Technology and Investment, 2026, 17(1), 36-76
https://www.scirp.org/journal/ti
ISSN Online: 2150-4067
ISSN Print: 2150-4059

A Three-Variable Contingent Claims Residential Mortgage Valuation Model

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How to cite this paper: Chatterjee, A. (2026). A Three-Variable Contingent Claims Residential Mortgage Valuation Model. *Technology and Investment, 17,* 36-76.
https://doi.org/10.4236/ti.2026.171005

Received: December 3, 2025
Accepted: January 25, 2026
Published: January 28, 2026

Abstract

The mortgage valuation literature is saturated with numerous studies using the contingent claims approach to value mortgages. So far, efforts are directed into two alternative model frameworks. First, the pricing of a prepayable mortgage is made possible by using short- and long-term interest rates as relevant variables. Second, a defaultable mortgage is valued with short-term interest rate and building value as explanatory variables. However, the comprehensive valuation of a defaultable and prepayable residential mortgage requires three variables. This study proposes a three-variable model, in which the pricing of a defaultable and prepayable residential mortgage is explained by short- and long-term interest rates, as well as the building value. The model incorporates both fixed-rate and adjustable-rate mortgages. Loan-level residential mortgage data is used for empirical analysis. Valuation results indicate a positive pricing spread between the primary market and the theoretically estimated value. Mean, median, and variance-based statistical studies suggest that the three-variable model is generally efficient in predicting primary mortgage prices, with efficiency being more pronounced for longer-term mortgages. Nonparametric Kernel Density regression analysis reveals that model efficiency increases with longer-term and larger mortgages.

Keywords

Residential Mortgages, Three-Variable Contingent Claims Mortgage Valuation Model, Explicit Finite Difference Method, Default, Prepayment, Adjustable-Rate Options

1. Introduction

The contingent claims model, introduced by Merton (1974), serves as a valuation framework that interprets corporate debt and equity as contingent claims on the firm's aggregate value. As an extension of the Black & Scholes (1973) option pricing model, this approach has become a significant instrument in corporate finance and risk assessment, especially when evaluating assets with uncertain future returns. Researchers have also employed the contingent claims valuation methodology to assess the value of mortgages. Dunn & McConnell (1980, 1981b) valued default-free, mortgage-backed GNMA securities as a function of the instantaneous risk-free interest rate and time to maturity. Following a similar approach, Cunningham & Hendershott (1984) examined the default option of a borrower, while Buser & Hendershott (1984) considered the prepayment option of a mortgage contract. On the other hand, Bhattacharya et al. (2019) propose a proportional hazards model for the simultaneous detection of default and prepayment behavior among borrowers.

Under restrictive assumptions, Collin-Dufresne & Harding (1999) develop a closed-form solution for pricing a prepayable fixed-rate residential mortgage whose value is defined as a function of the instantaneous risk-free rate and time. According to the authors, the single-factor representation is used only at a price by sacrificing the accuracy of mortgage cash flow estimation. Replicating Dunn & McConnell's (1980) work with single-factor representation (short-term interest rate) of mortgage pricing, Shilling (1995) rejects the one-variable general framework based on empirical findings. Besides full prepayment risk, Jones & Chen (2016) also considered the possibility of partial prepayment risk, as well as the potential for complete loan repayment through mortgage refinancing as a function of the short rate. Sadhwani et al. (2021) show that loan-specific and macroeconomic variables have a strong non-linear influence on borrowers' prepayment behavior.

In subsequent studies, the one-variable framework has been extended to two-variable models. Research in the two-variable framework has taken two separate directions. The resulting models have differed according to the choice of two of the three possible state variables: short- and long-term interest rates, and building value. One dual-variable model utilizes short- and long-term interest rates to explain the term structure for valuing a default-free, prepayable mortgage. Leading proponents of this framework include studies by Brennan & Schwartz (1982, 1983, 1985), McConnell & Muller (1988), Buser et al. (1990), Schwartz & Torous (1989a, 1989b, 1991), McConnell & Singh (1993, 1994), Van Bussel (1997), and Levin (1999). Archer & Ling (1995) and Chen & Yang (1995) report on the empirical validity of using alternative interest-rate-process models.

On the other hand, the building value as a state variable is required for valuing defaultable mortgages. In this setting, the term structure is modeled by the spot rate, while the building value (the second state variable) is used to capture the default characteristics of the asset. Since the advent of the subprime mortgage market crisis in 2007, mortgage default risk has taken on additional significance in the mortgage valuation literature. Foster & Van Order (1984, 1985), Epperson et al. (1985), Titman & Torous (1989), Kau et al. (1987, 1990a, 1990b, 1992, 1993, 1994, 1995), Schwartz & Torous (1992), Ambrose & Capone (1996, 1998), Capozza et al. (1998), and Hilliard et al. (1998) utilize similar two-state variable models. Downing et al. (2005) provide strong empirical evidence of the inclusion of building value as a state variable for defaultable and prepayable mortgages. Ever since the onset of the mortgage market crisis in 2007-2009, Ngene et al. (2016) and Schelkle (2018) observe that default risk has become increasingly crucial in mortgage valuation. Using improved data sources from the mortgage crisis, Foote & Willen (2018) addressed the causality between falling housing prices and rising foreclosures.

Harrison et al. (2002) investigate the exponential GARCH (EGARCH) conditional volatility estimates of house prices and interest rates, as well as their influence on mortgage termination decisions. The results complement earlier findings that house price volatility affects default option values, while interest rate volatility impacts prepayment decisions. This observation complements the findings of Sharp et al. (2008) and Yilmaz & Selcuk-Kestel (2019). Chernov et al. (2018) find that borrower prepayment probabilities are greatly affected by the Federal Reserve's quantitative easing programs in the aftermath of the 2007-2008 financial crisis. Conversely, Chen et al. (2021) observe that mortgage market liquidity and house price volatility during and after the financial crisis have a profound impact on default probability. Zhou et al. (2024) investigate the valuation issues of a mortgage contract that simultaneously considers full prepayment and default risk as linear functions of the risk-free interest rate and house prices. They find that the valuation of the mortgage decreases in both cases of prepayment and default options, with the decrease being greater in the case of default (due to the possibility of losing a portion of the outstanding principal and interest income) than in the case of prepayment (due to the expected loss of interest income).

Clapp et al. (2000) and Pavlov (2001) further include the difference between the optimal and actual bundle of housing services (describing the changes in individual circumstances over time as regional, neighborhood, and property characteristics change) as a stochastic variable to model mortgage-termination choices (prepay or default). In a similar vein, Buist & Yang (1998) and Yang et al. (1998) add borrower income as a variable to determine borrower decisions in early termination (default or prepayment). These studies, however, are not directed towards mortgage value determination. Dunn & McConnell (1981a), Hendershott & Van Order (1987), Kau & Keenan (1995), Vandell (1995), and Chatterjee et al. (1998) provide a comprehensive review of contingent claims mortgage valuation literature. Davidson & Levin (2014) show the systematic progression of contingent claims mortgage valuation literature over the years.

No prior research, however, comprehensively addresses the issues of valuing a residential mortgage originated with either a fixed-rate or an adjustable-rate interest rate, and that can be prepaid or defaulted on before the maturity of the loan contract. The complexity arises from the fact that a mortgage is typically prepaid and refinanced during periods of declining interest rates. However, declining interest rates do not preclude collapsing housing values. Therefore, when house prices fall below loan values, rational borrowers are often induced to default on their mortgage contracts. In a general valuation framework, Campbell & Cocco (2003) investigated the optimal choice between adjustable and fixed-rate mortgages under income uncertainty, borrower risk aversion, end-of-period house price variability, refinancing options, and borrower credit constraint. In a similar vein, Campbell & Cocco (2015) demonstrated that loan-to-value ratios and mortgage affordability play a crucial role in determining whether to select an adjustable-rate or fixed-rate mortgage, as well as their corresponding effects on default and prepayment decisions. The study's observations help explain the higher default rates on adjustable-rate mortgages during the 2007-2009 mortgage market crisis.

Our study aims to capture the unique aspects of adjustable-rate mortgages (ARMs) within a comprehensive model framework. An ARM is issued at a variable rate with periodic adjustments tied to the movements of a specific index rate, thereby hedging against interest rate risk. The discussion and appropriate modeling of adjustable-rate mortgages are necessitated due to the presence of variable-rate mortgages in the primary market data set. The Adjustable-rate mortgage (ARM) originated in response to significant interest rate fluctuations during the late 1970s and early 1980s. Prior to the 2008 financial crisis, ARMs frequently formed part of the subprime mortgage market, often being extended to borrowers with lower credit ratings. In the aftermath of the crisis, the prevalence of ARMs declined, accompanied by the introduction of more rigorous underwriting standards. Recently, ARMs have experienced renewed interest as borrowers seek to reduce their initial monthly payments, particularly in settings characterized by elevated interest rates. Through the numerical solution of their proposed model, Chiang & Sa-Aadu (2014) show that even in the aftermath of the 2008 financial crisis, pay option adjustable-rate mortgages (PO-ARM) keep relevance for borrowers as a way to mitigate liquidity and/or affordability constraints, and for those with a shorter horizon in living in the property.

Kau et al. (1995, 1990b, 1993) address the path dependency problem of valuing adjustable-rate mortgages by introducing the past contract rate as an auxiliary state variable, which highlights the critical importance of the index rate and periodic adjustment characteristics in valuing an ARM. Statman (1982), Dhillon et al. (1987), and Brueckner & Follain (1988) observe that an ARM is often booked at a lower rate than comparable fixed-rate mortgages. Kau et al. (1993) further observe that prepayment has a greater value, while default has a lower value for a fixed-rate mortgage than an ARM. Brueckner (1992, 1993, 1994) observes that inflation uncertainty, expected mobility of the borrower, and anticipated increase in market interest rate and borrower income increase the possibility of choosing an ARM, while Posey & Yavas (2001) conclude that high-risk borrowers prefer ARM.

A series of papers by Brennan & Schwartz (1982, 1983, 1985) demonstrates that the term structure is best explained by two interest rate variables: the short-term (spot) rate and the long-term (consol) rate. Therefore, a prepayable mortgage valuation depends upon these two interest rate processes. A third state variable not necessarily correlated with the interest rate process is needed to address prepayment and default issues simultaneously. We use building value as the third state to value a defaultable and prepayable mortgage that can also be utilized to value adjustable-rate mortgages with additional boundary conditions. Unlike the valuation of residential mortgages using the primary market loan data, Kariya et al. (2011) demonstrate the comprehensive application of a three-factor valuation model, employing the same three state variables, to comprehensively value mortgage-backed securities in the secondary market.

Under the proposed framework, all existing one- and two-variable models can be embedded within the proposed model. The study employs a backward-solving explicit finite difference methodology to value mortgages. Subsequently, we test our model with primary residential mortgage data to analyze its valuation efficiency. The strength of this paper lies in its use of loan-level residential mortgage data, unlike much of the mortgage valuation literature, which has employed simulations or aggregate data to test the efficiency of its models.

The remainder of this article is organized as follows. The following section presents the three-variable mortgage valuation model, along with appropriate terminal and boundary conditions for prepayment and default options, as well as conditions for adjustable-rate mortgages. Section 3 describes the valuation methodology for the model estimation. Section 4 presents the data and empirical findings, while Section 5 provides conclusions regarding the model's efficiency and primary market pricing.

2. The Contingent Claims Mortgage Valuation Model

The section begins with a description of the general framework for valuing derivative securities. A three-state variable residential mortgage valuation model is a restricted version of the general model. The stochastic processes of the state variables are also reported. Finally, the necessary terminal and boundary conditions for prepayment and default options are described to close the model.

2.1. General Framework

Cox et al. (1985) and Brennan & Schwartz (1985) provide the general theory of securities valuation, where a fundamental partial differential Equation (PDE) must be satisfied by the value of any derivative security. In this context, the valuation of a mortgage can be viewed as a special case of the general valuation of contingent claims.

It is assumed that the prices of all derivative securities are uniquely determined by time t and n-vector of state variables $S_1, S_2, \cdots, S_n$. The variables follow a joint stochastic process, which is described as

$$S=\left[S_1(t),S_2(t),\cdots,S_n(t)\right] \quad (1)$$

$$\mathrm{d}S_i=\alpha_i\mathrm{d}t+\beta_j\mathrm{d}Z_i \qquad \forall i=1,2,\cdots,n \quad (2)$$

The parameters of the process are α_i and β_j. Here, α_i represents the drift and β_j represents the instantaneous standard deviation of the i-th, dZ_i represents the increment of a Wiener process, such that

$$\mathrm{d}Z_i\mathrm{d}Z_j=\rho_{ij}\mathrm{d}t \quad (3)$$

It depicts the instantaneous correlation structure between the two Wiener processes, and the correlation coefficient is represented by ρ_{ij}. Consequently, the value of any security G can be written as a function of the state variables and time as follows:

$$G=G(S;t) \quad (4)$$

Using Ito's Lemma, the instantaneous change in the security value can be expressed as a diffusion process. The expression for the change in the value of G can be described as

$$\mathrm{d}G=X\mathrm{d}t+Y\mathrm{d}Z \quad (5)$$

Here, X represents the drift and Y represents the instantaneous standard deviation. The specific expressions for X and Y are written as

$$X\mathrm{d}t=\left[\sum_{i=1}^{n}\alpha_iG_i+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}G_{ij}\rho_{ij}\beta_i\beta_j+G_t\right]\mathrm{d}t \quad (6)$$

$$Y\mathrm{d}Z=\sum_{i=1}^{n}G_i\beta_i\mathrm{d}Z_i \quad (7)$$

Here, G_i and G_t represent the partial derivatives of the value of the security G with respect to the i-th state variable (S_i) and time (t) respectively. To ensure no arbitrage profit, the risk-return equilibrium condition by Merton (1973) is imposed on X and Y, such that

$$X=rG+\sum_{i=1}^{n}L_iG_i\beta_i-C \quad (8)$$

Here, r stands for the instantaneous short-term interest rate (the rate of return on risk-free investment), L_i is the market value of risk for the i-th state variable (S_i), while $C=C(S,t)$ is the instantaneous payout rate for the security. Equating the two values of the drift X from Equations (6) and (8), the expression for $G=G(S,t)$ is rewritten as

$$\sum_{i=1}^{n}G_i\left(\alpha_i-L_i\beta_i\right)+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}G_{ij}\rho_{ij}\beta_i\beta_j+G_t+C-rG=0 \quad (9)$$

This equation is referred to as the Fundamental Partial Differential Equation (PDE) for Contingent Claims by Cox et al. (1985). In principle, solving this PDE is sufficient to determine the value of any security, once the relevant terminal and boundary conditions for the security and the instantaneous payout function are defined.

2.2. Stochastic Processes

All information about the term structure of interest rates is assumed to be summarized by two state variables: r, the instantaneous risk-free short-term interest rate (the spot rate) and l, the yield on a bond with infinite maturity (the consol rate). We use the interest rate processes suggested by Brennan & Schwartz (1985), which are described as

$$\mathrm{d}r = \left(a_r + b_r\left(l - r\right)\right)\mathrm{d}t + \sigma_r r\mathrm{d}z_r \tag{10}$$

$$\mathrm{d}l = \left(a_l + b_l r + c_l l\right)\mathrm{d}t + \sigma_l l\mathrm{d}z_l \tag{11}$$

$$\mathrm{d}z_r\mathrm{d}z_l = \rho_{rl}\mathrm{d}t \tag{12}$$

Here, $\mathrm{d}z_r$ and $\mathrm{d}z_l$ are standardized Wiener processes, while ρ_{rl} is the instantaneous correlation coefficient between r and l. Also, σ_r and σ_l are the respective standard deviations, and a's, b's and c's are the coefficients for risk premium and speed-of-adjustment parameters. These specifications imply that the scales of unanticipated changes in both r and l are proportional to their current values. Furthermore, r reverts to the current value of l, while l itself varies stochastically over time.

All information about the values of building collateral is assumed to be summarized by the state variable B (the value of the mortgaged building). The stochastic process follows a lognormal diffusion process. Following Epperson et al. (1985) and Titman & Torous (1989), the specific form of the process can be written as

$$\mathrm{d}B = \left(\alpha - b'\right)B\mathrm{d}t + \sigma_B B\mathrm{d}z_B \tag{13}$$

$$\mathrm{d}z_r\mathrm{d}z_B = \rho_{rB}\mathrm{d}t \tag{14}$$

Here, α is the instantaneous total expected rate of return to the asset and b' is the continuous payout rate generated by the building. Accordingly, $\left(\alpha - b'\right)$ is the instantaneous mean rate of appreciation in property value, $\mathrm{d}Z_B$ is a standardized Wiener process, and ρ_{rB} is the instantaneous correlation coefficient between r and B. The instantaneous correlation between the building value process and the long-term yield process (ρ_{lB}) is assumed to be zero and (ρ_{rB}) is not restricted.

2.3. A Three-Variable (*r*-*l*-*B*) Model

The model proposes that the value of a mortgage is uniquely determined by time t and three state variables, which are as follows: r (the spot rate), l (the consol rate), and B (the value of the mortgaged building). Accordingly, the values of four assets—two default-free bonds (an instantaneous risk-free bond and a long-term bond with infinite maturity), the building, and the collateralized mortgage—can be determined as functions of three variables and time. Two interest rates determine the values of the bonds, while the mortgage value is calculated based on the two interest rates and the building's value. Therefore, two default-free bonds and the mortgage can be combined into an instantaneously risk-free dynamic portfolio. As illustrated by Cox et al. (1985), this no-arbitrage condition assures that the value of any derivative asset, such as a mortgage, can be derived from a partial differential Equation (PDE) with appropriate boundary and initial conditions.

The PDE is derived directly from the fundamental partial differential equation described in Equation (9). By letting the number of state variables equal three ($n = 3$), such that the price of the derivative security is uniquely determined by three state variables and time $\left[G = G\left(S_1, S_2, S_3; t\right)\right]$, Equation (9) can be rewritten as

$$\sum_{i=1}^{3} G_i\left(\alpha_i - L_i\beta_i\right) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} G_{ij}\rho_{ij}\beta_i\beta_j + G_t + C - rG = 0 \tag{15}$$

The valuation equation for the mortgage value $V\left(= V\left(r, l, B; t\right)\right)$ is obtained by replacing the state variables $\left(S_1, S_2, S_3\right)$ with r, l and B respectively. The value of the security G is replaced by the value of the mortgage (V). The instantaneous payout rate (C) is replaced by the continuous rate of mortgage payment (m). In addition, it is recognized that $\rho_{rr} = \rho_{ll} = \rho_{BB} = 1$ (the correlation coefficient of each variable with itself); $\rho_{rl} = \rho_{lr}$ and $\rho_{rB} = \rho_{Br}$ (the transpose of each correlation coefficient); $\rho_{lB} = \rho_{Bl} = 0$ (by assumption); and $V_{rl} = V_{lr}$, $V_{rB} = V_{Br}$ and $V_{lB} = V_{Bl}$ (the transpose of respective partial derivatives). The corresponding correlation coefficient and its transpose are equal for two scalar variables. Similarly, a partial derivative and its transpose of a scalar variable are equal. Following Titman & Torous (1989), the instantaneous correlation between l and B $\left[\rho_{lB} = \rho_{Bl}\right]$ is assumed to be zero. Substituting these relationships into Equation (15), the PDE is rewritten as

$$\begin{aligned}&\left(\alpha_r - L_r\beta_r\right)V_r + \left(\alpha_l - L_l\beta_l\right)V_l + \left(\alpha_B - L_B\beta_B\right)V_B + \frac{1}{2}\beta_r^2 V_{rr} + \frac{1}{2}\beta_l^2 V_{ll} \\ &+ \frac{1}{2}\beta_B^2 V_{BB} + \beta_r\beta_l\rho_{rl}V_{rl} + \beta_r\beta_B\rho_{rB}V_{rB} + V_t - rV + m = 0\end{aligned} \tag{16}$$

Specific expressions for the drift (α_r), and variance (β_r) of the spot-rate (r) are given in Equation (10). Similarly, expressions for α_l and β_l are shown in Equation (11), while the expressions for α_B and β_B are provided in Equation (13). Substituting these values into Equation (16), the PDE is rewritten as

$$\begin{aligned}&\left[a_r + b_r\left(l - r\right) - L_r\sigma_r r\right]V_r + \left[a_l + b_l r + c_l l - L_l\sigma_l l\right]V_l \\ &+ \left[\left(\alpha - b'\right)B - L_B\sigma_B B\right]V_B + \frac{1}{2}\sigma_r^2 r^2 V_{rr} + \frac{1}{2}\sigma_l^2 l^2 V_{ll} \\ &+ \frac{1}{2}\sigma_B^2 B^2 V_{BB} + \sigma_r\sigma_l r l\rho_{rl}V_{rl} + \sigma_r\sigma_B rB\rho_{rB}V_{rB} + V_t - rV + m = 0\end{aligned} \tag{17}$$

The valuation equation in Equation (17) contains three "market prices of risk" for r, l and B. However, as Brennan & Schwartz (1982, 1985) observe, the no-arbitrage condition ensures that the market prices of risk for long-term yield (l) and the building value (B) can be expressed in terms of their derivatives with respect to r, l and B. This is analogous to the result in Black & Scholes (1973), which states that it is not necessary to know the value of a stock or its risk price to price an option on the stock. On the other hand, it is impossible to eliminate the "market price of risk of *r*" because the instantaneous security is not a traded asset. The substitution of the respective partial derivatives of the market prices of the consol rate (L_l) and building value risks (L_B) into Equation (17) produce the following expressions:

$$L_l = \frac{\dfrac{\left(a_l + b_l r + c_l l\right)}{l} - \left(\sigma_l^2 + l - r\right)}{\sigma_l} \tag{18}$$

$$L_B = \frac{\alpha - r}{\sigma_B} \tag{19}$$

The values of L_l and L_B are plugged back into Equation (17). After rearranging terms, Equation (17) is rewritten as

$$\begin{aligned}
&\frac{1}{2}\sigma_r^2 r^2 V_{rr} + \sigma_r \sigma_l r l \rho_{rl} V_{rl} + \frac{1}{2}\sigma_l^2 l^2 V_{ll} + \sigma_r \sigma_B r B \rho_{rB} V_{rB} + \frac{1}{2}\sigma_B^2 B^2 V_{BB} \\
&+ \left[a_r + b_r\left(l - r\right) - L_r \sigma_r r\right] V_r + l\left(\sigma_l^2 + l - r\right) V_l + \left(r - b'\right) B V_B \\
&+ V_t - rV + m = 0
\end{aligned} \tag{20}$$

Here, $V\left(=V\left(r, l, B; t\right)\right)$ is the value of the mortgage $\left[t \in \left(0, T\right)\right]$. With appropriate terminal and boundary conditions, the valuation Equation (20) represents the three-state variable valuation model. The solution to this equation provides the value of the mortgage as a function of three state variables and time.

2.4. Default and Prepayment Options

Default rationally occurs when the unpaid balance exceeds the value of the mortgaged building. Kau & Keenan (1995) and Capozza et al. (1998) note that the default option is typically not exercised except at payment dates, as the borrower can continue to enjoy the property's services until a payment is due. Accordingly, at each payment date *k*, the value of the default option is evaluated as follows:

$$DEF_k = \begin{cases} DEF_{k+1} & \text{if } \left[\sum_{j=k}^{n} M_j - DEF_{k+1} - PRE_{k+1}\right] < B_k \quad \left(\text{no default}\right) \\ \sum_{j=k}^{n} M_j - B_k & \text{Otherwise} \qquad\qquad\qquad\qquad\qquad \left(\text{default}\right) \end{cases} \tag{21}$$

Here, DEF_k is the value of the default option to the borrowers at the period k, M_j is the monthly mortgage payment, PRE_{k+1} is the value of the prepayment option to the borrowers at the period $k+1$, B_k is the building value at the period k, and n represents the time-to-maturity of the mortgage. If the building value is greater than the mortgage value at the period k (unpaid balance, adjusted for the next period's default and prepayment options), the default option is carried over to the next period unexercised (DEF_{k+1}). Otherwise, the difference between the unpaid balance and the building value represents the value of the default option.

Similar to Kau et al. (1987, 1992), Kau et al. (1994) and Capozza et al. (1998), the paper hypothesizes that default results in immediate loss of the house (a foreclosure and immediate sale by the lender). The issues surrounding under-exercising the default option, mortgage delinquency, and delays in foreclosure, and how these influence the value of the default option, are not modeled in this study. These options are demonstrated in a series of works by Kau & Kim (1994), Vandell (1995), Ambrose & Capone (1996), Ambrose & Capone (1998), Pavlov (2001), and Downing et al. (2005).

Prepayment occurs when the loan's contract rate exceeds the prevailing refinancing rate. Kau et al. (1987, 1992), Kau et al. (1994) and Capozza et al. (1998) note that, unlike the default option, the prepayment option can be exercised between any two consecutive payment dates. Accordingly, at each payment date k, the prepayment option can take two positions, such as

$$PRE_k = \begin{cases} 0 & \text{(Borrower default)} \\ PRE_{k+1} \quad \text{Otherwise} & \text{(no default)} \end{cases} \tag{22}$$

At the payment date k, the prepayment option will have no value if a default occurs. Otherwise, it will be carried out unexercised to the next period. If prepayment occurs at any time t, the value of the prepayment option is described as

$$PRE_t = \sum_{j=t}^{n} M_j - \left[1 + c_k \left\{t - (k-1)\right\}\right] U_{k-1} \qquad \forall (k-1) < t \le k \tag{23}$$

Here, $\sum_{j=t}^{n} M_j$ represents the unpaid balance for the period t, c_k is the contract rate for the period k, U_{k-1} is the unpaid principal at the period $k-1$, and $\left[c_k \left\{t-(k-1)\right\} U_{k-1}\right]$ is the accrued interest on the unpaid balance for holding the mortgage during the period $(k-1)$ to t. The difference between the promised mortgage payment and the unpaid principal at the end of the period t represents the value of the prepayment option. Similar to the default option, the prepayment option framework in the paper does not consider suboptimal prepayment behavior (such as that associated with borrower relocation) that is largely independent of the mortgage value [Pavlov (2001)]. Azevedo-Pereira et al. (2003), Sharp et al. (2008), and Yilmaz & Selcuk-Kestel (2019) incorporate embedded default and prepayment options into mortgage contracts, allowing them to be simultaneously considered in the valuation process. In a similar vein, this study incorporates default and prepayment possibilities through boundary conditions. The construction of the default and prepayment options presented here aligns with the simultaneous choices of these options by borrowers, as proposed by Bhattacharya et al. (2019). Similarly, using the Cox-Ingersoll-Ross (CIR) type uniformly parabolic partial differential Equation (PDE) framework, Jones & Chen (2016) model the optimal prepayment behavior as a free boundary condition.

2.5. Boundary Conditions for Adjustable-Rate Mortgages

The most essential characteristic of an ARM is the variable nature of its contract rate. The pool of mortgage data under study also includes ARMs. Therefore, the variable nature of the contract rate of an ARM is captured under a terminal condition. The valuation of ARMs, thus, is conditional upon that additional terminal condition.

The special characteristic of an ARM is that the contract rate is adjusted at an annual interval. On each adjustment date, a new contract rate is calculated. The caps required that the new contract rate never exceed the original contract rate by more than the life-of-loan cap, and that the new contract rate never deviates from the previous contract rate by more than the yearly cap. Subject to these restrictions, the new contract rate can be expressed as:

$$c_k = \begin{cases} I_k + m & \text{if } c_k - c_{k-1} < x \\ c_0 + y & \text{if } c_k - c_0 > y \\ c_{k-1} + x & \text{if } c_k - c_{k-1} > x \end{cases} \tag{24}$$

where c_k, during the period k, is the current period's contract rate; I_k, during the period k, is the current index rate; m is the margin; x and y represent the yearly cap and life-of-loan cap, respectively. After the new contract rate at the adjustment period k is established, the mortgage payments M_k for the next twelve months, have been determined.

Under these circumstances, the unpaid balance at each subsequent period will be determined by the unpaid balance from the last period and the mortgage payment made in the previous period at the prevailing contract rate. Therefore, the terminal condition for valuing an ARM can be expressed as:

$$\sum_{j=k}^{n} M_j = \sum_{j=k+1}^{n} M_j + M_k \tag{25}$$

Here, the prevailing contract rates for the periods k and $k+1$ are c_k and c_{k+1}, respectively. $\sum_{j=k}^{n} M_j$ and M_k, the unpaid balance and mortgage payment for the period k, are determined by c_k. $\sum_{j=k+1}^{n} M_j$, the unpaid balance for the period $k+1$, is determined by c_{k+1}. At each adjustment period, the unpaid balance and the mortgage payment are determined by the prevailing contract rate. This framework retained the arguments espoused by Kau et al. (1995, 1990b) and Campbell & Cocco (2003) to aid this study. Lin & Ho (2005) demonstrate the unifying valuation framework for both adjustable-rate and fixed-rate mortgages in the presence of prepayment possibilities.

2.6. Terminal and Boundary Conditions

This section describes the initial boundary condition, several free boundary conditions, and the terminal boundary condition to close the model. In essence, the specifications are utilized to capture the extreme values in all state variables (r, l, B and t). The initial boundary condition states that at origination $(t = 0)$, the value of the mortgage must be equal to the value of the loan, minus the number of points paid at origination. Therefore, the mortgage value at origination is described as

$$V_0 = \sum_{j=0}^{n} M_j - DEF_0 - PRE_0 = (1-\delta) L \tag{26}$$

V_0 and L represent the value of the mortgage and the loan amount at origination, δ is the points paid at origination, while DEF_0 and PRE_0 are the corresponding default and prepayment options. The free boundary conditions are needed to capture the extreme values of the two interest rates and the building value. If the instantaneous risk-free and the consol rates are zero ($r=0$ and $l=0$), the corresponding PDE becomes a function of only the building value and time,

$$\frac{1}{2}\sigma_B^2 V_{BB} - b' B V_B - V_t = 0 \tag{27}$$

In a similar fashion, the infinite values of the interest rates ($r=\infty$ and $l=\infty$) produce a mortgage value equal to zero at the limit,

$$\lim_{r,\, l \to \infty} V(r, l, B; t) = 0 \tag{28}$$

If the building value is zero $(B=0)$, the prepayment option ceases to exist, and the value of the default option at any time would be equivalent to the total mortgage payment due at that instant, that is,

$$\begin{aligned} &PRE_t = 0 \\ &DEF_t = \sum_{j=t}^{n} M_j \quad (\forall\ k-1 < t < k) \end{aligned} \tag{29}$$

On the other hand, the infinite building value $(B=\infty)$ would produce a PDE as a function of only the instantaneous risk-free rate, the consol rate, and time. Also, the value of the default option is zero at the limit,

$$\begin{aligned} &\frac{1}{2}\sigma_r^2 r^2 V_{rr} + \sigma_r \sigma_l r l \rho_{rl} V_{rl} + \frac{1}{2}\sigma_l^2 l^2 V_{ll} + \left[a_r + b_r (l-r) - L_r \sigma_r r\right] V_r \\ &+ l\left(\sigma_l^2 + l - r\right) V_l + V_t - rV + m = 0 \\ &\text{and} \\ &\lim_{B \to \infty} DEF_t = 0 \end{aligned} \tag{30}$$

Last, the terminal boundary condition $(t=T)$ is expressed as follows:

$$V(r, l, B; T) = V_n = 0 \tag{31}$$

The time span $[0, T]$ denotes the term-to-maturity of each mortgage, and the expression states that the mortgage is fully amortized at the maturity date. Essentially, these boundary conditions, together with default, prepayment, and adjustable-rate mortgage options, provide the parameters that a PDE must satisfy. For accurate mortgage pricing, the values of these options and boundary conditions are incorporated in the model. This implies that we must solve the partial differential Equation (Equation (20)) subject to the boundary conditions described in Equations (21) through (31). The iterative procedure for solving the PDE subject to the boundary conditions is described in the next section.

3. Valuation Methodology

First, the parameters of three state variable processes (r, l and B) are estimated. The estimated values of these parameters are then plugged back into the valuation PDE (Equation (20)). Second, the PDE is numerically solved using the explicit finite difference method to determine the value of the mortgage $\left[V\left(r, l, B; t\right)\right]$.

3.1. Parameter Estimates of State Variable Processes

The methodology for estimating stochastic processes, as proposed by Brennan & Schwartz (1982) and Sharp et al. (2008), is extended to estimate interest rates and building value processes. Discrete approximations replace all stochastic processes involving state variables. In the estimation process, the instantaneous rate of return on the building (α) is approximated as a fraction of the one-period lagged instantaneous (spot) interest rate $\left(a_B r_{t-1}\right)$. Cunningham & Hendershott (1984) and Hendershott & Van Order (1987) show that the building value appreciation rate $\left(\alpha - b'\right)$ exactly equals to $\left(r - b'\right)$ in a tax-less world, and the end results hold even in a world with taxes. Therefore, the stochastic processes in Equations (10), (11), and (13) are approximated as

$$\frac{r_t - r_{t-1}}{r_{t-1}} = \frac{a_r}{r_{t-1}} + b_r \frac{l_{t-1} - r_{t-1}}{r_{t-1}} + \varepsilon_{rt} \tag{32}$$

$$\frac{l_t - l_{t-1}}{l_{t-1}} = \frac{a_l}{l_{t-1}} + b_l \frac{r_{t-1}}{l_{t-1}} + c_l + \varepsilon_{lt} \tag{33}$$

$$\frac{B_t - B_{t-1}}{B_{t-1}} = a_B r_{t-1} - b' + \varepsilon_{Bt} \tag{34}$$

Here, t represents the current period. These equations are estimated by an iterative Aitken (1935) procedure. Dhrymes (1971) points out that this procedure yields the maximum likelihood estimator. The discrete approximation equations have the general form of

$$Y_t = AX_t + E_t \tag{35}$$

Y_t and X_t represent the respective arrays of estimated dependent and independent variables, A describes the corresponding coefficient values, while E_t is the error term. The specific expressions for the corresponding equations are described as

$$\begin{aligned} Y_t &= \left[\frac{r_t - r_{t-1}}{r_{t-1}}; \frac{l_t - l_{t-1}}{l_{t-1}}; \frac{B_t - B_{t-1}}{B_{t-1}}\right] \\ A &= \left[\left(a_r, b_r\right); \left(a_l, b_l, c_l\right); \left(a_B, b'\right)\right] \\ X_t &= \left[\left(\frac{1}{r_{t-1}}, \frac{l_{t-1} - r_{t-1}}{r_{t-1}}\right); \left(\frac{1}{l_{t-1}}, \frac{r_{t-1}}{l_{t-1}}, 1\right); \left(r_{t-1}\right)\right] \\ E_t &= \left[\varepsilon_{rt}; \varepsilon_{lt}; \varepsilon_{Bt}\right] \end{aligned} \tag{36}$$

The iterative procedure provides the respective maximum likelihood estimators for a's, b's and c's. The corresponding standard deviations and correlation coefficients are calculated from the estimated error terms. The expressions for the standard deviations and correlation coefficients are as follows:

$$\begin{gathered}\sigma_r = Std\left(\varepsilon_{rt}\right); \quad \sigma_l = Std\left(\varepsilon_{lt}\right); \quad \sigma_B = Std\left(\varepsilon_{Bt}\right)\\ \rho_{rl} = \frac{Cov\left(\varepsilon_{rt}, \varepsilon_{lt}\right)}{\left[Std\left(\varepsilon_{rt}\right)\right]\left[Std\left(\varepsilon_{lt}\right)\right]}; \quad \rho_{rB} = \frac{Cov\left(\varepsilon_{rt}, \varepsilon_{Bt}\right)}{\left[Std\left(\varepsilon_{rt}\right)\right]\left[Std\left(\varepsilon_{Bt}\right)\right]}\end{gathered} \tag{37}$$

The advantage of this iterative procedure lies in its ability to estimate each of the state variable processes separately. Brennan & Schwartz (1982) describe the method to estimate the market price of risk of r, (L_r). In this procedure, the value of L_r is determined by minimizing the pricing prediction errors between actual and model prices through successive iterations.

3.2. Numerical Solution of the Valuation Equation (PDE)

The explicit finite difference method used here extends the methodology suggested by Hull & White (1990, 1993, 1994), Hilliard et al. (1998), Azevedo-Pereira et al. (2000), and Sharp et al. (2008) to the three-variable model of this study. The numerical solution of a PDE with more than one state variable is possible if the following two conditions are met: 1) the instantaneous standard deviation of each variable must be constant, and 2) the variables must be uncorrelated with each other. Two sets of transformations resolve these conditions.

Three state variables (r, l and B) are transformed into three new state variables Θ_1, Θ_2 and Θ_3 so that the instantaneous standard deviations remain constant. The specific expressions for the three corresponding processes are described as

$$\mathrm{d}\Theta_1 = q_1 \mathrm{d}t + k_1 \mathrm{d}z_1 \tag{38}$$

$$\mathrm{d}\Theta_2 = q_2 \mathrm{d}t + k_2 \mathrm{d}z_2 \tag{39}$$

$$\mathrm{d}\Theta_3 = q_3 \mathrm{d}t + k_3 \mathrm{d}z_3 \tag{40}$$

The specific expressions for q's and k's are as follows:

$$\begin{gathered}q_1\left(r,t\right) = \frac{\delta\Theta_1}{\delta t} + \left(a_r + b_r\left(l-r\right) - L_r\sigma_r r\right)\frac{\delta\Theta_1}{\delta r} + \frac{1}{2}\sigma_r^2 r^2 \frac{\delta^2\Theta_1}{\delta r^2}\\ q_2\left(l,t\right) = \frac{\delta\Theta_2}{\delta t} + l\left(\sigma_l^2 + l - r\right)\frac{\delta\Theta_2}{\delta l} + \frac{1}{2}\sigma_l^2 l^2 \frac{\delta^2\Theta_2}{\delta l^2}\\ q_3\left(B,t\right) = \frac{\delta\Theta_3}{\delta t} + B\left(r - b'\right)\frac{\delta\Theta_3}{\delta B} + \frac{1}{2}\sigma_B^2 B^2 \frac{\delta^2\Theta_3}{\delta B^2}\\ k_1 = \sigma_r r \frac{\delta\Theta_1}{\delta r}; \quad k_2 = \sigma_l l \frac{\delta\Theta_2}{\delta l}; \quad k_3 = \sigma_B B \frac{\delta\Theta_3}{\delta B}\end{gathered} \tag{41}$$

In multiple-variable model situations, Wiener processes $\mathrm{d}z_1, \mathrm{d}z_2$ and $\mathrm{d}z_3$ are instantaneously correlated with one another. More specifically, ρ_{rl} represents the correlation between $\mathrm{d}z_1$ and $\mathrm{d}z_2$, while the correlation between $\mathrm{d}z_1$ and $\mathrm{d}z_3$ is defined by ρ_{rB}. The correlation between $\mathrm{d}z_2$ and $\mathrm{d}z_3$ is assumed to be zero in the model specification. Therefore, three variables Θ_1, Θ_2 and Θ_3 are transformed into four new state variables $\varphi_1, \varphi_2, \varphi_3$ and φ_4 that are instantaneously uncorrelated with each other. The four corresponding stochastic processes are described as

$$\mathrm{d}\varphi_1 = \left(k_2 q_1 + k_1 q_2\right)\mathrm{d}t + k_1 k_2 \sqrt{2\left(1+\rho_{rI}\right)}\mathrm{d}z_4 \quad (42)$$

$$\mathrm{d}\varphi_2 = \left(k_2 q_1 + k_1 q_2\right)\mathrm{d}t + k_1 k_2 \sqrt{2\left(1-\rho_{rI}\right)}\mathrm{d}z_5 \quad (43)$$

$$\mathrm{d}\varphi_3 = \left(k_3 q_1 + k_1 q_3\right)\mathrm{d}t + k_1 k_3 \sqrt{2\left(1+\rho_{rB}\right)}\mathrm{d}z_6 \quad (44)$$

$$\mathrm{d}\varphi_4 = \left(k_3 q_1 + k_1 q_3\right)\mathrm{d}t + k_1 k_3 \sqrt{2\left(1-\rho_{rB}\right)}\mathrm{d}z_7 \quad (45)$$

These new variables are mutually independent. Therefore, the possible unconditional movements of these four variables, along with their associated probabilities (as specified by the corresponding partial differential equations, or PDEs), can be determined individually. The second transformation increases the states from three to four state variables. The transformation has been done to remove the correlation among the variables. Because of the linearity of this transformation (where three variables have been linearly transformed into four variables), the corresponding covariance matrix is investigated for possible singularity. The matrix turns out to be non-singular, justifying the validity of the second transformation. As an example, the one-dimensional PDE for the first new state variable (φ_1) is expressed as

$$V_{i-1,j} = \frac{1}{1+r\Delta t}\left[P_{j,j-1}V_{i,j-1} + P_{j,j}V_{i,j} + P_{j,j+1}V_{i,j+1} + M_i\right] \quad (46)$$

Here, i denotes a drift in the time interval $\left[t_i = t_0 + i\Delta t\right]$, while j denotes a drift in the variable $\left[\varphi_j = \varphi_0 + j\Delta\varphi\right]$. M_i is the continuous rate of mortgage payment at time i, r is the instantaneous risk-free (discount) rate, and Δt is the unit time interval. V_i denotes the value of the mortgage at the time i and P_j denotes the corresponding probability. In this backward solving method, the valuation starts at the maturity date (T), when the value of the mortgage equals zero for fully amortizing loans. The value of the mortgage at any given time t can be obtained by repeatedly working backwards from time T to time t in steps of Δt. For each variable, the value of the mortgage at time $(i-1)$ is associated with three possible unconditional movements: $j-1$, j and $j+1$ at time i. In this situation, $P_{j,j-1}, P_{j,j}$ and $P_{j,j+1}$ are interpreted as the probabilities of moving from $\varphi_{1,j}$ to $\varphi_{1,j-1}, \varphi_{1,j}$ and $\varphi_{1,j+1}$ respectively. The analysis is suitable for any one of the four stochastic variables, since they are independent of each other. Similar expressions can be found for $V_{i-1,m}, V_{i-1,n}$ and $V_{i-1,s}$, representing the value of the mortgage associated with other corresponding state variables (φ_2, φ_3 and φ_4). Therefore, the PDE corresponding to any variable is modeled by using a one-dimensional lattice with three branches coming out of each node. Specific expressions of one-dimensional PDEs ($V_{i-1,j}, V_{i-1,m}, V_{i-1,n}$ and $V_{i-1,s}$) and their associated probabilities $\left(P_j\text{'s}\right)$ in each of these four new variables are given in the Appendix.

The PDEs for more than one variable are now easier to form, as all four stochastic variables are independent of each other. Specifically, the three-state variable (r, l, B) model is estimated by using the new variables: $\varphi_1, \varphi_2, \varphi_3$ and φ_4. The probability of any given point being reached is the product of the unconditional probabilities associated with corresponding movements in φ_1 to φ_4. It is modeled by using a four-dimensional lattice with eighty-one branches ($3\times3\times3\times3=3^4$) at each node. A four-dimensional lattice is the product of four unrelated one-dimensional (each with three nodes) lattices. Therefore, the value of the mortgage at the time $(i-1)$ is associated with eighty-one alternative values of the mortgage at time i. After all probabilities are appropriately defined, the resultant PDE for four variables, which must be satisfied for the value of the mortgage, is expressed as

$$\begin{aligned} V_{i-1;\ j,m,n,s} = \frac{1}{1+r\Delta t}\Big[& P_{j;\ j-1,m-1,n-1,s-1} V_{i;\ j-1,m-1,n-1,s-1} + \cdots + P_{j;\ j,m,n,s} V_{i;\ j,m,n,s} \\ & + \cdots + P_{j;\ j+1,m+1,n+1,s+1} V_{i;\ j+1,m+1,n+1,s+1} + M_i \Big] \end{aligned} \tag{47}$$

Therefore, the value of the mortgage $(V_{i-1;\ j,m,n,s})$ at t_{i-1} can be related to eighty-one alternative values of the mortgage at the time t_i and the mortgage payment M_i for Δt time.

4. Data and Empirical Findings

4.1. Primary Mortgage Data

The mortgage data are obtained from several mid-South small to medium-sized mortgage originators through a questionnaire and several documents provided by individual borrowers. A survey was sent to 171 institutions located in Mississippi, Arkansas, and Louisiana. The survey design strikes a balance between consistency and flexibility. The objective is to ensure consistent responses despite the use of different bookkeeping systems and initial application forms at separate institutions. To improve feasibility and provide a consistent interpretation of responses, the study extracts data items primarily from original mortgage documents rather than from respondents. In total, information is available on 374 loans, spanning a period of over nine years, from 2011 to 2019. The mortgages, qualifying for inclusion in the study, have the following criteria: 1) that they are first mortgage loans to individuals granted for owner-occupied residential properties, and 2) that the loans are booked on one of the twenty randomly selected business dates from each quarter in odd years beginning in 2011 and ending in 2019.

The survey generates responses from seventy-one institutions, with each response representing a single mortgage loan package. A random number table has been employed to select random dates and months. First, twenty random months are chosen, with one month randomly picked from each quarter during the survey period. Similarly, twenty random dates are selected, with each date representing a month. The booking dates represent one percent (1%) of all business dates during the period. Consequently, each loan in the sample corresponds to approximately 100 loans originated.

The most significant number of loans booked occurred in 2019. That is followed by 2015, 2017, 2013, and 2011. The mortgages have a total original loan amount of $223,070,609 and a total outstanding balance of $158,380,132, representing seventy-one (71%) percent of the original loan. The average original amount borrowed is $169,004, while the average loan outstanding is $119,993. The summary statistics, which include the number of loans booked, average LTV (loan-to-value) ratio, average contract rates, and average points, are provided in Table 1.

Table 1. Summary of loans booked on different dates.

Dates	Loans Booked	Average LTV Ratio	Average Contract Rate	Average Points
02/09/2011	7	0.6867 (0.1864)	4.8786 (0.5411)	2.2857 (1.3591)
04/18/2011	5	0.7542 (0.1621)	5.0500 (0.6025)	1.0000 (0.8367)
09/22/2011	11	0.7854 (0.1735)	4.5375 (0.5064)	2.1000 (1.3565)
11/16/2011	7	0.7026 (0.1744)	5.2321 (0.5533)	3.2143 (1.8489)
01/24/2013	3	0.7388 (0.0405)	4.4667 (0.4497)	2.0000 (1.4142)
05/14/2013	5	0.7211 (0.1217)	4.0500 (0.3522)	1.5000 (1.0243)
07/09/2013	18	0.7343 (0.1815)	4.0476 (0.2834)	2.1765 (1.4583)
12/04/2013	15	0.6827 (0.2185)	3.7747 (0.5439)	1.0667 (0.7173)
03/03/2015	21	0.7603 (0.1139)	4.2553 (0.4158)	1.8095 (1.5026)
05/27/2015	13	0.7909 (0.1274)	3.5312 (0.3776)	1.3846 (0.9869)
07/17/2015	24	0.7823 (0.1168)	4.1156 (0.2279)	2.0312 (1.1852)
10/28/2015	27	0.7215 (0.1963)	3.8982 (0.4601)	2.0555 (1.5206)
03/06/2017	13	0.7919 (0.1019)	4.7500 (0.1506)	1.9615 (0.8009)
04/26/2017	36	0.7865 (0.1643)	4.5632 (0.4008)	1.8184 (1.0738)
08/03/2017	4	0.7830 (0.0622)	4.1250 (0.4714)	1.2500 (1.4142)
12/19/2017	26	0.7790 (0.1793)	4.1900 (0.3145)	1.4615 (1.1145)
02/12/2019	79	0.7680 (0.1847)	4.6668 (0.5011)	1.5367 (1.0441)
06/04/2019	30	0.8011 (0.2750)	4.6633 (0.4314)	1.3750 (1.2242)
08/27/2019	14	0.7022 (0.2297)	4.9714 (0.3125)	1.0000 (0.7216)
10/11/2019	8	0.7462 (0.1021)	4.6719 (0.6559)	1.4219 (0.7764)
Total	**374**	**0.7444 (0.1703)**	**4.4219 (0.4276)**	**1.6581 (1.1258)**

Standard deviations are provided in parentheses.

The average LTV ratios for mortgages are generally between 70% and 80%. For all loans pooled together, the average LTV ratio is 0.74, though several individual mortgages report high LTV ratios. This observation is supported by the fact that in the aftermath of the financial crisis of 2008, stringent origination requirements for mortgages led smaller financial institutions (such as the ones approached for mortgage data) to avoid issuing fixed-rate loans with high LTV ratios. Duca et al. (2016) observe a significant decrease in the LTV ratio immediately after the onset of the mortgage market crisis. In terms of mortgage valuation, Titman & Torous (1989) and Kau et al. (1992) report that the value of the default option in the presence of the prepayment option has little impact on mortgage valuation if the LTV ratio is lower than 0.80. Krainer et al. (2009) emphasize the critical importance of LTV ratios in assessing default probabilities for both fixed-rate and adjustable-rate mortgages. Brennan & Schwartz (1985) suggest that model valuation in the presence of a default option is extremely sensitive to the building value. Consequently, the building value and the accompanying low LTV ratios are viewed as an indirect indicator of mortgage default probability. During the sample period, the average contract rates vary directly with T-bill and T-bond rates. Average points, however, fail to produce any systematic pattern.

4.2. Parameter Estimation Results

The estimation of spot rate, consol rate, and building value parameters (r, l and B) are approximated as follows: The yield-to-maturity on one-month Certificates of Deposit proxies the spot rate, the coupon yield on a 30-year maturity T-bond is used for the consol rate, while the average new building value in the mid-south region proxies the building value. Monthly CD rates and T-bond rates are obtained from the Federal Reserve Bulletin, while monthly average building values are obtained from various issues of Current Construction Reports.

Parameter estimates are obtained using the discrete approximation procedure from the previous section, and the estimated values are then plugged back into the three state variable processes—the results of the stochastic processes estimations are given below. The t-statistics of parameter estimates are shown in parentheses below the estimated values.

$$\mathrm{d}r = \begin{bmatrix} 0.0001 & + & 0.0528 & (l-r) \\ (0.1551) & & (1.0789) & \end{bmatrix} \mathrm{d}t + 0.091 r \mathrm{d}z_r \tag{48}$$

$$\mathrm{d}l = \begin{bmatrix} 0.0022 & - & 0.0023 & r - & 0.0178 & l \\ (1.4021) & & (-0.1166) & & (-0.6643) & \end{bmatrix} \mathrm{d}t + 0.0392 l \mathrm{d}z_l \tag{49}$$

$$\mathrm{d}B = \begin{bmatrix} \alpha - & 0.0051 \\ & (-0.1087) \end{bmatrix} B\mathrm{d}t + 0.1604 B \mathrm{d}z_B \tag{50}$$

$$\mathrm{d}z_r\,\mathrm{d}z_l = 0.5234\mathrm{d}t \tag{51}$$

$$\mathrm{d}z_r\,\mathrm{d}z_B = 0.0158\mathrm{d}t \tag{52}$$

In the short-rate process, the estimated value of b_r is positive but statistically insignificant. The positive value is consistent with the assumption that the short-rate process is mean-reverting. It further implies that the short rate tends to regress towards the current value of the long rate. This finding supports the findings of Merton (1974) and Brennan & Schwartz (1982, 1985). The estimated value of a_r is also positive. The observation implies that the change in the short rate at $r = 0$ is positive for all current values of r and l. This finding is also consistent with that of Brennan & Schwartz (1982, 1985) and Sharp et al. (2008). The estimated values of the volatility parameter (σ_r) is as expected. The value of the Durbin-Watson (D-W) statistic is *1.4521*; with a value less than 2, the presence of positive autocorrelation remains inconclusive.

In the long-rate process, the absolute value of a_l is approximately equal to the absolute value of b_l. The signs of b_l and c_l are both negative, contradicting the findings of Brennan & Schwartz (1982, 1985). It is argued that the period of this study (January 2011 through December 2019) is recognized as a period of low interest rates, characterized by sustained quantitative easing by the Federal Reserve, which continuously bought long-term securities, such as government bonds and MBSs. Additionally, during the period of our study, the Federal Reserve deliberately adopted a policy of "Shaping the Yield Curve" by selling short-term Treasuries and buying longer-term ones, thereby further suppressing long-term rates relative to short-term rates, which can directly cause a reversion (negative values) of long-rate parameters. We further observe that, while the Fed's policy causes the volatility of short rates to return to pre-crisis levels relatively quickly, the volatility of long rates remains high for some time (a hindrance to the mean reversion of the short-rate to the long-rate). In contrast, the Brennan and Schwartz studies are based on data from the 1970s and early 1980s, which were characterized by high inflation and elevated interest rates with a steep, positively sloped yield curve. Furthermore, the estimates of a_r and b_r are of far greater importance since they enter directly into the valuation equation. The estimated value of σ_l is as expected. The estimated value of the correlation parameter (ρ_{rl}), however, indicates a strong positive correlation between the two processes. The value of the D-W statistic (*1.2274*) is less than the lower bound of the critical value.

The estimated value of the payout rate (b') is negative. This finding, though contrary to popular intuition, does not refute the assumption of log-normality of the building value process. Region-specific factors are suspected to be responsible for this contradiction. The estimated value of the volatility parameter (σ_B) is positive and conforms to the observations in Titman & Torous (1989). The estimated value of the correlation parameter (ρ_{rB}) is observed as positive. This is contrary to the popular belief that interest rate and building value movements are inversely related. However, the value is relatively smaller than the positive implied correlation parameter values observed by Titman & Torous (1989). Chen et al. (2021) observe that the co-movement of building value and interest rate volatility has a significant impact on the mortgage default probabilities. In line with this, it is worth noting that the period under study exhibits a high level of default in the aftermath of the mortgage market crisis, which justifies a positive correlation between the two variables.

4.3. Valuation Results

Table 2 reports the mortgage valuation results for the three-variable model. For different loan size classes, results are presented for the average model price per $100 of primary mortgage value for all mortgages in the sample. The average model prices relative to the primary mortgage value are further classified into two distinct subgroups: short-term (1 - 15 years) loans and long-term (16 - 30 years) loans, as well as between adjustable-rate and fixed-rate mortgages. Small loans are classified as those between $0 and $100,000, while any loan exceeding $100,000 is classified as a large loan. In essence, a three-way classification of the valuation results (smaller loans versus larger loans, short-term mortgages versus long-term mortgages, and adjustable-rate mortgages versus fixed-rate mortgages) is presented here.

Table 2. Three variable model valuation results.

Average Loan Size Groups ($)	Time-to-Maturity	Number of Loans Booked	Average Model Prices per $100 of Original Loan Amount ($)
0 - 50000.00	**All Loans**	**65**	**116.714**
	Long-term	11	107.676
	Short-term	54	118.555
	Fixed rate	42	109.384
	Adjustable rate	23	112.947
50000.01 - 100000.00	**All Loans**	**152**	**107.009**
	Long-term	97	104.602
	Short-term	55	111.143
	Fixed rate	102	112.311
	Adjustable rate	50	114.732
100000.01 - 150000.00	**All Loans**	**91**	**104.928**
	Long-term	69	103.801
	Short-term	22	108.469
	Fixed rate	67	110.055
	Adjustable rate	24	112.271
150000.01 - 200000.00	**All Loans**	**26**	**105.259**
	Long-term	18	103.299
	Short-term	8	109.669
	Fixed rate	17	108.368
	Adjustable rate	9	110.035

Continued

	All Loans	**24**	**104.576**
	Long-term	19	104.109
200000.01 - 250000.00	Short-term	5	106.351
	Fixed rate	15	108.381
	Adjustable rate	9	110.785
	All Loans	**10**	**103.613**
	Long-term	8	102.494
250000.01 - 300000.00	Short-term	2	108.102
	Fixed rate	4	106.943
	Adjustable rate	6	107.287
	All Loans	**6**	**103.874**
	Long-term	5	102.871
Over 300000.00	Short-term	1	108.889
	Fixed rate	3	105.922
	Adjustable rate	3	107.021
	All Loans	**374**	**107.819**
	Long-term	227	104.246
Total	Short-term	147	113.226
	Fixed rate	245	108.153
	Adjustable rate	129	110.181
	All Loans	**153**	**104.801**
	Long-term	116	103.643
Large Loans	Short-term	37	108.434
	Fixed rate	97	106.934
	Adjustable rate	56	108.311
	All Loans	**217**	**110.027**
	Long-term	112	104.935
Small Loans	Short-term	105	114.882
	Fixed rate	148	110.046
	Adjustable rate	69	112.989

Short-term loans refer to loans with a term-to-maturity of between 0 and 15 years. Long-term loans constitute loans with a term-to-maturity of between 16 and 30 years. Small loans are defined as loans with original loan values between $0 and $150,000. Large loans are defined as loans with original loan values over $150,000. A fixed-rate mortgage features a fixed loan rate throughout the mortgage term. In contrast, the loan rate on an adjustable-rate mortgage is adjusted periodically, based on the underlying index rate and various caps.

The results indicate that the average price estimated by the model exceeds $100 for all primary loan size groups. The investigation of average model prices further reveals that the average pricing spread decreases as loan size increases. The observation suggests an inverse relationship between the average model price (as well as the average spread amount) and the loan size. It justifies the fact that smaller loans are often used for refinancing purposes, especially in light of the mortgage market crisis. Though proportional, the imposition of higher transaction costs (margin points, origination fees, prepayment penalty clauses, and past due penalties) on smaller loans as a percentage of the loan amount has been common during the period under study (especially with stringent loan qualification criteria imposed through the Dodd-Frank Act of 2010). Many other real estate fees, such as recordation fees, appraisal fees, and title insurance fees, are fixed in nature and do not scale proportionally as the loan size decreases. Since mortgage transaction costs are not explicitly estimated through additional boundary conditions in the model, they appear as an exogenous cost that further inflates the model value for smaller loans. As a validation, the investigation of the data set reveals that the average margin charged for smaller-denomination loans was 30 basis points higher than that of larger loans.

Further investigation of average model prices reveals that the ratio of average model prices to short-term loan values is systematically higher than the ratio of average model prices to long-term loan values. This finding is valid for all size classes, thus extending the research of Titman & Torous (1989). Similar to the relationship between average loan price and loan size, the observation strongly suggests an inverse relationship between the average model price (as well as the average spread amount) and the term-to-maturity of a mortgage. Roll (1994) notes that the explanatory power of contingent claims models increases significantly with longer-term maturities. In the context of mortgage valuation models, the increase in explanatory power reduces the pricing spread, further implying that the differences between average model prices and loan values are more pronounced for short-term loans. Unlike an accounting approach to directly estimating the transaction cost, the valuation procedure implicitly estimates the transaction cost. Without its explicit estimation in the valuation process (by imposing additional appropriate terminal and boundary conditions), any disproportionate increase in transaction costs inflates the model-estimated value and, consequently, the pricing spread (the spread is defined as the model price over every $100 of the original loan amount). The findings are similar to those observed for smaller mortgage loans.

The average model value analysis for adjustable-rate and fixed-rate mortgages reveals that model values of adjustable-rate mortgages are consistently higher than those of comparable fixed-rate mortgages across all size classes. This is explained by the fact that the contingent claims valuation of adjustable-rate mortgages is based on additional boundary conditions, which incorporate the fluctuating nature of mortgage interest rates and can lead to a wider variation between the model and actual values of the mortgages. Equations (24) and (25) model the ARMs through boundary conditions. Equation (24) incorporates the variability of the mortgage rate, subject to four underlying variables (the index rate that can be changed daily based on Secured Overnight Financing Rate (SOFR), the margin above the index rate, which is determined by the borrower's credit score, the yearly-, and life-of-loan caps as determined by the loan contract) that enter directly into the valuation process. It is worth noting that adjustable-rate adjustments tend to have a positive bias when upward adjustments of the prevailing interest rate occur more frequently than downward adjustments. Equation (25), on the other hand, reflects the changing nature of the valuation as the underlying interest rate from Equation (24) changes. Since the valuation trajectory changes from one period to the next due to frequently upward bound changes in the underlying interest rate, the resultant model value tends to inflate more above the actual value. Tong (2007) posits that adjustable-rate mortgages (ARMs) are more vulnerable to prepayment and default risks than fixed-rate mortgages (FRMs) throughout the life of the loan. Chiang & Sa-Aadu (2014) demonstrate that ARMs are closely aligned with borrowers' mobility horizons (borrowers tend to match their choice of mortgage based on their mobility expectations) and are vulnerable to fluctuations in mortgage interest rates. Consequently, ARMs are more susceptible to changes in mortgage rates and house prices than FRMs. Two distinct trends emerge when comparing the estimated values from adjustable-rate and fixed-rate models. First, the average model value differences between adjustable-rate and fixed-rate mortgages decrease as the loan size increases. Second, the average model value differences decrease as the term-to-maturity (moving from short-term to long-term loans) increases.

Several factors seem to justify the existence of a systematic pricing spread between the model and actual values. In most years during the period under study, the mortgage market has experienced long-term interest rates that are higher than the corresponding short-term interest rates, indicative of a standard yield curve. The underlying economic behavior based on this phenomenon can inflate the model value over time, as both long- and short-term rates enter directly (without substitution) into the valuation process. The supporting evidence is found in Brennan & Schwartz (1985) and Titman & Torous (1989), who observe that model valuation is more sensitive to the consol rate (l) and building value (B) than the spot rate (r). The zero correlation between the long-term interest rate and the building value (ρ_{lB}) in the model is considered another contributing factor to the spread. Furthermore, the frequently higher long-term interest rate values (higher than the short-rate values) in the model contribute to the pricing spread. The observations by both Brennan & Schwartz (1985) and Titman & Torous (1989) support this finding. The probable regional-specific bias in pricing the building value process parameters can also be a causal factor in the positive pricing spread.

4.4. Statistical Test Results

The study employs mean-, median-, and variance-based tests to investigate the model efficiency in predicting primary mortgage values. The analysis requires testing the null hypotheses of equality of means, medians, and variances between actual primary market values and model-estimated values. The following three hypotheses and their alternatives are tested,

$$\begin{aligned} H_0 &: \quad \mu_{Model} = \mu_{Actual} \\ H_1 &: \quad \mu_{Model} \neq \mu_{Actual} \\ H_0 &: \quad M_{Model} = M_{Actual} \\ H_1 &: \quad M_{Model} \neq M_{Actual} \\ H_0 &: \quad \sigma^2_{Model} = \sigma^2_{Actual} \\ H_1 &: \quad \sigma^2_{Model} \neq \sigma^2_{Actual} \end{aligned} \tag{53}$$

Here, μ represents the mean, M represents the median, while σ^2 represents the variance. In line with the indications from valuation results that inverse relationships exist between the spread amounts and loan size, as well as between the spread amounts and term-to-maturity, the analysis is extended to each of the loan size classes and to the sub-groups of long-term and short-term loans. The mean-, median-, and variance-based test results are provided in Table 3.

Table 3. Statistical results for model efficiency.

Test Methodology	Time-to-Maturity and Loan Size	Test Statistic	Probability of Significance
Equality of Means $\left(H_o : \mu_{Model} = \mu_{Actual}\right)$			
t-test	**Total**	**0.280774**	**0.7790**
	Long-term	0.057084	0.9545
	Short-term	0.732586	0.4644
	Large Loans	0.404241	0.6863
	Small Loans	0.322627	0.1218
	Fixed-rate	0.486291	0.5267
	Adjustable-rate	0.592015	0.3829
ANOVA F-test	**Total**	**0.078834**	**0.7790**
	Long-term	0.003259	0.9545
	Short-term	0.536683	0.4644
	Large Loans	0.163412	0.6863
	Small Loans	1.432092	0.1218
	Fixed-rate	0.829738	0.5267
	Adjustable-rate	1.035871	0.3829

Continued

Equality of Medians $\left(H_0 : M_{Model} = M_{Actual}\right)$			
Wilcoxon/Mann-Whitney z-test	**Total**	**0.767479**	**0.4428**
	Long-term	0.222341	0.8240
	Short-term	1.141257	0.2538
	Large Loans	1.374914	0.1692
	Small Loans	1.434030	0.1255
	Fixed-rate	1.213279	0.2537
	Adjustable-rate	1.387245	0.1892
Median χ^2-test	**Total**	**0.353591**	**0.5521**
	Long-term	0.036697	0.8481
	Short-term	0.222222	0.6374
	Large Loans	1.581699	0.2085
	Small Loans	0.775120	0.3786
	Fixed-rate	0.492378	0.4132
	Adjustable-rate	0.532295	0.1972
Adjusted Median χ^2-test	**Total**	**0.270718**	**0.6029**
	Long-term	0.009174	0.9237
	Short-term	0.125002	0.7237
	Large Loans	1.307190	0.2529
	Small Loans	0.612440	0.4339
	Fixed-rate	0.183125	0.4127
	Adjustable-rate	0.229138	0.2539
Kruskal-Wallis χ^2-test	**Total**	**0.589296**	**0.4427**
	Long-term	0.049605	0.8238
	Short-term	1.304084	0.2535
	Large Loans	1.892165	0.1690
	Small Loans	2.092946	0.1255
	Fixed-rate	1.583745	0.1937
	Adjustable-rate	1.734216	0.1431
Van der Waerden χ^2-test	**Total**	**0.795441**	**0.3725**
	Long-term	0.081359	0.7755
	Short-term	1.554738	0.2124
	Large Loans	2.040787	0.1312
	Small Loans	2.226134	0.1085
	Fixed-rate	1.762944	0.1877
	Adjustable-rate	2.049278	0.1139

Continued

Equality of Variances $\left(H_0 : \sigma^2_{Model} = \sigma^2_{Actual}\right)$			
F-test	**Total**	**1.001680**	**0.9873**
	Long-term	1.002218	0.9870
	Short-term	1.073137	0.6736
	Large Loans	1.068165	0.6849
	Small Loans	1.021984	0.8755
	Fixed-rate	1.053216	0.7629
	Adjustable-rate	1.091568	0.4327
Siegel-Tukey z-test	**Total**	**0.628877**	**0.5294**
	Long-term	0.226903	0.8205
	Short-term	0.572409	0.5670
	Large Loans	1.245693	0.2129
	Small Loans	0.370863	0.7107
	Fixed-rate	0.487294	0.4219
	Adjustable-rate	0.537245	0.2281
Bartlett χ^2-test	**Total**	**0.000254**	**0.9873**
	Long-term	0.000266	0.9870
	Short-term	0.177465	0.6736
	Large Loans	0.164665	0.6849
	Small Loans	0.024530	0.8755
	Fixed-rate	0.096521	0.6357
	Adjustable-rate	0.113265	0.3378
Levene F-test	**Total**	**0.000144**	**0.9904**
	Long-term	0.000028	0.9957
	Short-term	0.056218	0.8127
	Large Loans	0.031584	0.8591
	Small Loans	0.007957	0.9290
	Fixed-rate	0.009327	0.8837
	Adjustable-rate	0.011229	0.7836

Continued

Brown-Forsythe F-test	**Total**	**0.000534**	**0.9816**
	Long-term	0.000149	0.9903
	Short-term	0.037330	0.8469
	Large Loans	0.020572	0.8860
	Small Loans	0.008947	0.9247
	Fixed-rate	0.009298	0.7952
	Adjustable-rate	0.012396	0.5319

*** represents significance at 1% level, ** represents significance at 5% level, and * represents significance at the 10% level. Short-term loans refer to loans with a term-to-maturity of between 0 and 15 years. Long-term loans are those with a term-to-maturity of between 16 and 30 years. Small loans are defined as loans with original loan values between $0 and $150,000. Large loans are defined as loans with original loan values over $150,000. A fixed-rate mortgage features a fixed loan rate throughout the mortgage term. In contrast, the loan rate on an adjustable-rate mortgage is adjusted periodically, based on the underlying index rate and various caps.

The Mean-, Median-, and Variance-based Tests of Equality: The mean-based analyses (both t-tests and ANOVA F-tests) indicate that the results fail to reject the null hypothesis of equality of means in every category, suggesting model efficiency. The t-value is calculated by comparing the difference between the two sample means to the variability within the samples. An F-test for equality of means compares the means of two or more groups by testing if they are significantly different. It works by calculating an F-statistic, which is the ratio of the variance between groups to the variance within groups.

Five different nonparametric test statistics are generated to test the equality of two medians (or distributions) of actual and model-estimated values. Conover (1999) and Sheskin (2003) provide detailed descriptions of various nonparametric test procedures. Wilcoxon Signed Rank z-Test, Median and Adjusted Median χ^2-Tests, Kruskal-Wallis χ^2-Test, and Van der Waerden χ^2-Test are performed for the total loans, and for each of the loan-class and maturity-class sub-samples. The Wilcoxon Signed Rank z-Test calculates the test statistic by comparing the values of the two classification variables relative to their medians. Median χ^2-Test is a rank-based ANOVA test comparing the number of observations above and below the overall median in each series. The Adjusted Median χ^2-Test reports statistics similar to Yates' continuity correction. The Kruskal-Wallis χ^2-Test is a one-way ANOVA by ranks viewed as a multivariate extension of the Wilcoxon/Mann-Whitney z-test. Van der Waerden (normal scores) χ^2-Test differs from the Kruskal-Wallis test in that the ranks of observations in the standard scores test are smoothed by converting them into normal quantiles. Each of the test results from each sample fails to reject the null hypothesis of equal median and concludes that primary mortgage values are efficiently estimated.

The parametric and nonparametric variance-based analysis tests the null hypothesis that the variances in two series are equal against the alternative that they are different. The results from the F-test, Siegel-Tukey z-test, Bartlett χ^2-test, Levene F-test, and Brown-Forsythe F-test show that the null hypothesis of equal variances cannot be rejected for each of the data sets under investigation. The F-test calculates the F-statistic as the ratio of variances derived from two groups. The Siegel-Tukey z-test is based on the hypothesis that two series are independent and have equal medians, where the ranking of observations alternates from the lowest to the highest value for every other rank. Bartlett-Test compares the logarithm of the weighted average variance with the weighted sum of the logarithms of the variances of the two series. The Levene F-Test is based on an analysis of variance of the absolute difference from the mean. At the same time, the Brown-Forsythe F-Test modifies the Levene test, where the absolute mean difference is replaced with the absolute median difference.

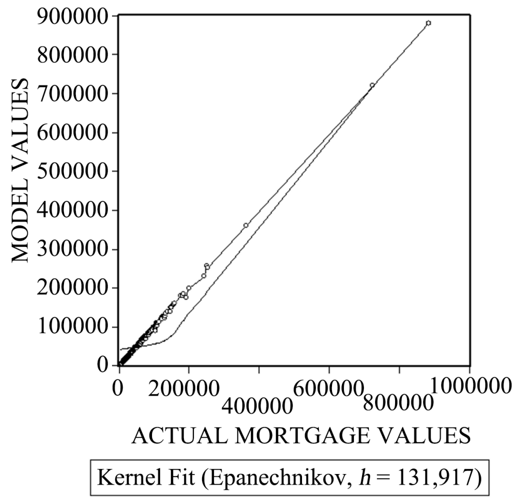
The overall mean, median, and variance-based test results complement our earlier findings, which indicate that model values are efficient predictors of primary mortgage values. Further investigation of the p-values from test results suggests that they are larger (acceptance of the null hypothesis is more pronounced) for the long-term and larger loan sub-categories, providing indirect evidence of increased model efficiency. Detailed descriptions of the implementation methods for these various tests can be found in Conover (1999), Sheskin (2003), and Gibbons & Chakraborti (2020).

Nonparametric Kernel Density Regression: Hypothesis testing comprehensively proves the model's efficiency across all sizes and maturity classes. The tests, however, fail to demonstrate the extent to which the degree of model efficiency increases when evaluating larger and longer maturity loans. Valuation results, as well as earlier observations by Titman & Torous (1989) and Roll (1994), strongly indicate that model efficiency increases as the loan size and/or term-to-maturity of a loan increase. This study uses a kernel density regression technique to estimate and evaluate such behavior of the contingent claims model. The kernel nonparametric regression fits the local polynomial of the second series in group Y on the first series in group X. Maxam & LaCour-Little (2001) and LaCour-Little et al. (2002) provide an excellent generalized description of the nonparametric kernel multivariate regression framework and its application in real estate valuation literature. The complete system of the kernel regression with different underlying functions (to estimate the joint density of two random variables) is expressed as

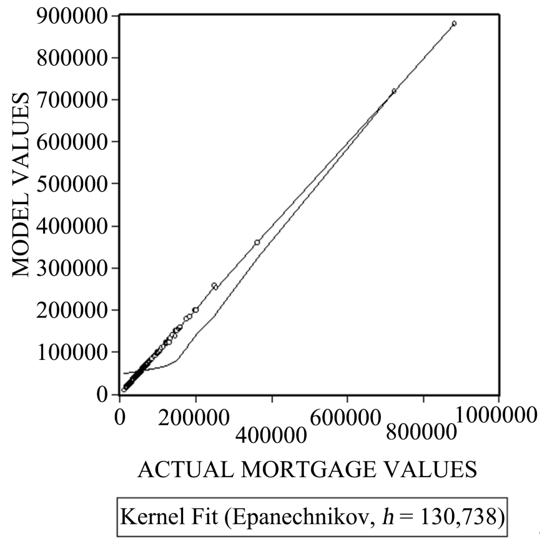
$$\begin{aligned} \hat{f}(x) &= \sum_{i=1}^{N}\left(Y_i - \beta_0 - \beta_1(x - X_i) - \cdots - \beta_k(x - X_i)^k\right)^2 K\left(\frac{x - X_i}{H}\right) \\ K(x) &= \frac{3}{4}\left(1 - u^2\right) I\left(|u| \le 1\right) \\ \hat{H} &= 0.15\left(X_U - X_L\right) \end{aligned} \tag{54}$$

In the kernel estimator equation, $f(x)$ represents the estimator that fits Y (the model value) against X (the actual value) at each value x, by choosing the parameters β to minimize the weighted sum-of-squared residuals. In the regression, the minimizing estimates of β differ for each x. Here, N is the number of observations, $K(\bullet)$ is the appropriate kernel function that integrates to one, k represents the order of the polynomial to determine the form of the local regression, and H characterizes the bandwidth or smoothing parameter.

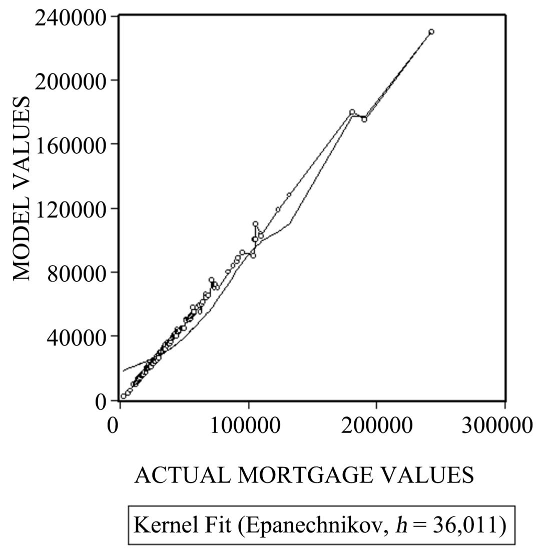
The kernel function $K(x)$ represents the probability (the kernel) density function that is used to assign weight to the observations in each local regression. This study employs an Epanechnikov Kernel to ensure a smooth and efficient function that converges within the data interval (integrates to one). We select the Epanechnikov Kernel over the Gaussian Kernel procedure due to its emphasis on minimizing the Mean Integrated Squared Error (MISE) in density estimation between the model-generated and actual mortgage values. The Epanechnikov procedure satisfies the objective of this study by minimizing the difference between the estimated model and actual mortgage values through graphical analysis and testing the efficiency of the three-variable contingent claims model. Chu et al. (2017) provide validation of Epanechnikov Kernel usage in assessing the efficiency of model estimation. In the kernel function equation, u represents the argument of the kernel function and I is the indicator function that takes a value of one if its argument is valid, and zero otherwise. In the same vein, the bandwidth H determines the weights to be applied to the observations in each local regression. In the bandwidth equation, X_U and X_L represent the upper bound and lower bound in the range of X_i. The representation of the bandwidth in this fashion ensures that the larger the value of H (the range), the smoother the kernel fit. In essence, the distance between an arbitrary observation x and all other joint observations (Y_i, X_i) in the data set is scaled (weighted) by the bandwidth H and assigned a probability density through the kernel function $K(x)$. The weighted average of these density functions is the estimate of the joint density $f(x)$ at the given x. The kernel estimations of the total loans and of the long-term and short-term loans sub-samples are provided in **Figure 1**.



(a)



(b)



(c)

Figure 1. Kernel density regression estimates between model output and actual mortgage values. (a) Kernel-fit between model output and actual mortgage values (total loans), (b) Kernel-fit between model output and actual mortgage values (long-term loans), (c) Kernel-fit between model output and actual mortgage values (short-term loans).

The graphical representation of the total loan analysis clearly shows that the kernel estimation provides a better fit when it evaluates larger mortgages. It complements the earlier findings in the valuation section, which indicate that the three-variable model estimation process becomes more efficient as larger loans are evaluated. The separate analyses of long-term and short-term loans further reveal that a smoother kernel fit occurs (reflected by a significantly larger bandwidth value) for long-term loans compared to short-term loans. It supports the earlier observation that the degree of model efficiency is more pronounced for long-term loans.

5. Conclusion

We propose a three-variable contingent claims model that adds to the ongoing mortgage valuation research. The model values residential mortgages that can be long-term or short-term, adjustable-rate or fixed-rate, and can either be prepaid or defaulted before the maturity of the loan contract. We use short- and long-term interest rates, as well as the building's value, as state variables to estimate the mortgage's value. The spot and the consol rates explain the prepayment characteristic, while the short-term interest rate and the building value capture the default character of the mortgage. The estimation procedure involves solving a three-variable partial differential equation, incorporating prepayment and default options, and utilizing specific conditions of an adjustable-rate mortgage as the necessary terminal and boundary conditions. Accordingly, the study employs a backward-solving explicit finite difference methodology to estimate the valuation equation of the model.

An empirical investigation on primary residential mortgage data is conducted to examine the valuation efficiency of the model. As a prelude to obtaining the model-estimated values of mortgages, the stochastic processes of three variables are projected using a discrete approximation. In the process, the parameter estimates of the two interest rate processes are observed to be quite accurate. Valuation results, obtained using the finite difference method, indicate a systematic presence of a positive pricing spread between the model-estimated and actual primary market values of all mortgages over the entire period under study. In line with Titman & Torous (1989), Roll (1994), and Downing et al. (2005), the results further reveal that the pricing spread decreases as the size and term-to-maturity of a loan increase. These trends are further evident in comparisons between adjustable-rate and fixed-rate mortgages, as well as between short-term and long-term mortgages. The uniqueness of this study lies in the fact that the proposed model incorporates all three relevant variables (short-term and long-term interest rates, and the building value) to comprehensively value mortgages that can include varied mortgage characteristics, such as defaultable and prepayable, short-term and long-term, adjustable-rate and fixed-rate mortgages.

The valuation model's efficiency in estimating primary mortgage prices is evaluated through various mean-, median-, and variance-based tests. The investigation assesses the null hypotheses that the means, medians, and variances derived from actual and theoretically estimated prices are equal. The analysis reveals that our model is overall efficient in predicting primary mortgage prices, with efficiency more pronounced for longer-term and larger mortgages. The study employs a nonparametric kernel density regression analysis to determine the extent to which the degree of model efficiency increases with larger and longer maturity loans. A similar trend is pervasive in comparison between adjustable-rate and fixed-rate mortgages. The graphical representation clearly shows that kernel estimation provides a better fit when evaluating larger and longer-term mortgages. In essence, the proposed model, along with its terminal and boundary conditions and numerical estimation procedure, offers a comprehensive valuation framework that can be utilized to efficiently and accurately value mortgages based on actual primary mortgage origination data.

Two limitations of this study must be noted in conclusion. Future extensions of the study should opt for model refinement to capture suboptimal borrower behavior in default and prepayment decisions. The suboptimal behavior by the borrower stems from making financially poor choices, such as not refinancing when mortgage rates in the market are falling, making refinancing economically beneficial (suboptimal prepayment), or overpaying for the property by continuing the mortgage beyond financial means, thereby risking default to avoid foreclosure (suboptimal default). Along with borrowers' financial inattention, factors like rapid interest rate shifts, volatile house prices, personal setbacks (such as job loss), low house equity, and lender limits on prepayment or default can all contribute to these suboptimal behaviors. These incidents have become more pronounced during and in the aftermath of the mortgage market crisis. Tuysuz (2024) provides an elaborate account of the factors that can lead a borrower to make suboptimal decisions. Relying on the behavioral economics framework, Odorović (2017) demonstrates that the cognitive biases exhibited by borrowers can lead to these suboptimal behaviors. In this context, contingent claims valuation frameworks, especially those used to model post-financial-crisis mortgage data, should include suboptimal borrower behavior in additional terminal and boundary conditions. This is a task for the future discourse.

A second direction that any future contingent-claims mortgage valuation study should take is to incorporate both aggregate-level loan data and loan-level mortgage data into the valuation process, thereby complementing the valuation framework of this study. The use of loan-level data does not refute the robustness of the valuation model presented in this study; however, employing aggregate, higher-level mortgage loan data in testing the model can certainly supplement the efficacy of the framework and provide a macro-level view of understanding market-wide trends. On the one hand, the loan-level data enable the incorporation of loan-specific principal balances, mortgage rates, borrower credit scores, loan-to-value ratios, mortgage delinquency statuses, and individual borrower default or prepayment behaviors into the mortgage valuation model to accentuate precise modeling. On the other hand, loan-level data are subject to geographical and borrower-specific characteristic biases, influenced by the location of mortgage origination and the borrower's characteristics, which hinder the generalization of the valuation results. The valuation model in this study is tested on a limited mortgage dataset originating from the deep South region of the United States and emanating from smaller financial institutions. The model efficiency can certainly improve if applied to a larger, higher-order dataset from a broader geographic area or to countrywide aggregate data, free from location or lender-specific biases. Extensive aggregate mortgage loan data are available from both the government and private sources. The Federal Housing Finance Agency (FHFA) provides appraisal, loan performance, demographic, and property characteristics data through the National Mortgage Database (NMDB) and the Uniform Appraisal Dataset (UAD). The Federal Financial Institutions Examination Council (FFIEC) and the Consumer Financial Protection Bureau (CFPB) publish the mortgage lending data. In the private sector, the Mortgage Bankers Association (MBA) provides lender revenues and expenses data, while ATTOM offers aggregate ownerships, aggregate loan positions, sales, foreclosure, and valuations data. Mortgage Capital Trading (MCT) presents insight into the aggregate loan sale data. Bogin et al. (2019) demonstrate how the use of localized aggregate house price indices, which can capture submarket trends in mortgage valuation, could have avoided incorrect, opaque pricing and risk underestimation during the mortgage market crisis. The availability of aggregate-level mortgage loan data, combined with empirical evidence that their use in valuing mortgages can improve model accuracy, certainly warrants future research efforts in this direction.

On a final note, the study provides a valuable tool to practitioners in the real estate market, such as primary mortgage lenders and their associated risk assessors, the secondary mortgage market originators, including GSEs (Fannie Mae and Freddie Mac), market investors in Mortgage-backed Securities (MBSs) and Collateralized Mortgage Obligations (CMOs), and tertiary players in the credit-default-swap (CDS) market. Accurate mortgage pricing can provide significant assistance in risk management within the loan market. It reduces lenders' and investors' risk through precise collateralization, determines viable loan terms, and ensures the loan's viability in the secondary market. The appropriate pricing mechanism guides real estate agents, surveyors, and property appraisers in determining the true value of a property, enabling them to make objective assessments. Ultimately, for buyers and sellers, identifying the true property value helps establish the appropriate sales price and equity assessment in property transactions.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

The PDE of the first state variable (φ_1) in Equation (46) is reproduced here. The variable is a part of the four new state variables ($\varphi_1, \varphi_2, \varphi_3$ and φ_4) that are instantaneously uncorrelated to each other.

$$V_{i-1,j} = \frac{1}{1+r\Delta t}\left[P_{j,j-1}V_{i,j-1} + P_{j,j}V_{i,j} + P_{j,j+1}V_{i,j+1} + M_i\right] \tag{46}$$

The specific expressions of possible probability movements $\left(P_j\text{'s}\right)$ in a unit time interval (Δt, from $i-1$ to i) associated with the first state variable φ_1 can be written as

$$\begin{aligned} P_{j,j-1} &= \Delta t\left[\frac{-k_2q_1 + k_1q_2}{2\Delta\varphi_1} + \frac{k_1^2k_2^2\left(1+\rho_{rl}\right)}{\Delta\varphi_1^2}\right] \\ P_{j,j} &= 1 - \Delta t\left[\frac{2k_1^2k_2^2\left(1+\rho_{rl}\right)}{\Delta\varphi_1^2}\right] \\ P_{j,j+1} &= \Delta t\left[\frac{k_2q_1 + k_1q_2}{2\Delta\varphi_1} + \frac{k_1^2k_2^2\left(1+\rho_{rl}\right)}{\Delta\varphi_1^2}\right] \end{aligned} \tag{A.1}$$

In a similar fashion, the value of the mortgage $\left(V_{i-1,m}\right)$ and the expressions of corresponding probabilities $\left(P_j\text{'s}\right)$ associated with the second state variable φ_2 are written as

$$\begin{aligned} V_{i-1,m} &= \frac{1}{1+r\Delta t}\left[P_{j,m-1}V_{i,m-1} + P_{j,m}V_{i,m} + P_{j,m+1}V_{i,m+1} + M_i\right] \\ P_{j,m-1} &= \Delta t\left[\frac{-k_2q_1 - k_1q_2}{2\Delta\varphi_2} + \frac{k_1^2k_2^2\left(1-\rho_{rl}\right)}{\Delta\varphi_2^2}\right] \\ P_{j,m} &= 1 - \Delta t\left[\frac{2k_1^2k_2^2\left(1-\rho_{rl}\right)}{\Delta\varphi_2^2}\right] \\ P_{j,m+1} &= \Delta t\left[\frac{k_2q_1 - k_1q_2}{2\Delta\varphi_2} + \frac{k_1^2k_2^2\left(1-\rho_{rl}\right)}{\Delta\varphi_2^2}\right] \end{aligned} \tag{A.2}$$

The expressions for $V_{i-1,n}$ and corresponding probabilities $\left(P_j\text{'s}\right)$ associated with φ_3 can be written as follows:

$$\begin{aligned} V_{i-1,n} &= \frac{1}{1+r\Delta t}\left[P_{j,n-1}V_{i,n-1} + P_{j,n}V_{i,n} + P_{j,n+1}V_{i,n+1} + M_i\right] \\ P_{j,n-1} &= \Delta t\left[\frac{-k_3q_1 + k_1q_3}{2\Delta\varphi_3} + \frac{k_3^2k_1^2\left(1+\rho_{rB}\right)}{\Delta\varphi_3^2}\right] \\ P_{j,n} &= 1 - \Delta t\left[\frac{2k_3^2k_1^2\left(1+\rho_{rB}\right)}{\Delta\varphi_3^2}\right] \\ P_{j,n+1} &= \Delta t\left[\frac{k_3q_1 + k_1q_3}{2\Delta\varphi_3} + \frac{k_3^2k_1^2\left(1+\rho_{rB}\right)}{\Delta\varphi_3^2}\right] \end{aligned} \tag{A.3}$$

Finally, the expressions for $V_{i-1,s}$ and P_j's associated with the fourth state variable $\left(\varphi_4\right)$ are expressed as

$$\begin{aligned} V_{i-1,s} &= \frac{1}{1+r\Delta t}\left[P_{j,s-1}V_{i,s-1} + P_{j,s}V_{i,s} + P_{j,s+1}V_{i,s+1} + M_i\right] \\ P_{j,s-1} &= \Delta t\left[\frac{-k_3q_1 - k_1q_3}{2\Delta\varphi_4} + \frac{k_3^2k_1^2\left(1-\rho_{rB}\right)}{\Delta\varphi_4^2}\right] \\ P_{j,s} &= 1-\Delta t\left[\frac{2k_3^2k_1^2\left(1-\rho_{rB}\right)}{\Delta\varphi_4^2}\right] \\ P_{j,s+1} &= \Delta t\left[\frac{k_3q_1 - k_1q_3}{2\Delta\varphi_4} + \frac{k_3^2k_1^2\left(1-\rho_{rB}\right)}{\Delta\varphi_4^2}\right] \end{aligned} \tag{A.4}$$

These four instantaneously uncorrelated one-dimensional (each with three nodes) lattices ($V_{i-1,j}, V_{i-1,m}, V_{i-1,n}$ and $V_{i-1,s}$) are used to create the four-dimensional lattice (as a product of four lattices) in Equation (37). The lattice represents the value of the mortgage $\left(V_{i-1;\,j,m,n,s}\right)$ at t_{i-1}, related to eighty-one alternative values of the mortgage at the time t_i and the mortgage payment M_i.