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A Mathematical Formulation of the Valuation of Short Sales and Put Options on Real Estate

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Abstract

The purpose of this article is to construct mathematical valuation models of short sales and put options on real estate with declining values. For 1) farmland, the underlying assumption is that prices decline gradually. The first formulation is for the risk-averse investor, who takes minimum gains with the decline in real-estate prices. The investor's sentiments are represented by the Arrow-Pratt coefficient of risk-aversion. The pricing distribution is for a Levy-Ito decomposition. The second formulation is for the moderate risk-taker whose price expectations are modeled by a Bessel function with a Levy-Ito decomposition as price function. This strategy offers modest gains. The third formulation is for the risk-taker with price expectations modeled by an exponential distribution and a pricing function with a Levy-Khintchine formula that has sharp gains. For 2) decaying shopping malls, prices decline initially, followed by upward movement as the decaying malls are repurposed into apartments and cultural centers. The risk-averse investor's Arrow-Pratt coefficient of risk-aversion is subject to a Lebesgue integral of falling, then rising prices, to yield modest gains. The moderate risk-taker's Bessel function is also subject to the Lebesgue integral with puts that are exercised to yield gains. The risk-taker's exponential distribution is subject to a Lebesgue integral. However, the overconfidence of this investor results in prices increasing before put exercise, so that no exercise occurs. For 3) mines and quarries, a freely falling price distribution is used, as no recovery in prices is anticipated. Riemann integrals model the price distribution for each type of investor, as they show a vertical decline in prices. The short-sale strategy of the risk-averse investor and the put purchase strategy of the moderate risk-taker yield gains, while the put purchase strategy of the risk-taker results in losses. The paper advances knowledge by developing profit and loss strategies for three negatively priced investments.

Keywords

Put Options, Real Options, Real Estate Options, Levy-Khintchine Formula, Lebesgue Integral, Riemann Integral

1. Introduction

Investors with long-term investment horizons may consider investing in real estate. Iacoviello and Ortalo-Magne (2003) set forth that investors may increase their portfolio wealth by hedging property price risk using derivatives. The purpose of this paper is to provide mathematical models for the valuation of derivatives on declining commercial real estate values. The first category of negatively priced commercial real estate is farmland. Farmland in the United States is used to grow crops such as corn, wheat, and soybeans. Corn may be processed into ethanol which is an additive to gasoline. Alternatively, livestock including cattle, poultry, and hogs may be raised on farmland. Basha et al. (2022) documented the falling values of Midwestern US farmland in 2020 following the interest rate hikes of 2015-2018. Net farm income in 2020 was 13% below 2013 income due to the reduction on loan availability leading lenders to increase rates on the limited supply of available loans. It follows that successive interest rate increases in 2024 reduced the values of farmland.

The second category of declining real estate values arises from abandoned shopping malls. Shopping malls were the hub of community activity from 1980-2010, as their variety of stores selling clothing, arts, crafts, luggage, furniture, household items, and food, attracted shoppers of all ages. Shim and Eastlick (1998) set forth that customer values for social affiliation or identification with the social groups present at the mall strongly influenced a positive attitude toward shopping attributes, such as social interaction with fellow shoppers, attraction to store displays, and affinity to the bright lights and modern amenities of malls. The advent of online sales drove shoppers indoors, where the click of a mouse permitted them to purchase the identical goods available for sale at the mall, obviating the need to lose time to travel to the mall. Social affiliation has become less important than acquiring goods in a timely fashion.

The third category of declining real estate values originates from mines and quarries that have been stripped of all of their mineral resources. Martinez-Lopez et al. (2021) identified the Sierra Minera de Cartagena-La Union region of Spain as having many years of mining activity until 1990. Metallic sulfides such as lead, zinc, and iron have been found in the soil and groundwater of this region. Their review describes potential toxic elements (PTEs) contained in mining ponds with dissolved metals and sulfates polluting the water. Pyrite oxidation created new toxic minerals which were transferred to four plant species and animal species such as amphipods and sea urchins. It follows that land in regions with toxic chemicals in the groundwater and soil carries significant health risk. Contaminated land cannot be used for construction of homes, hospitals, and schools. As wasteland, such land experiences declining values.

Investors who wish to gain from declining real estate values may do so in two ways. Short sales of real estate consist of borrowing to purchase real estate with declining values. After the price declines, the investor repays the loan with cheap land that has already declined in value. This strategy results in gains to the investor. Alternatively, the investor may purchase put options on real estate. A put option confers the right to sell the property at the exercise price, which if greater than the value of the land, grants a gain to the investor of (Exercise price – Land Price). Multiple rounds of put trading yielding gains in every round may confer significant gains to the put buyer. We sense a research gap in that there is no mathematical formulation of the gains to the short seller or put buyer on real estate. This study is directed to rectifying this omission.

This study advances knowledge in three ways. First, it builds on early theoretical formulations of the valuation of options on real estate. Syz (2008) derived the price of a call option on real estate as a Black-type formula. Van Bragt et al. (2015) priced options on real estate assuming the existence of a commercial property index as the underlying asset. Fabozzi et al. (2012) created a real estate option pricing model with stochastic interest rates on the mortgages financing the real estate investment. While these models may be used to value real estate with rising or falling prices, this study is focused on the valuation of put options on real estate with continuously declining prices. Second, we select situations in which market conditions guarantee price declines. There is no possibility of losses on short sales or put purchases due to rising real estate prices. Thus, our model effectively hedges against price risk. Third, we account for price gains due to the volatility of the three-month time period between put option purchase and price settlement. Put options have a spot price on the day of purchase and a term price three months into the future based on expectations of future prices. Our model includes both prices, unlike certain other models which combine the two prices into a single unit despite their conceptual differences (see Geltner & Fisher's, 2007 critique).

2. Review of Literature

2.1. Background of Real Estate Investing

Investments in real estate may be either in residential real estate or commercial real estate. Residential real estate consists of single-family homes or multifamily condominium units. Fabozzi et al. (2019) set forth that the home is one of the largest investments in household portfolios. As house values increase, the appreciating value of the home(s) build investment wealth. Commercial real estate, with investments in office buildings, apartments, and retail stores, builds investor wealth as a steady stream of rents is capitalized to increase the value of the properties.

The fundamental problem with real estate investment is price risk due to illiquidity. Illiquidity suggests that properties may remain unsold for long periods of time, during which prices may fluctuate, assuming a trajectory that differs from the original projected prices. For example, a fully-occupied shopping mall with strong cash flows from rental income, may suddenly experience vacancies, due to businesses being unable to compete with online sales. Rental income is depleted, with renters being unable to provide the cash flow for mall owners to meet mortgage payments. Property values decline sharply. Short selling can mitigate price risk, although high down payments and transaction costs may limit its usage. The investor borrows in order to purchase the property at the prevailing price. After the price declines, the investor repays the loan with cheaper real estate.

Derivatives can manage property illiquidity and hedge price risk with a small investment. Instead of investors supplying a large amount of capital for purchase of the property, they can purchase put options on the property. As mortgage rates rise, property values decrease due to property remaining unsold as investors are unable to obtain mortgages. Investors may purchase put options which permit them to sell the real estate at high prices. Then, the property is purchased at the reduced price, to achieve a gain. Multiple rounds of trading can yield significant gains. Early options research was devoted to creating put option strategies to mitigate mortgage price risk. Case and Shiller (1996) devised put strategies, theorizing that periods of price declines from mortgage rate increases, were followed by periods of price increases. In their model, put writers sell put options to earn option premiums from put buyers who never buy puts on real estate due to price increases. Curcio et al. (2015) developed put options that permit put buyers to gain as mortgage interest rates rise, and property values fall.

2.2. Models of Put Strategies for Real Estate

The Syz (2008) options model on real estate valued call options using a modification of the Black-Scholes formula with options being purchased on an underlying housing index. Put-call parity established the price of the put option corresponding to the call option with the same value of the housing index. Our concern about this model is its assumption that real estate prices follow a price trajectory similar to securities as assumed by the Black-Scholes model. Van Bragt et al. (2015) extended the Syx model by estimating the housing price index as the weighted average of lognormal variables. Ciurlia and Gheno (2009) reasoned that since housing prices were sensitive to both interest rates and volatility, that option pricing models must take these two factors into account. They decomposed put option prices into a spot price based on today's value and a term price at maturity based on price fluctuations during a three-month period. Interest rates on mortgages influenced spot prices, while volatility influenced maturity prices. Fabozzi et al. (2012) proposed a variation to the Ciurlia and Gheno (2009) model, by adding the case of stochastic interest rates. Fabozzi et al. (2019) crafted an ARMA-EGARCH model that valued both call options and put options on real estate. Upon testing with U.K. housing index data, put options yielded the higher return.

3. Findings

3.1. Investment in Farmland

3.1.1. Short Selling by the Risk-Averse Investor

The sentiments of the risk-averse investor are modeled by the Arrow-Pratt coefficient of risk-aversion. The coefficient of risk-aversion assumes the form $A(c)$,

$$A(c) = -u''(c)/u'(c) = 1 \tag{1}$$

where,

$A(c)$ = an individual's propensity to avoid risk.

$u(c)$ = utility function of payoffs.

Figure 1 shows the risk-averse investor's desire for real estate with decreasing prices. Demand is flat until it slopes upwards vertically in line *OP*, as higher and higher returns for additional risk are contrary to the risk-aversion of this investor's psyche. At point *S*, the risk-averse investor borrows to purchase farmland. Point *S* is the maximum price at which farmland can be sold. Then, price of farmland declines to point *B*, at which the investor repays the loan in cheaper land, making a gain. The objective of the investor is to maximize the utility of the gain. Maximization of the gain is an increase in the utility of wealth from the short-sale transaction, or the reduction in risk-aversion from the increase in gains from short-selling. This function is the first derivative of Equation (1),

$$\text{Max}\frac{\mathrm{d}}{\mathrm{d}x}A(c) = \left\{-u'(c)u(c) - \left[u(c)\right]^2\right\} \Big/ u'(c)^2 \tag{2}$$

Subject to

$$-\prod\left(R(-1,1)\int\left(\mathrm{e}^{i\theta x} - 1\right)v(\mathrm{d}x) + \int\left(\mathrm{e}^{i\theta x} - 1 - i\theta x\right)\mu(\mathrm{d}x)\right. \tag{3}$$

Equation (3) is the constraint to the maximization of short-sale gain. It describes the price distribution of farmland with continually declining prices. Equation (3) is a Levy-Ito decomposition of Brownian motion with drift, θ. Prices decrease in independent increments with discontinuous jumps, represented as *ABCD*. A negative sign has been laced before the first item of the equation to denote falling prices. The second derivative of the Lagrangian function of Equation (1) and Equation (2) defines the minimum price, or maximum gain from short selling of farmland.

Taking Lagrangians,

$$\begin{aligned}\text{Max}\frac{\left\{-u'(c)u(c) - \left[u(c)\right]^2\right\}}{u'(c)^2} &- -\prod\left(R(-1,1)\int\left(\mathrm{e}^{i\theta x} - 1\right)v(\mathrm{d}x)\right. \\ &- L\int\left(\mathrm{e}^{i\theta x} - 1 - i\theta x\right)\mu(\mathrm{d}x)\end{aligned} \tag{4}$$

Or

$$\begin{aligned}\text{Max}\frac{\left\{-u'(c)u(c) - \left[u(c)\right]^2\right\}}{u'(c)^2} &+ \prod\left(R(-1,1)\int\left(\mathrm{e}^{i\theta x} - 1\right)v(\mathrm{d}x)\right. \\ &- L\int\left(\mathrm{e}^{i\theta x} - 1 - i\theta x\right)\mu(\mathrm{d}x)\end{aligned} \tag{5}$$

Taking first derivatives of Equation (5),

$$\text{Max}\frac{\left\{-u'(c)u(c)-\left[u(c)\right]^2\right\}}{u'(c)^2}+\prod\left(R(-1,1)\left(e^{i\theta x}-1\right)v-L'\left(e^{i\theta x}-1-i\theta x\right)\mu \quad (6)$$

Note that the utility function of the risk-averse investor does not change as the investor does not derive addition utility of wealth from significant declines in farmland prices. The pricing distribution changes due to the presence of jumps or sharp declines in farmland prices.

Taking second derivatives of Equation (5) to obtain the maximum gain, minimum price short sale function,

$$\text{Max}\frac{\left\{-u'(c)u(c)-\left[u(c)\right]^2\right\}}{u'(c)^2}+\prod\left(R(-1,1)\left(e^{i\theta x}\right)v-L''\left(e^{i\theta x}-i\theta x\right)\mu \quad (7)$$

where,

$+\prod\left(R(-1,1)\left(e^{i\theta x}\right)v$ = Brownian motion of the movement of a random variable *x*, of farmland prices, with drift *v*.

μ = mean of the distribution of farmland prices.

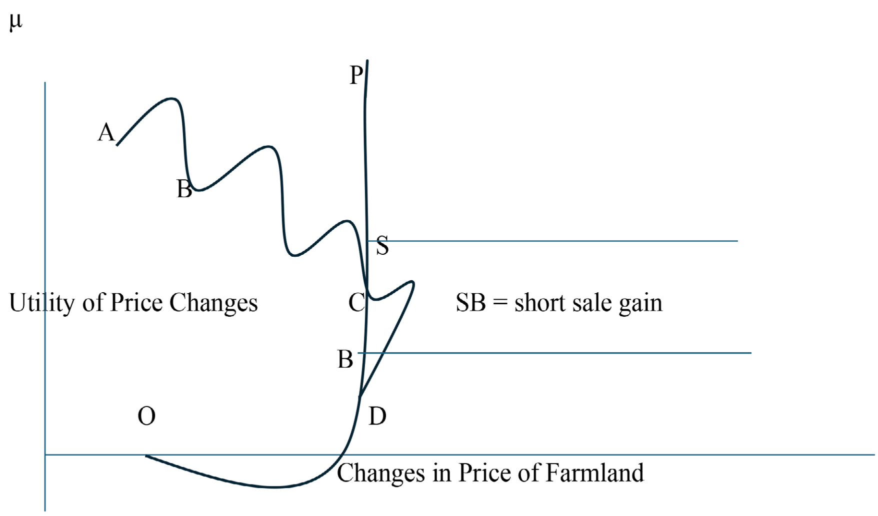

Figure 1. The gain to the risk-averse short sale investor in farmland.

Figure 1 shows the risk-averse investor's desire for real estate with decreasing prices. Demand slopes upwards vertically in line *OP*, as higher returns for additional risk do not yield higher utility. At point *S*, the risk-averse investor borrows to purchase farmland. Then, the price of farmland declines to point *B*, at which the investor repays the loan in cheaper land, making a gain.

The risk-averse investor does not invest in put options, deeming the strategy to be too risky.

3.1.2. Put Buying by the Moderate Risk Taker

The moderate risk-taker wishes to earn higher returns than the limited returns obtained from two rounds of short selling, which is the maximum amount of short selling permitted by regulation. The excess gains may be obtained from buying put options on farmland. The investor sentiment for gain from negatively priced farmland may be modeled by a Bessel function. The Bessel function accommodates revisions of price expectations as put buying progresses into multiple rounds. In the first round, the investor purchases the put option by paying a premium to the put writer. As farmland prices decline, the put option is exercised to sell the farmland at the higher exercise price. The gain is the (Exercise price - Farmland sale price). More puts are purchased at the lower farmland sale price. Prices decline, put options are exercised and further gains are taken. The moderate risk-taker uses the waves of the Bessel function to revise price expectations as gains decrease with rising put premiums in successive rounds of trading. Gains in later rounds of put trading are less than gains in earlier rounds of put trading. In Equation (7), the Bessel function is defined by the Hanssen-Bessel function (Weisstein, 1996), which is a sum of integrals, where the integrals represent the revisions of expectations of gain with a cosine function with higher expectation of gain followed by an exponential function with incremental gain. The first constraint in Equation (8) is a Levy-Ito decomposition of a Levy jump process with Laplace transforms to represent the revision of prices due to the high volatility of put prices during the three-month term period before maturity, when prices of farmland can vary so extensively, that final farmland prices may yield gains that are higher or lower than originally anticipated.

The formulation is as follows.

$$\text{Max}\frac{1}{\prod\cos\int(nr-x\sin r)\mathrm{d}r}-\frac{1}{n}\int \mathrm{e}^{inr-x\sin r}\mathrm{d}r \tag{8}$$

where r is a random variable representing the moderate risk-taker's preferences for farmland. The cosine function represents certainty of price predictions, while the exponential function consists of drift.

Subject to

$$-\prod\left(R(-1,1)\int\left(\mathrm{e}^{i\theta x}-1\right)v(\mathrm{d}x)+\int\left(\mathrm{e}^{i\theta x}-1-i\theta x\right)\mu(\mathrm{d}x)+(s-s_0)\mathrm{e}^{s-s_0}\beta(t)\mathrm{d}t \tag{9}$$

where,

$-\prod\left(R(-1,1)\int\left(\mathrm{e}^{i\theta x}-1\right)v(\mathrm{d}x)+\int\left(\mathrm{e}^{i\theta x}-1-i\theta x\right)\mu(\mathrm{d}x)\right.$ = Levy-Ito decomposition of put option prices,

$(s-s_0)\mathrm{e}^{s-s_0}\beta(t)\mathrm{d}t$ = Laplace transform of put price revisions, with s representing put prices, and β representing revisions at increments of time.

Taking Lagrangians,

$$\begin{aligned}&\text{Max}\frac{1}{\prod\cos\int(nr-x\sin r)\mathrm{d}r}-\frac{1}{n}\int \mathrm{e}^{inr-x\sin r}\mathrm{d}r\\&-L\Big[\prod\left(R(-1,1)\int\left(\mathrm{e}^{i\theta x}-1\right)v(\mathrm{d}x)\right.\\&+\int\left(\mathrm{e}^{i\theta x}-1-i\theta x\right)\mu(\mathrm{d}x)+(s-s_0)\mathrm{e}^{s-s_0}\beta(t)\mathrm{d}t\Big]\end{aligned} \tag{10}$$

Taking first derivatives of Equation (10),

$$\text{Max}\frac{1}{\prod-\sin\left(nr-x\sin r\right)}-\frac{1}{n}\mathrm{e}^{inr-x\sin r}$$
$$-L'\left[\prod\left(R\left(-1,1\right)\left(\mathrm{e}^{i\theta x}-1\right)v+\left(\mathrm{e}^{i\theta x}-1-i\theta x\right)\mu+\left(s-s_0\right)\mathrm{e}^{s-s_0}\beta\left(t\right)\right]\tag{11}$$

Taking the second derivative of Equation (10) to obtain the minimum farmland price with maximum gain from put purchases.

$$\text{Max}\frac{1}{\prod-\cos\left(n-x\cos r\right)}-\frac{1}{n}\mathrm{e}^{inr-x\sin r}$$
$$-L''\left[\prod\left(R\left(-1,1\right)\left(\mathrm{e}^{i\theta x}-1\right)+\left(\mathrm{e}^{i\theta x}-i\theta x\right)\mu+\left(s-s_0\right)\mathrm{e}^{s-s_0}\beta'\left(t\right)\right]\tag{12}$$

Figure 2 depicts the Bessel function of investor expectations *OS* intersecting with the Levy jump process with Laplace transforms for price revisions. The point *T* is the location of put option purchases. After property prices decline the put is exercised at point *U*, to obtain a gain of *TU*.

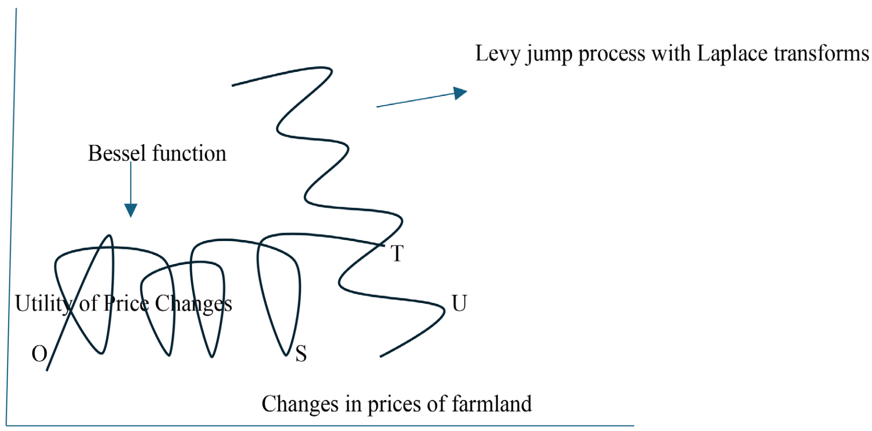

Figure 2. Put buying by the moderate risk-taker.

Figure 2 depicts the Bessel function of investor expectations *OS* intersecting with the Levy jump process with Laplace transforms for price revisions. The point *T* is the location of put option purchases. The put is exercised at point *U*, to obtain a gain of *TU*.

3.1.3. Put Buying by the Risk-Taker

The risk-taker has an insatiable demand for negatively priced farmland. As the risk-taker is not concerned about the risk of losing invested funds, he or she will seek to maximize gains through multiple rounds of trading. The desire to earn more gains is modeled by an upward-sloping exponential distribution. In Figure 3, the curve *OT* depicts the exponential distribution of investor sentiment. As prices of farmland decrease, sharp jumps of rapidly falling prices ensue, as it is these rapid declines that maximize gains for the put buyer. The Levy-Khintchine formula is used to represent sharp jumps in reduced farmland prices. Falling put prices are presented as *PQRS* jumps in Figure 3. Points *P, R,* and *S*, present successive put purchases and the accumulation of put gains.

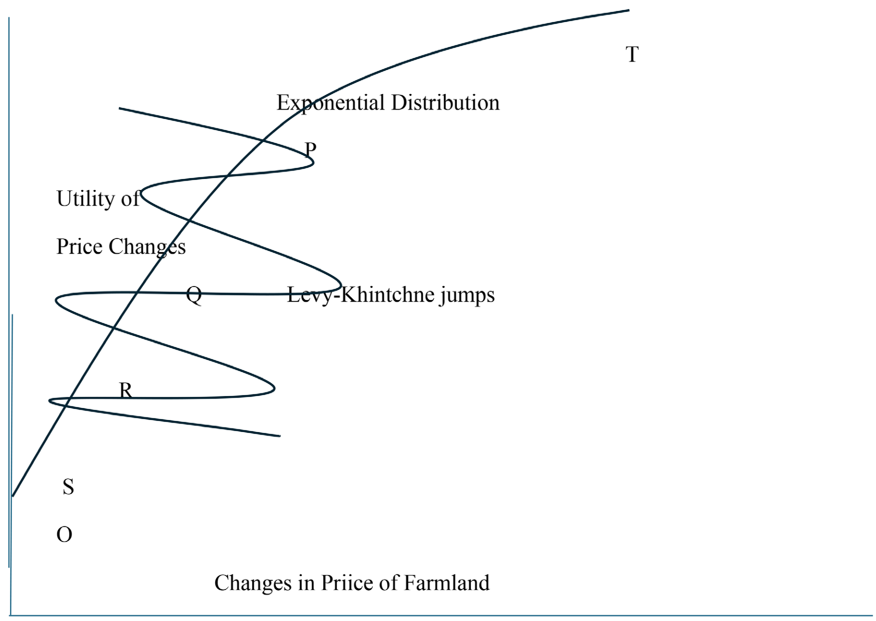

Figure 3. The optimal prices of the risk-taker's investment in farmland.

Figure 3 shows the exponential distribution *OT*, for the risk-taking put buyer. *PQRS* shows the jump process for continuously declining put option prices. Gains can be taken at any point *P*, *Q*, *R*, *S*, at which the exponential distribution intersects with the jump process.

The formulation is presented below.

Maximize the exponential distribution of investor sentiments, subject to the Levy Khintchine formula for the price distribution,

$$\text{Max } \exp\prod\left(-x\lambda_i\right)+\lambda_0/\left(\lambda_1\cdots\lambda_n\right) \tag{13}$$

where λ = is the inverse scale of the exponential distribution, and x is a random variable.

Subject to

$$\exp\left(t\left(ai\theta-0.5\sigma^2+\int\left(\mathrm{e}^{i\theta x}-1i\theta x1_{x<1}\right)\prod \mathrm{d}x\right)\right) \tag{14}$$

Θ = incremental change in farmland prices.

Taking Lagrangans,

$$\begin{aligned}&\text{Max } \exp\prod\left(-x\lambda_i\right)+\lambda_0/\left(\lambda_1\cdots\lambda_n\right)\\&\quad -L\left[\exp\left(t\left(ai\theta-0.5\sigma^2+\int\left(\mathrm{e}^{i\theta x}-1i\theta x1_{x<1}\right)\prod \mathrm{d}x\right)\right)\right]\end{aligned} \tag{15}$$

Taking first derivatives of Equation (15),

$$\begin{aligned}&\text{Max } \exp\prod\left(-x\lambda_i\right)+\lambda_0/\left(\lambda_1\cdots\lambda_n\right)\\&\quad -L'\left[\exp\left(t\left(ai\theta-0.5\sigma^2+\left(\mathrm{e}^{i\theta x}-1i\theta x1_{x<1}\right)\prod\right)\right)\right]\end{aligned} \tag{16}$$

Taking the second derivative of Equation (15), yields the maximum gain to the risk-taking put buyer,

$$\text{Max } \exp\prod\left(-x\lambda_i\right)+\lambda_0/\left(\lambda_1\cdots\lambda_n\right)-L''\left[\exp\left(ai\theta-\sigma+\left(e^{i\theta x}-i\theta x 1_{x<1}\right)\prod\right)\right] \quad (17)$$

3.2. Derivatives on Real Estate in Decaying Shopping Malls

The advent of online shopping has reduced foot traffic in shopping malls to the extent that shopping malls are closing each year. In 1986, the United States had 25,000 shopping malls. This figure has declined to 1150 in 2024, with 1170 malls closing every year from 2017-2022. Certain malls, upon experiencing losses from declining rent, repurpose their malls into apartments. These apartments replace rental units, by offering amenities such as art galleries, theaters, entertainment arcades, and culinary centers in the space of closed retail stores. For example, the Highland Park Mall in Austin, Texas has 80 acres of such amenities. The pricing distribution for decaying malls is most appropriately modeled as a Lebesgue integral (Lebesgue, 1972) with declining curve that turns upwards, as the decline in real estate prices ends with the creation of apartments, so that prices increase, i.e. the rents from new apartment tenants and art galleries, theaters, etc., drives an increase in mall prices.

3.2.1. The Risk-Averse Investor's Short-Sale Strategy in a Decaying Mall

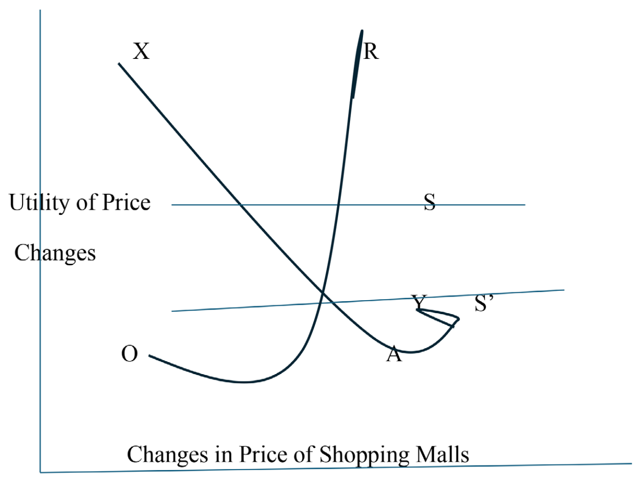

Figure 4. The short sale strategy of the risk-averse investor in malls.

The investor's expectation of mall prices is a *OR*, with an expectation of modest profits from 2 rounds of short-selling. The investor borrows ownership in the mall at point *S*. After prices decline, he or she repays the borrowed real estate in lower priced real estate at point *S'*. *SS'* is the gain from short-selling.

Figure 4 depicts the short-sale strategy of the risk-averse investor. The investor's expectation of mall prices is a *OR*, with an expectation of modest profits from 2 rounds of short-selling. The investor borrows ownership in the mall at point *S*. After prices decline, he or she repays the borrowed real estate in lower priced real estate at point *S'*. *SS'* is the gain from short-selling. This is the first round of short-selling. After 2 rounds, short-selling ends, and the investor leaves the market. The investor does not pursue further rounds of trading in the real-estate market, as the rise in mall prices at point A will result in losses with certainty. A risk-averse investor's concern for losses will prevent further short selling in case price increases extinguish all of the profits.

$$\text{Max } A(c) = -u''(c)/u'(c) = 1 \tag{18}$$

where,

$A(c)$ = an individual's propensity to avoid risk.

$u(c)$ = utility function of payoffs.

Subject to,

$$\int f\text{d}\mu + i\int g\text{d}\mu > 0 \tag{19}$$

where f, and g are real-valued functions of the Lebesgue integral, each of which is integrable.

Taking Lagrangians of Equation (19),

$$\text{Max } A(c) = -u''(c)/u'(c) - 1 - L\left[\int f\text{d}\mu + i\int g\text{d}\mu\right] \tag{20}$$

Taking the first derivative of Equation (20),

$$\text{Max } -u''(c)/u''(c) - L'\left[f\text{d}\mu + i(g)\right] \tag{21}$$

Taking the second derivative of Equation (20), to achieve maximum profit,

$$\text{Max } -u''(c)/u''(c) - L'\left[f'\text{d}\mu + i(g')\right] \tag{22}$$

3.2.2. Put Buying by the Moderate Risk-Taker

The moderate risk-taker has a Bessel function with increasing sentiment of gains from put option purchases on decaying shopping malls. Figure 5 shows the Bessel function with revisions of sentiment as more information becomes available. At point T, the moderate risk-taker purchases put options with the intent to achieve gains. At point A, prices of mall real estate decline to their minimum value. The investor exercises the put option, selling the real estate at a price that is higher than the market price. This sale results in gains TA. No further put buying is possible as prices will rise with the rental of apartments, art galleries, and culinary centers.

The formulation is as follows,

$$\text{Max} \frac{1}{\prod \cos\int(nr - x\sin r)\text{d}r} - \frac{1}{n}\int \text{e}^{inr - x\sin r}\text{d}r \tag{23}$$

Subject to,

$$\int f\text{d}\mu + i\int g\text{d}\mu > 0 \tag{24}$$

where f, and g are real-valued functions of the Lebesgue integral, each of which is integrable.

The Lebesgue integral converges monotonically to point A, so that Equation (25) specifies the monotonic convergence theorem,

$$f_k(x) \le f_{k+1} \tag{25}$$

Taking Lagrangians,

$$\begin{aligned} &\text{Max}\frac{1}{\prod \cos\int(nr - x\sin r)\mathrm{d}r} - \frac{1}{n}\int \mathrm{e}^{inr - x\sin r}\mathrm{d}x \\ &-L\left[\int f\mathrm{d}\mu + i\int g\mathrm{d}\mu + f_k(x) - f_{k+1}\right] \end{aligned} \quad (26)$$

Taking the first derivative of Equation (26),

$$\text{Max}\frac{1}{\prod -\sin(nr - x\sin r)} - \frac{1}{n}\mathrm{e}^{inr - x\sin r} - L'\left[f + i\left[g + f_k(x) - f_{k+1}\right]\right]$$

$$\text{Max}\frac{1}{\prod -\cos(n - x\cos r)} - \frac{1}{n}\mathrm{e}^{inr - x\sin r} - L'\left[f + ig'\mathrm{d}\mu + f_k'(x) - f_{k+1}'\right] \quad (27)$$

Equation (27) represents point A, the intersection of the Bessel function and minimum price of the shopping mall.

3.2.3. Put Buying by the Risk-Taker

The risk-taker is excessively confident about making gains so that he or she may purchase a put option on the real estate of decaying malls, delay exercising the option in the hope of securing maximum gains by exercising at the minimum price of the mall. In Figure 5, the minimum is achieved at point *A*. If exercise is delayed to point *A*, prices have risen so that all gains from the put buying strategy have been reduced to zero. Any exercise at point *O*, will result in losses, so the risk-taker will not exercise the option. The risk-taker loses the premium paid for the put option. This paper models the risk-taker's sentiments as an exponential distribution *PS*, that depicts rising sentiments. The pricing distribution is a Lebesgue integral *FAO*.

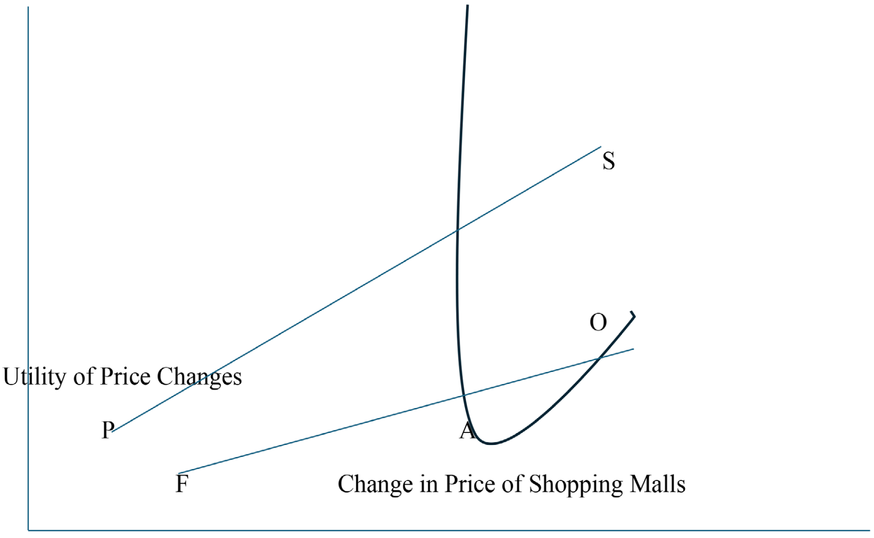

Figure 5. The optimal price of the risk-taking investor in shopping malls.

If exercise is delayed to point *A*, prices have risen so that all gains from the put buying strategy have been reduced to zero. Any exercise at point *O*, will result in losses, so the risk-taker will not exercise the option. The risk-taker loses the premium paid for the put option. This paper models the risk-taker's sentiments as an exponential distribution *PS*, that depicts rising sentiments. The pricing distribution is a Lebesgue integral *SAO*.

The formulation is presented below.

Maximize the exponential distribution of investor sentiments, subject to the Lebesgue integral for the price distribution,

$$\text{Max } \exp \prod \left(-x\lambda_i\right) + \lambda_0 / \left(\lambda_1 \cdots \lambda_n\right) \tag{28}$$

where λ = is the inverse scale of the exponential distribution, and *x* is a random variable.

Subject to,

$$\int f \mathrm{d}\mu + i \int g \mathrm{d}\mu > 0 \tag{29}$$

where *f*, and *g* are real-valued functions of the Lebesgue integral, each of which is integrable.

The Lebesgue integral converges monotonically to point A, so that Equation (30) specifies the monotonic convergence theorem,

$$f_k\left(x\right) \le f_{k+1} \tag{30}$$

Taking Lagrangians,

$$\text{Max } \exp \prod \left(-x\lambda_i\right) + \frac{\lambda_0}{\lambda_1 \cdots \lambda_n} - L \int f \mathrm{d}\mu + i \int g \mathrm{d}\mu - 0 - f_k\left(x\right) - f_{k+1} \tag{31}$$

Taking the first derivative of Equation (31),

$$\text{Max } \exp \prod' \left(-x\lambda_i\right) + \frac{\lambda_0}{\lambda_1 \cdots \lambda_n} - L' \left[f + i g \mathrm{d}\mu - 0 - f_k\left(x\right) - f_{k+1} \right] \tag{32}$$

Taking the second derivative of Equation (31),

$$\text{Max } \exp \prod'' \left(-x\right) + \frac{\lambda_0}{\lambda_1 \cdots \lambda_n} - L'' \left[f + i g \mathrm{d}\mu - f_k\left(x\right) - f_{k+1} \right] \tag{33}$$

The negative function (33) shows losses sustained by failure to exercise the option.

3.3. Investment in Unused Mines and Quarries

We present three formulations for the price distribution with unused mines and quarries with continuously declining prices. As noted, these locations have severe environmental hazards, so that building upon them is impossible. This results in continuously declining prices.

3.3.1. Short Selling by the Risk-Averse Investor

The risk-averse investor's sentiments are modeled by the Arrow-Pratt coefficient of risk-aversion. The pricing distribution is a downward-sloping Reimann integral, as prices of quarries and mines are unlikely to recover. Figure 6 shows the relatively flat and rising parallel lines *AB* and *CD* that show short-sale gains. *OP* is the pricing distribution. At point *B*, the investor borrows real estate, and sells it. After prices decline, he or she repays the borrowed real estate with cheaper real estate at point *D*. This action earns a gain, *BD*.

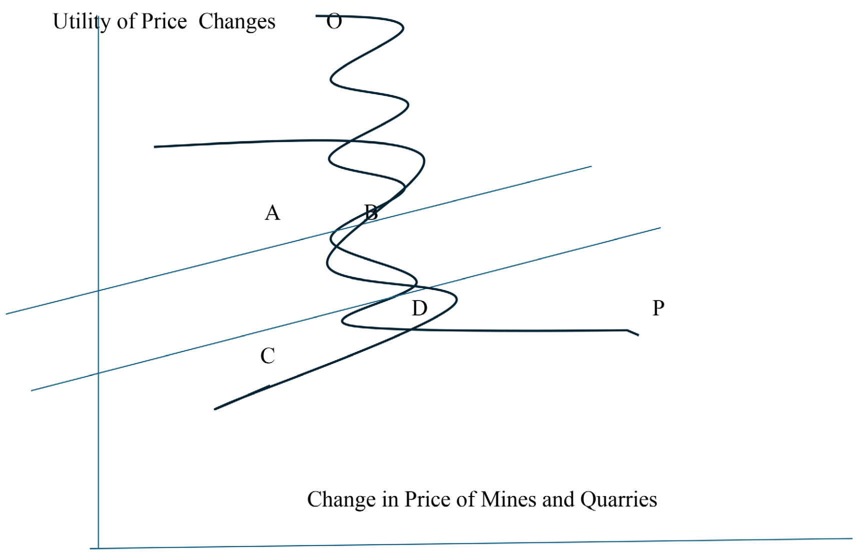

Figure 6. Short-sale gains for mines and quarries.

Figure 6 shows the relatively flat and rising parallel lines *AB* and *CD* that show short-sale gains. *OP* is the pricing distribution. At point *B*, the investor borrows real estate, and sells it. After prices decline, he or she repays the borrowed real estate with cheaper real estate at point *D*. This action earns a gain, *BD*.

The formulation is given below.

$$\text{Max } A(c) = -u''(c)/u'(c) = 1 \tag{34}$$

where,

$A(c)$ = an individual's propensity to avoid risk.

$u(c)$ = utility function of payoffs.

Subject to,

The Reimann integral, which is the sum of all areas under the curve in the first quadrat, or the total of incremental price reductions to obtain the lowest short sale purchase price at point *D*.

$$\sum f(t_i)(x_i - x) - s \tag{35}$$

Taking Lagrangians,

$$\text{Max } -u''(c)/u'(c) - 1 - L\left[\sum f(t_i)(x_i - x) - s\right] \tag{36}$$

Taking the first derivative of Equation (36)

$$\text{Max } -u''(c)/u''(c) - L'\left[\sum f(t_i)(x_i - x) - s\right] \tag{37}$$

Taking the second derivative of Equation (36),

$$\text{Max } -u''(c)/u''(c) - L''\left[\sum f(t_i)(-x) - s\right] \tag{38}$$

Equation (38) is point *D*, at which short sale gain is maximized.

3.3.2. Put Buying by the Moderate Risk-Taker

The moderate risk-taker employs the Bessel function, *AB*, in **Figure 7**, to express sentiments of cautious optimism, as he or she revises expectations of gain as information becomes available. This function intersects with the downward-sloping Reimann integral at point *B*, at which put options are purchased. The put option is exercised to sell the property at its maximum price. Upon prices declining, the real estate is purchased, at a lower price, ensuring a gain at point *C*.

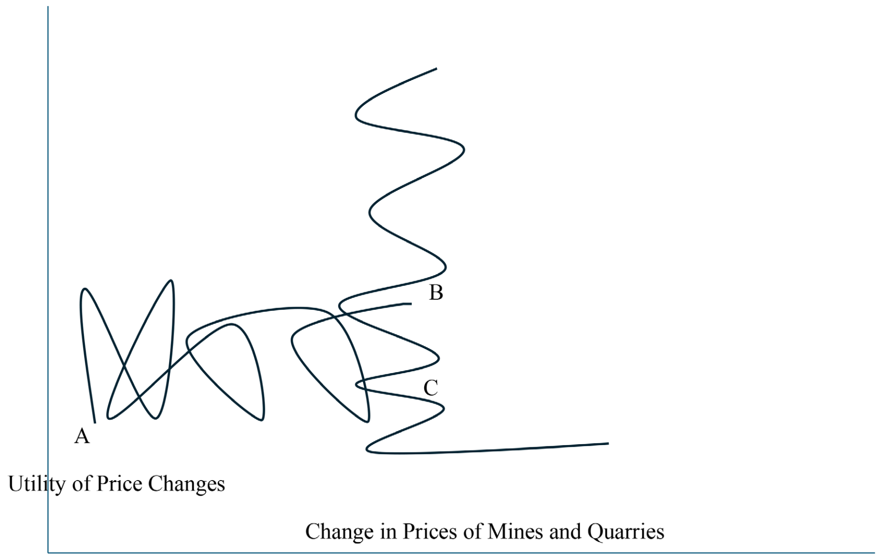

Figure 7. Put-buying by moderate risk-takers.

The moderate risk-taker employs the Bessel function, *AB*, in **Figure 7**, to express sentiments of cautious optimism, as he or she revises expectations of gain as information becomes available. This function intersects with the downward-sloping Reimann integral at point *B*, at which put options are purchased. The put option is exercised to sell the property at its maximum price. Upon prices declining, the real estate is purchased, at a lower price, ensuring a gain at point *C*.

$$\text{Max}\frac{1}{\prod\cos\int(nr-x\sin r)\mathrm{d}r}-\frac{1}{n}\int\mathrm{e}^{inr-x\sin r}\mathrm{d}r \tag{39}$$

Subject to,

The Reimann integral, which is the sum of all areas under the curve in the first quadrat, or the total of incremental price reductions to obtain the lowest short sale purchase price at point *D*.

$$\sum f(t_i)(x_i-x)-s \tag{40}$$

Taking Lagrangian,

$$\text{Max}\frac{1}{\prod\cos\int(nr-x\sin r)\mathrm{d}r}-\frac{1}{n}\int\mathrm{e}^{inr-x\sin r}\mathrm{d}x-L\left[\sum f(t_i)(x_i-x)-s\right] \tag{41}$$

Taking the first derivative of Equation (41),

$$\text{Max}\frac{1}{\prod\left(-\sin x\left(nr-x\cos r\right)\text{d}r\right.}-\frac{1}{n}\text{e}^{inr-x\sin r}-L'\left[\sum f\left(t_i\right)\left(x_i-x\right)-s\right] \tag{42}$$

Taking the second derivative of Equation (41),

$$\text{Max}\frac{1}{\prod'\left(-\cos x\left(nr+x\sin r\right)\text{d}r\right.}-\frac{1}{n}\text{e}^{inr-x\sin r}-L''\left[\sum f\left(t_i\right)\left(x_i-1\right)-s\right] \tag{43}$$

Equation (43) is the minimum point *C*, at which put option gains are maximized for the moderate risk-taker.

3.3.3. Put Buying by the Risk-Taker

The risk-taker is excessively confident of high returns from a put-buying strategy. This optimism is modeled by an exponential distribution *AB*. The price distribution is the downward-sloping Reimann integral *PQ*. At point *P*, the risk-taker purchases put options. Then, the risk-taker waits, till prices reach the minimum point *B'*. The put is exercised at point *B'*, so that the put buyer can sell the real estate at a price significantly higher than the market price, thereby assuring a gain. These relationships are depicted in Figure 8.

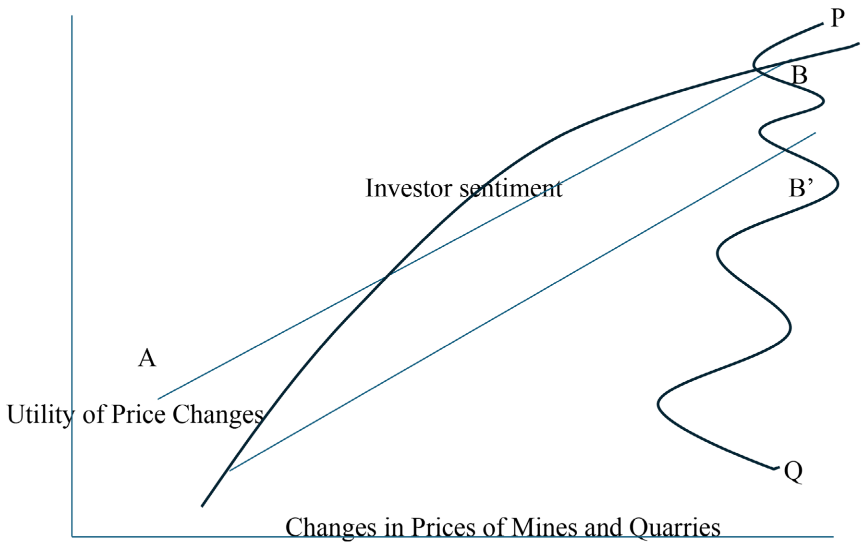

Figure 8. Put-buying by risk-takers

This optimism is modeled by an exponential distribution *AB*. The price distribution is the downward-sloping Reimann integral *PQ*. At point *P*, the risk-taker purchases put options. Then, the risk-taker waits, till prices reach the minimum point *B'*. The put is exercised at point *B'*.

The formulation is as follows.

$$\text{Max}\exp\prod\left(-x\lambda_i\right)+\lambda_0/\left(\lambda_1\cdots\lambda_n\right) \tag{44}$$

where λ = is the inverse scale of the exponential distribution, and x is a random variable.

Subject to,

The Reimann integral, which is the sum of all areas under the curve in the first quadrat, or the total of incremental price reductions to obtain the lowest short sale purchase price at point D.

$$\Sigma f(t_i)(x_i - x) - s \tag{45}$$

Taking Lagrangians,

$$\text{Max } \exp \prod(-x\lambda_i) + \frac{\lambda_0}{\lambda_1 \cdots \lambda_n} - L\left[\Sigma f(t_i)(x_i - x) - s\right] \tag{46}$$

Taking first derivatives of Equation (46),

$$\text{Max } \exp \prod{}'(-1\lambda_i) + \frac{\lambda_0}{\lambda_1 \cdots \lambda_n} - L'\left[\Sigma f(t_i)(x_i - x) - s\right] \tag{47}$$

Taking second derivatives of Equation (46),

$$\text{Max } \exp \prod{}''(-\lambda_i) + \frac{\lambda_0}{\lambda_1 \cdots \lambda_n} - L''\left[\Sigma f(t_i)(x_i - 1) - s\right] \tag{48}$$

Equation (48) represents the maximum gain to the risk-taker from put purchases on mines and quarries.

4. Conclusion

4.1. Theoretical Implications

This paper provides mathematical formulations of short sales strategies and put purchase strategies for three types of investors, i.e. the two extreme cases of the risk-averse investor and the risk-taker. The modeling of the intermediate position of the moderate risk-taker employed a Bessel function, which permitted revisions of investor sentiments as prices were revealed. This formulation accounts for the moderate attitude to risk of the moderate risk-taker.

The paper employs complex formulations for both investor sentiments and price distributions. Investor sentiments include the Arrow-Pratt coefficient of risk-aversion, the Bessel function, and the exponential distribution. Pricing distributions include the Levy-Khintchine formula, the Lebesgue integral, and the Riemann integral. The measures are both traditional and novel. The Arrow-Pratt coefficient of risk aversion is a traditional measure of risk-aversion, while the use of the Lebesgue integral and Riemann integral in put strategies for real-estate investing is novel.

This paper presents alternative formulations that distinguish between Lebesgue integrals and Riemann integrals by drawing on the literature on these two functions (Bartle, 1995; Bauer, 2001). The Lebesgue integral describes limits under the integral sign. As it describes a curve that is vertical, reaches a minimum point, and then increases, the Lebesgue integral forms limits first at the minimum point, and then at the end point of the upward sloping curve. There is a sequence of integrals, which change direction, as described by the Lebesgue integral. The Riemann integral cannot establish limits under the integral sign, so it is modeled by a continuously downward sloping curve as in the declining prices of mines and quarries. There are no end points with changing signs, so the Riemann integral can effectively model continuously declining prices.

4.2. Practical Implications

Investors may earn returns by making investments in real estate. Yet, real estate investments have the negative characteristic of illiquidity as funds invested real cannot be recovered in the sort-term as it can take as long as 2 - 5 years to sell commercial real estate. Further, investment outcomes may be uncertain in that certain properties, such as resorts and upscale shopping districts experience rising property values, while others, such as repositories of toxic waste experience freely falling values. This paper presents derivatives as a less risky real estate investment, as investors can participate in rising returns with low investment, even with negatively priced property. The short sale and derivative strategies modeled in this paper provide nine formulations of real estate strategies with negatively priced property.

Seven strategies using puts make gains for risk-averse investors, moderate risk-takers, and risk-takers. There are only two loss-making strategies, both for the risk-taker. This result should caution mutual fund managers to limit investment in highly risky aggressive growth real estate properties, as tarrying too long to exercise put options on declining property values results in significant losses. A closer examination of the losses shows that 1) occurred when the risk-taker invested in decaying malls assuming that prices would fall indefinitely. Yet, prices fell, then rose as the mall was repurposed into apartments. The upturn in prices ensured losses for the put buyer. 2) Investment in abandoned quarries and mines with environmental hazards may never find a buyer for the property, so that the put option cannot be exercised, and expires worthless.

4.3. Research Limitations

The study examines three types of risk-takers. There may be other risk-takers, segmented by age. Future research should examine the put strategies set forth by GenX, GenY, GenZ, baby boomers, and other age groups. Other types of real estate with declining values should be explored in future research. They include rural areas, and small towns with a single large employer, such as an automobile firm, that has left the town. Schools close, the tax base is eliminated, and retail stores that supported the workers will close, devastating property values.

The assumption of continuously declining prices for properties with former mines and quarries may not hold in practice. If there is extensive environmental cleaning to decontaminate the soil, the property may be repurposed to farmland, entertainment venues, or housing developments. In these cases, property values will rise, rendering the put purchase strategy to be unprofitable. Future research should examine the case of increasing prices of former mines and quarries and the gains made by call option valuation model that provides gains to the three different types of investors from rising property prices.

Another limitation is that short sale price movements are based on continuously falling prices. Future research should explore cases in which probabilities are used to weight short sale prices to account for uncertainty in the magnitude and direction of short sale prices.

Another limitation is the emphasis on mathematical formulation, without empirical investigation. This paper is an initial attempt to create a theoretical formulation of relationships between investor sentiment and negatively priced property. As the theory is developed at this point in time, future research may use the Commercial Real Estate database to extract put prices which can be regressed in a panel data fixed effects model on property prices for mines, abandoned shopping malls, and farmland. This will provide empirical validation of the effect of put values showing pessimistic investor sentiment for risk-averse investors, and (1-put value) showing optimistic investor sentiment for risk-takers.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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