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A Mathematical Formulation of the Valuation of Silver ETFS: The Role of Investor Sentiment

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Abstract

Silver ETFs are exchange-traded funds of silver mining stocks. This paper posits that investor sentiment toward risk influences silver ETF returns. The paper creates formulations of invest sentiment toward risk with risk-averse investors investing modestly in silver ETFs with optimal returns at the intersection of the Arrow-Pratt coefficient of risk-aversion and silver ETF returns. Moderate risk-takers make revised estimates of silver returns as modeled by a Laplace transform, which intersects with the Bessel function of silver ETF returns to yield the optimal returns. Risk-takers follow an exponential distribution of continuously rising return expectations, which yield an optimal return upon intersection with the Bessel function of silver ETF returns. Theoretical formulations were subjected to empirical validation using silver ETF returns regressed on proxies for investor sentiment, also known as investor risk aversion. The paper concludes with theoretical implications and practical implications. Theoretically, the study advances knowledge by presenting mathematical formulations of investor sentiment's varying influence on silver ETF returns. From a practical standpoint, ETF managers can vary investor recommendations about investing in silver ETFs, depending upon whether investors are risk-averse, moderate risk-takers, or risk-takers. There is a paucity of similar studies with both a mathematical emphasis and empirical validation.

Keywords

Precious Metals, Silver ETFs, Risk-Aversion, Silver ETF Price-Volume Relationships, Bessel Function, Investor Sentiment

1. Introduction

Silver is a precious metal and a commodity. Unlike gold, silver (often sold as silver

ETFs) is less effective as a hedge against inflation. Shobha (2024) demonstrated empirically that when safe haven currencies, such as the dollar, pound, or yen, decline in value, investors flee to silver ETFs as silver acts as a substitute for declines in values of safe haven currencies. However, silver has a complementary effect on weak currencies. When weak currencies such as special drawing rights decline, investors do not flee to silver, viewing silver as being subject to the same geopolitical and market risks as the weak currencies. They surmise that silver products will also decline in value, so that they will not make a suitable investment.

Investor sentiment about precious metals is the behavioral forecast made by investors on future prices of gold, silver, and platinum. For example, Lian et al. (2022) observed the significant influence of the MSCI World Country Index and the VIX Volatility Index on silver futures prices. This finding supports the effect of investor sentiment on silver prices as the MSCI World Country Index shows the effect of the sentiments of foreign investors, while the VIX Volatility Index shows the volatility of investments based on the differing perceptions of different investors. Investors may exhibit risk-aversion, whereby they select the safest investments, with the minimum of price fluctuations in silver prices. The Arrow-Pratt coefficient of risk-aversion provides a measure of the rate of change in relative risk-aversion with gradually increasing levels of risk, suggesting that investors in silver ETFs who are cautious about risk will invest in silver ETFs during periods of minimal geopolitical risk, and minimal market volatility. A second category of investor sentiment is moderate risk-taking. The moderate risk-taker will make investments in silver products to the extent that such returns yield positive returns. There may be many revisions of the amount of silver investment as the investor receives additional information about the profitability of the investment. The investor may revise price expectations downward if negative information about silver price forecasts is received. Conversely, the investor may revise price expectations upward if positive information about price expectations is received. The third type of investor is the risk-taker. The risk-taker is not restricted by risking risk, so that he or she will continue to invest in silver even during periods of high geopolitical risk and market risk in order to capitalize on the prospect of securing high returns.

The purpose of this study is to develop formulations of optimal silver ETF prices for risk-averse investors, moderate risk-takers, and risk-takers. Optimal ETF prices were obtained by intersecting mathematical formulations of each type of investor sentiment with silver ETF prices. These formulations were validated empirically, using current United States ETF data, along with proxies for risk-aversion and geopolitical risks, such as GDP, and market risk, such as the VIX Volatility Index. The study makes three significant contributions. First, it theoretically establishes investor sentiment as a predictor of silver ETF prices through the use of linear programming models. Second, it provides empirical support for the relationship between risk aversion, geopolitical risk, and market prices with silver

ETFs. Third, it distinguishes gold from silver, by showing that silver is a weak hedge against uncertainty.

2. Review of Literature

2.1. Background of Silver ETFs

Silver is a precious metal. It is mined at many worldwide locations, and can be sold as a single commodity in the form of bullion, silver jewelry, or as silver products used in industrial production. Silver Exchange Traded Funds (ETFs) are collections of silver mining stocks and silver products sold by brokerages. Ivanov (2013) found that silver ETFs closely track silver prices using tracking error and price deviation metrics. This finding suggests that silver ETF prices may be used as proxies for silver bullion prices. In short, tracking silver ETF prices is equivalent to measuring silver prices. Further, Ivanov (2013) proved empirically that ETFs have moved price discovery for silver to the ETF market, so that investors predicting future silver prices may find it to be more productive to examine silver ETF prices, rather than silver futures prices. Bernstein (2009) attributes the strength of ETFs in predicting silver prices to the strong demand for silver ETFs. Investors in silver ETFs seek high returns that match market volatility (Lian et al., 2022), and returns on less stable currencies (Shobha, 2024). Gold is a more effective inflation hedge as its prices are countercyclical to market index prices, while silver prices move in conjunction with security index prices.

Precious metals add diversification to ETFs as their movements may be subject to geopolitical risk in the form of idiosyncratic shocks. Chitale et al. (2017) investigated sources of idiosyncratic shocks in silver ETFs. They found that varying weather conditions, political risks, and production impediments could suddenly elevate or depress silver ETF prices. For example, earthquakes or violent storms in a silver-mining region, will reduce the supply of silver, spiking prices. In addition, Streck et al. (2024) posited that macroeconomic indicators such as Gross Domestic Product (GDP) and interest rates may have different effects on silver prices than other precious metals, such as gold and platinum due to silver's lesser value and greater correlation with the financial markets.

2.2. Investor Attitudes toward Risk

Chen (2022) examined the traditional Markowitz model, one of the cornerstones of investment theory. The model assumes identical investor attitude to risk, indicating that investors will select portfolios with maximum return for the same level of risk. However, empirical results have shown departure from this model presumably due to investors having personal preferences for other portfolios. For example, an entrepreneurial risk-taker may prefer a portfolio with higher risk than that supported by the mean-variance frontier of the Markowitz model.

Donadelli et al. (2025) set forth a consumption-based capital asset pricing model (CAPM). This model set forth that investors' choices of assets were based on both asset returns and investor preferences at different points in time. This concept is relevant to our study in that investors may select silver ETFs partly to achieve returns, and partly to satisfy personal preferences for precious metals. Such preferences were psychological, varying over time. The study by Bednoui and BenMoubaruk (2017) follows from this concept of CAPM that is based on investor preferences. Bednoui and BenMoubarouk (2017) estimated utility functions that showed departures from the traditional CAPM assumption of uniform investor risk. They identified different investor attitudes to risk captured in different betas that showed different changes in risk perceptions. The three types of investor attitudes to risk included 1) hyperbolic absolute risk aversion, 2) the constant relative risk-aversion, and 3) exponential utility function. Hyperbolic absolute risk aversion is the reluctance to assume additional risk to achieve higher returns. Such investors create utility functions in which the beta value < 1. By definition, beta is the sensitivity of firm risk to the overall market. With hyperbolic absolute risk-aversion, beta is so close to 0 that market changes in returns have little effect on the willingness of the investor to accept additional risk to earn higher returns. This study identifies this investor as the risk-averse investor, The second type of investor is a moderate risk-taker. Constant relative risk-aversion is associated with a beta that is close to 1. The investor takes some risk to take advantage of growth opportunities, yet the amount of risk accepted is similar to that accepted by the typical investor in the market. Finally, the exponential utility function reflects investors who concentrate on earning high returns, regardless of risk. The utility function is strongly upward sloping, suggesting that the investor will purchase additional quantities of the investment with the expectation of high returns. Beta > 2 indicates that market changes in returns will result in greater investor demand for the investment, raising price expectations by the investor of continuously higher returns.

2.3. Investor Attitudes towards Silver ETFs

There are few studies describing investor attitudes towards silver ETFs. This paragraph describes two studies in which silver ETFs exhibited abnormal returns, which investors may capitalize upon to boost the returns of their portfolios. The Shahid et al. (2020) study pertains to U.S. investors, while the Dias et al. (2024) provide information for MENA (Middle East and North Africa) investors. Shahid et al. (2020) set forth the adaptive market hypothesis to explain the predictability of silver and other precious metals. This hypothesis envisions silver prices as fluctuating, failing to be absorbed in market prices rapidly. This theory is counter to the efficient market hypothesis, which views asset prices as experiencing very short disruptions, after which they are absorbed into market prices, In this regard, silver prices could remain abnormally high or low for a certain length of time, as they are not reflected in regular market prices of commodities or securities. Some evidence was found using U. S. precious metals data. Dias et al. (2024) found evidence of such abnormal precious metal prices during periods of macroeconomic instability in MENA (Middle East and North Africa) countries.

In sum, prices of silver ETFs may correlate with market prices during stable economic periods, and may temporarily exhibit abnormal returns during periods of macroeconomic instability. Silver ETFs are less effective than gold as a hedge against inflation due to their correlation with geopolitical risk, and oil prices. They act as a safe haven upon the depreciation of strong currencies, though fail to be a safe haven upon the devaluation of unstable currencies. This paper links investor sentiment about risk-aversion to create optimal silver ETF price functions for risk-averse investor, moderate risk-takers, and risk-takers.

3. Findings and Analysis

3.1. The Risk-Averse Investor

Figure 1 shows the optimal silver ETF price, *P,* at the intersection of the Arrow-Pratt relative risk aversion function and the Bessel function of Silver ETF prices.

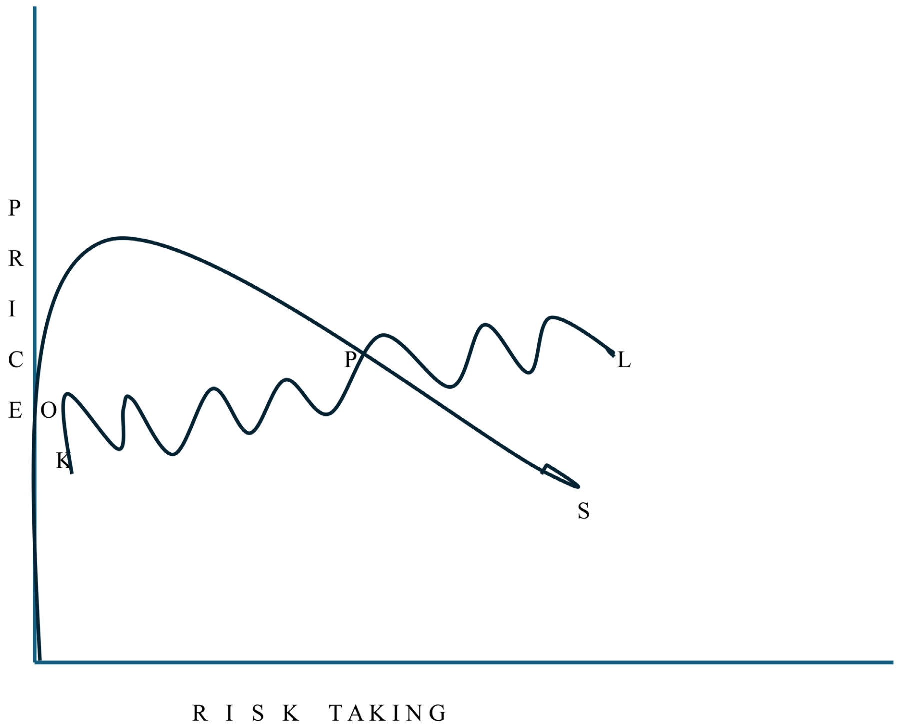

Figure 1. Optimal silver ETF prices for the risk-averse investor.

The optimal price of Silver ETFs for the risk-averse investor lies at P, the intersection of the Arrow-Pratt relative risk aversion function, *OS*, and the Bessel function of silver ETF prices, *KL*.

The risk-averse investor has a convex utility function, *OS,* wherein there is less utility for risky silver ETFs as risk aversion increases due to rising risk. The risk-averse investor only adds silver ETFs to his or her portfolio as risk increases. As risk increases, this investor becomes increasingly reluctant to make additional investments in silver ETFs due to the decreasing utility of additional investment with uncertain returns. Hence, the utility function, *OS*, is convex. This function intersects with the Bessel function of silver ETF prices, *KL*. KL is a classic Bessel function of symmetrical cylindrical shape often considered to be a wave function. It is an oscillating sine or cosine function that that decay to $x^{0.5.}$. Silver ETF prices are assumed to be of wave form with their rising and falling values with geopolitical risk, unstable currency prices, and market volatility. For example, weak currency prices rise and fall regularly, so that silver's correlation with them causes silver ETF prices to fluctuate. The objective function and constraints of this formulation are listed below.

Maximize

$$-cu''(c)/u'(c) \tag{1}$$

Equation (1) shows the Arrow-Pratt, measure of relative risk aversion. c consumption, $u(c)$ = utility of consumption, $u'(c)$ = rate of change in the utility of consumption, $u''(c)$ = maximum utility of consumption. The objective function maximizes the benefit from an additional unit of silver ETF at a diminishing rate over time, as the risk-averse investor grows increasingly resistant to investing in risky silver ETFs.

Subject to,

$$\sum m = 0 \; to \; \alpha\left(\frac{-1}{m!\int(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m}\right) > 0 \tag{2}$$

where, Equation (2) is the Bessel function solution to a symmetrically cylindrical wave differential equation (Dutka, 1995). The values of m and n are constants, while α is a nonnegative integer whose positive value yields a finite Bessel function.

$$\frac{1}{n}\int 0 \; to \; n \cos(nrxsinr) > 0 \tag{3}$$

where r is the angle between consecutive waves in the first order Bessel function.

Taking Lagrangians,

$$\begin{aligned} &-cu''(c)/u'(c) - L\_1(0 \; to \; \alpha\left(\frac{-1}{m!\int(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m}\right) - 0 \\ &-L\_2\left(\frac{1}{n}\int 0 \; to \; n\cos(nrxsinr) - 0\right) \end{aligned} \tag{4}$$

Taking first derivatives of Equation (4),

$$-cu''(c)/u'(c) - L\_'1(0 \; to \; \alpha\left(\frac{-1}{(m+\alpha+1)}\left(\frac{1\backslash}{2}\right)^{2m}\right) - L\_'2\left(\frac{1}{n}(-r - x\cos r)\right) \tag{5}$$

Taking second derivatives of Equation (4),

$$-cu''(c)/u'(c) - L\_''1(0 \; to \; \alpha\left(\frac{-1}{(\alpha)}\left(\frac{1\backslash}{2}\right)^{m}\right) - L\_''2\left(\frac{1}{n}(-1 + xsiner)\right) \tag{6}$$

Equation (6) defines point P, the optimal price of silver ETFs for the risk-averse investor.

3.2. The Moderate Risk-Taker

Figure 2 depicts the optimal price of silver ETFs, *Q*, for the moderate risk-taker.

The moderate risk-taker will accept some risky silver ETFs in his or her portfolio. If risk-aversion increases, the moderate risk-taker will forego excessive caution in favor of silver ETFs with geopolitical risks, and correlation with unstable currencies. Therefore, the curve of investor sentiment will slope downward with increasing risk-aversion, as shown in **Figure 2**. The moderate risk-taker will continuously revise the decision to invest in silver ETFs as new information about silver stock prices becomes available. The information enters silver ETFs at a time interval, *t*, called the time domain, with a frequency, *s,* termed the frequency domain. The time domain and the frequency domain together form a Laplace transform (Widder, 1941) of investor sentiment for the moderate risk-taker. This Laplace transform of investor sentiment *OS*, intersects the aforementioned Bessel function (see Section 3.1) of silver ETF prices *KM* to yield the optimal silver ETF price for the moderate risk-taker, *Q*.

Figure 2. Optimal silver ETF prices for the moderate risk-taker.

Figure 2 shows the optimal price of silver ETFs for the moderate risk taker, Q, at the meeting point of the Laplace transform of investor sentiment, *OS*, and the silver ETF Bessel price function, *KM*.

The mathematical formulation of the optimal silver ETF price follows.

Maximize

$$\frac{1}{2\pi ilimT} \to \alpha \int e^{st} F(s) ds \tag{7}$$

Subject to,

$$\sum m = 0 \; to \; \alpha(\frac{-1}{m!\int(m+\alpha+1)}(\frac{x}{2})^{2m}) > 0 \tag{8}$$

where, Equation (8) is the Bessel function solution to a symmetrically cylindrical wave differential equation. The values of m and n are constants, while α is a nonnegative integer whose positive value yields a finite Bessel function.

$$\frac{1}{n}\int 0 \; to \; n\cos(nrxsinr) > 0 \tag{9}$$

where r is the angle between consecutive waves in the first order Bessel function.

Taking Lagrangians of Equation (4),

$$\frac{1}{2\pi ilimT} \rightarrow \alpha\int e^{st}F(s)ds - L_1[\frac{1}{n}\int 0 \; to \; n\cos(nrxsinr) - 0] \tag{10}$$

Taking first derivatives of Equation (10),

$$\frac{1}{2\pi ilimT} \rightarrow \alpha e^{st}F'(s) - L_1[\frac{1}{n}\int 0 \; to \; n - \sin(nrxsinr)] \tag{11}$$

Taking second derivatives of Equation (10)

$$\frac{1}{2\pi ilimT} \rightarrow \alpha e^{st}F'(s) - L_2[\frac{1}{n} - \cos(nrxcosr)] \tag{12}$$

Equation (12) represents the optimal silver ETF price for the moderate risk taker that maximizes returns assuming revisions of silver ETF prices, based upon new information received periodically, subject to a Bessel function of symmetrically cylindrical silver ETF prices.

3.3. The Risk-Taker

The risk-taker's personality is such that he or she seeks continuously higher returns from silver ETFs, without regard to risk. As risk aversion among other investors increases, the risk-taker accumulates larger quantities of silver ETFs with the expectation of continuously higher returns. Figure 3 depicts this demand for rising silver ETFs at high prices as an exponential distribution of positive investor sentiment, *OS*. The exponential distribution assumes that information about geopolitical risks and market volatility due to negative reports, inflation, or wars, are independent Poisson processes that influence expectations of silver ETF prices. Risk takers are indifferent to these risks, as they are on a quest to achieve high returns, regardless of risk. The investor sentiment function intersects with the Bessel function of silver ETF prices, *KM*, at point *R*, the optimal silver ETF price for the risk taker.

Figure 3 depicts the optimal silver ETF price, *R*, for the risk taker, at the intersection of an upward-sloping exponential distribution of price expectations of higher returns from silver ETFs and *KM*, the Bessel function of cylindrically symmetrical silver ETF prices.

The mathematical formulation of the risk-taker's optimal silver ETF price is given below.

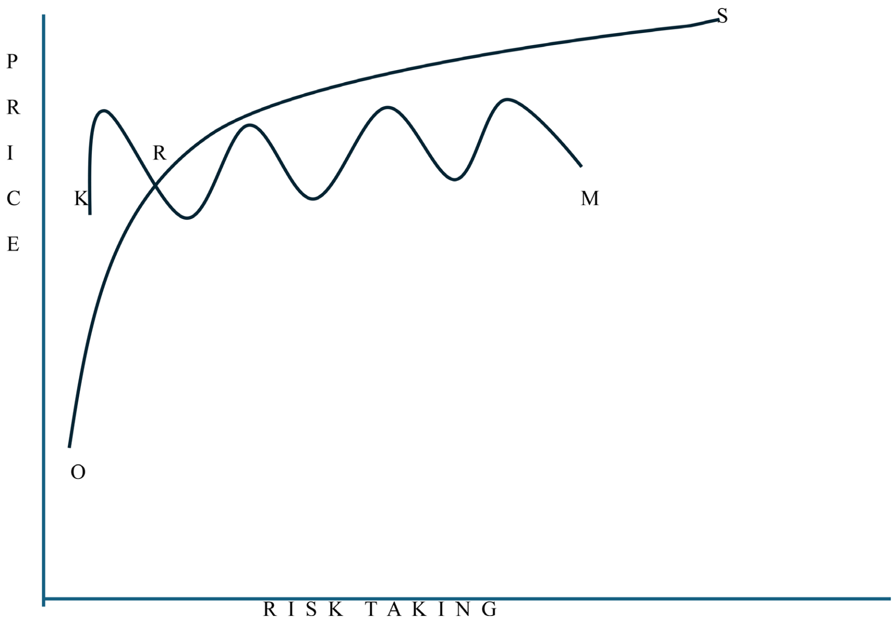

Figure 3. Optimal silver ETF prices for the risk-taking investor.

Maximize

$$\Sigma(1/(n-k)\lambda\Sigma(\frac{1}{(n-k)\lambda}+\frac{\Sigma 1}{(n-k)\lambda^2}+(\frac{\Sigma 1}{(n-k)\lambda})^2 \quad (13)$$

where,

λ = rate at which information about geopolitical risks and market volatility enters silver ETF prices,

n, k = frequency with which the investor's price expectations are adjusted in response to new information being embedded in silver ETF prices,

Subject to,

Equation (14) is the Bessel function solution to a symmetrically cylindrical wave differential equation. The values of m and n are constants, while α is a nonnegative integer whose positive value yields a finite Bessel function (Dutka, 1995)

$$\frac{1}{n}\int 0 \; to \; n\cos(nrxsinr) > 0 \quad (14)$$

We add a gradient vector to define the upward slope of price expectations,

$$\nabla > 0 \quad (15)$$

The gamma function underlying the Bessel function is expressed in the following constraint (Abramowitz & Stegun, 1964).

$$\frac{(-1)^m}{m!\prod(m+\alpha+1)(\frac{x}{2})^{2m}} > 0 \quad (16)$$

where the $\prod$ function is a gamma distribution of the factorial function $m!$, embedded in the Bessel function.

Taking Lagrangians,

$$\Sigma(1/(n-k)\lambda\Sigma(\frac{1}{(n-k)\lambda}+\frac{\Sigma 1}{(n-k)\lambda^2}+(\frac{\Sigma 1}{(n-k)\lambda})^2-L_1(\frac{1}{n}\int 0\ to\ n\cos(nrxsinr) \\ -0-L_2(\nabla-0)-L_2(\frac{(-1)^m}{m!\prod(m+\alpha+1)(\frac{x}{2})^{2m}}-0) \tag{17}$$

Taking first derivatives of Equation (17),

$$\Sigma(1/(n-k)\lambda'\Sigma(\frac{1}{(n-k)\lambda'}+\frac{1}{(n-k)\lambda'^2}+(\frac{1}{(n-k)\lambda'})^2 \\ -L_1'(\frac{1}{n}\cos(nrxsinr)-L_2'(\nabla')-L_2'(\frac{(-1)^m}{m!\prod'(m+\alpha+1)(\frac{x}{2})^{2m}}) \tag{18}$$

Taking second derivatives of Equation (17),

$$(\frac{1}{\lambda^n(n-k)\lambda^{n-1}}(\frac{1}{(n-k)\lambda^{;'}}+\frac{1}{\lambda(n-k)\lambda}+2(\frac{1}{(n-k)\lambda''})^1 \\ +L_1''(\frac{1}{n}\text{sine}(nrxsinr)-L_2''(\nabla'')-L_2''(\frac{(-1)^m}{m!\prod''(m+\alpha+1)(\frac{x}{2})^m}) \tag{19}$$

Equation (19) represents the optimal silver ETF price for the risk taker.

Three types of ETFs were listed in the data source. They include equity (low risk, high risk-aversion), commodities (moderate risk, moderate risk-aversion), and leveraged commodities (high risk, low risk-aversion). The rationale for the risk classification is that equity ETFs are composed of lesser risk silver mining stocks. Commodities are higher risk than stocks due to higher volatility, so are classed as moderate risk. Finally, leveraged commodities use more debt than commodities, making them the riskiest form of ETFs. Risk-aversion scores of 9 and 10 were awarded to equity ETFs, 5-8 for commodities, and 1 - 4 for leveraged commodities.

4. Data and Methodology

Data Collection

The following Silver ETF data (volume and closing prices) were downloaded on December 10, 2025. They include 1) IShares Silver Trust, 2) abrdn Physical Silver Shares, 3) Global Silver Miners, 4) Amplify Junior Silver, 5) ProShares Ultra Silver, 6) IShares MSCI Global Finance metals Miners, 7) Sprott Silver Mining, 8) UBS AG, 9) ProShares Ultra Short Silver, 10) Kurv Silver Enhanced, 11) Amplify Junior Silver Miners Covered Call, and 12) Themes Silver. The inclusion rule for silver securities is that they be ETFs, not silver mining stocks, and that the ETFs must be U.S.-based. Data is single cross-section, not a time series. Equation (1) shows the regression equation tested. Coding rules required the measurement of returns as percentages, risk-aversion as scores, and size in dollar millions. T tests for small samples adjusted for small samples.

$$Y_i=\alpha_i+\beta_1X_{1i}+\beta_2X_{2i}+\varepsilon \tag{1}$$

Y = Holding Period Returns,

α = Constant,

X_1 = Risk Aversion,

X_2 = Size (Total Assets).

5. Empirical Validation

The theoretical analysis in Section 3 was subject to empirical validation. The website of silver ETFs was accessed at https://www.etf.com/topics/silver. A total of 12 silver ETF data was collected. ETF size was measured by assets under management which ranged from $16.84 million (for Themes Silver Mines) to $47.32 billion (for iShares Silver Trust). ETF returns were measured by the 3-month trailing return, which acted as a proxy for price. Risk Aversion was measured by the type of ETF, on scores of 4, 8, and 10. The lower the risk of the fund, the higher the risk aversion. Commodity equity funds had the highest risk aversion, giving them a score of 4. Leveraged commodity equity funds had more debt than commodity equity funds, so lower risk aversion at a score of 8. Equity silver mines had the lowest risk aversion, i.e., highest risk, giving them a score of 10. **Table 1** lists descriptive statistics for these variables.

Table 2 shows the correlation matrix for the variables listed in **Table 1**.

Table 1. Descriptive statistics.

Variable	Mean	Variance	Skewness	Kurtosis
Size	5.96	176.29	3.23	10.82
Risk Aversion	8.0	6.54	−0.94	−0.85
Return	46.73	382.29	−1.07	1.88

Table 2. Correlation matrix of predictors and criterion.

Variable	Size	Return	Risk Aversion
Size	1	0.16	0.27
Return		1	0.94***
Risk Aversion			1

***$p < 0.001$.

Table 3. Results of regression of Silver ETF returns on ETF size and risk aversion

Variable	Coefficient	T statistic	Significance
Constant	34.33	1.90	0.08
Risk Aversion	4.26	2.12	0.05*
Size	−0.80	−2.05	0.07
N, R^2	12, 0.93		

*$p < 0.05$.

Table 3 shows the results of the regressions of silver ETF prices, i.e. Returns on Risk Aversion, and ETF Size. Risk Aversion significantly explained the variation in silver ETF prices (coefficient = 4.26, t = 2.12, $p < 0.05$). This result suggests that returns on silver ETFs increase with risk aversion, so that risk-averse investors may experience the highest returns from investment in silver ETFs.

5. Conclusions

5.1. Discussion of Findings

This paper has created theoretical formulations leading to the optimal price of silver ETFs for three types of investors, i.e. the risk-averse investor, the moderate risk-taker, and the risk-taker. The three types of investors are distinguished from each other on the basis of attitude toward risk, or risk-aversion. Risk-averse investors reduce the investment in silver ETFs as returns and risk rise, moderate risk-takers only increase risk to the extent that they can earn positive returns, and risk-takers take undue risk with the expectation of achieving very high returns. The theory was subjected to empirical validation. Silver ETF returns (proxy for silver ETF prices) were regressed on risk-aversion, based on the type of silver ETF under consideration. Risk-averse investors were found to achieve the highest returns from silver ETFs, as the increase in risk-aversion resulted in higher silver ETF returns. Therefore, the increase in demand for lower-risk silver ETFs resulted in the best performance of silver ETFs.

The paper's main contribution lies in its formulation of silver ETF prices as a function of investor sentiment in terms of attitude to risk. While silver ETFs provide diversification in investor portfolios, such investment depends upon investor attitude to risk. Bednoui and BenMoubarouk (2017) estimated utility functions that showed departures from the traditional CAPM assumption of uniform investor risk. They identified different investor attitudes to risk captured in different betas that showed different changes in risk perceptions. The three types of investor attitudes to risk included 1) hyperbolic absolute risk aversion, 2) the constant relative risk-aversion, and 3) exponential utility function. Hyperbolic absolute risk aversion is the reluctance to assume additional risk to achieve higher returns. Such investors create utility functions in which the beta value < 1. We add to this result by finding that the highest returns for silver ETFs are obtained by investors who have hyperbolic absolute risk aversion. Such risk-averse investors add a small percentage of silver ETFs to their portfolios. They are unlikely to keep increasing the percentage allocation of silver ETFs as that would increase beta > 1, or decrease the coefficient of risk aversion in the regression of silver ETF returns on risk-aversion.

5.2. Theoretical Implications

Both the Markowitz (1952) model and CAPM (Sharpe, 1964) assume that investors have uniform risk preferences. This paper challenges that assertion. This paper sets forth that for investments in precious metals like silver ETFs, different levels of risk tolerance by different investors result in different optimal prices. Empirically, this study's finding is that risk-averse investors achieve the highest returns as they recognize that silver ETFs can be very risky. Risk-averse investors will limit investment in silver ETFs to the level at which returns rise continuously. If silver returns suddenly decline, such investors will immediately cease to invest in silver.

Future research may modify the Markowitz (1952) model and the CAPM model to include risk-aversion as a predictor of security returns. First, the Markowitz model traditionally consists of risk-return portfolios with the optimal portfolios with lowest risk and highest return lying on the mean-variance frontier. The mean-variance frontier may be redrawn using risk aversion. Risk-aversion will result in the rejection of portfolios at higher levels of risk. **Figure 4** shows *OS*, the curve of traditional mean-variance portfolios without risk-aversion. *PX* is the modified mean-variance frontier with risk-aversion.

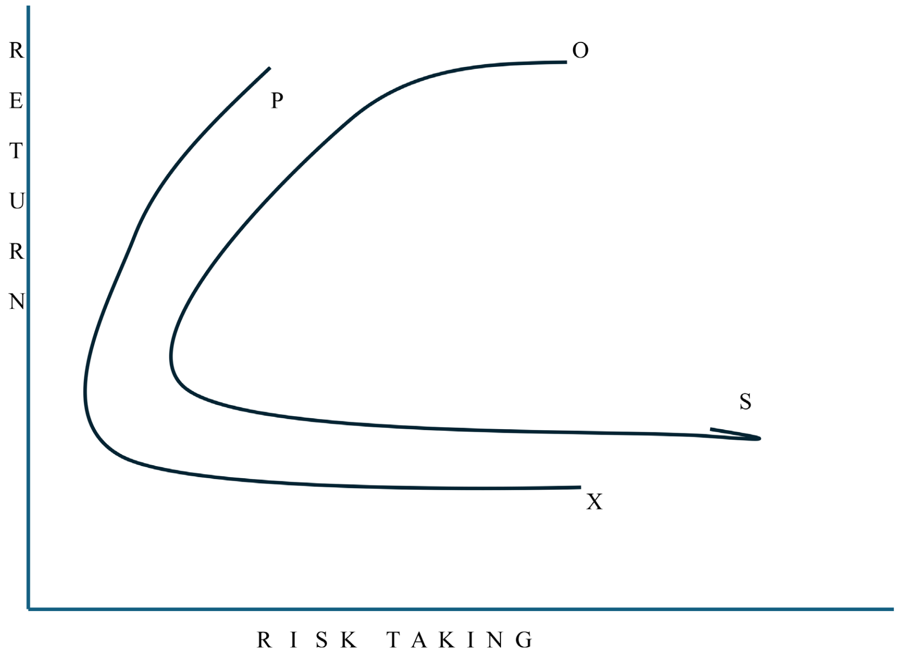

Figure 4. Mean-variance portfolio frontiers in the Markowitz model.

Figure 4 creates mean-variance frontiers in the Markowitz model. OS is the mean-variance frontier in the traditional Markowitz model. PX is the mean-variance recommended by this study.

Figure 5 shows the CAPM model which plots security returns on the security's beta coefficient (measure of systematic risk). The Security Market Line shows the original optimal portfolio at point *M*. If we add the modified Markowitz portfolio with risk-aversion, that portfolio will be tangent to the Security Market Line, or the optimal price *T* is the price of the portfolio with minimum return for a level of risk-aversion.

Figure 5 shows *M*, the optimal risk-return combination for a two-asset portfolio in the traditional CAPM model. The modified CAPM model adds risk aversion to the Security Market Line, so that the optimal portfolio at point *T*, has lower risk than the traditional CAPM portfolio, *M*.

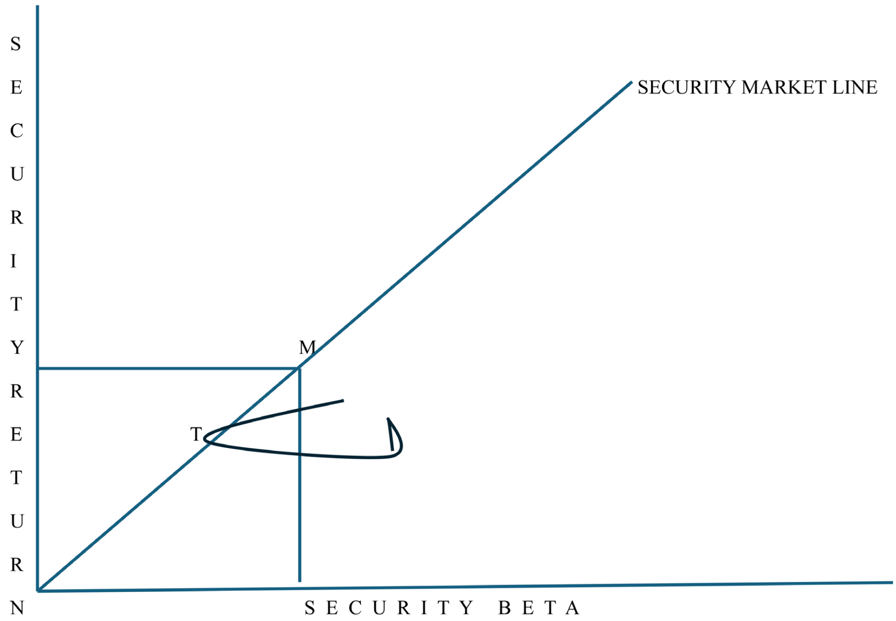

Figure 5. Optimal risk-return combinations of CAPM portfolios.

5.3. Research Limitations

The study was conducted using silver ETFs in the United States. It should be replicated for ETFs offered in other countries, in order to obtain a more comprehensive assessment of the predictors of silver ETF prices.

Silver prices are correlated with geopolitical risks, market volatility, and weak currency prices during economic downturns. The study may be replicated in environments with high geopolitical risks such as war or political instability. The study could be done in environments with hyperinflation, where the surge in prices makes silver a safe haven for investors. Finally, the study should be relicated in countries with unstable currencies, particularly during recessions, and other economic downturns.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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