 1.00 s. This coincides with the period depicted in **Figure 3**.

### 3. Conclusions

The motivation for crafting this report is to show a simple approach to the concept of a bifilar pendulum [7]-[9] utilizing the usage of a CAS, namely *Mathematica* harvesting numeric and graphic aspects of the relevant quantities. The analysis of a bifilar pendulum needs to be included in the popular college texts, and its associated Lagrangian metrology needs to be included in the literature. The *Mathematica* codes are embedded in this report so the interested reader may practice. The presented simple case may be generalized by making assumptions that, *e.g.*, the vertical string might not be vertical see **Figure 6**. *i.e.*, each string may create a different angle with vertical, or the disk may have a different COM other than the center of the disk and the strings may have different lengths, the suggested scenarios might be challenging but are do-able. The *Mathematica* codes given may

be helpful! Interested readers in *Mathematica* coding might find [10]-[12] resourceful.



**Figure 6.** Asymmetric bifilar pendulum.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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