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Categorization of Threshold Phenomena in Mathematical Models of Demand for Addictive Goods

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Abstract

A lack of mathematical economic demand models for addictive goods that account for nonconvex consumer preferences with thresholds resulting in discontinuous demand curves (frequently with bifurcations) is often attributed to the need that economic variables must be relevant to incentivize behavioral changes in addicts. Once addiction thresholds are reached, and economic variables such as prices or taxes become less efficient or completely inefficient in their corrective role, addressing underlying causal addiction issues becomes maybe the only remedy to the problem. The objective of this paper is to develop a mathematical categorization of threshold phenomena and subsequently suggest a different way to model demand for addictive goods from what has been a standard practice to date. Non-exhaustive set of threshold phenomena including a discontinuous threshold phenomenon (DTP), a singular-point threshold phenomenon (STP), and quasi-threshold phenomenon (QTP) are defined mathematically and presented visually. While economic variables may play a preventative role in addictive goods consumption, prescriptive component in addiction economics research can be accommodated in the models such as these developed here. It is more realistic and useful when coming modeling of addictive behavior comes from positive, rather than normative, framework and analysis. In such cases, the transfer of knowledge acquired through addiction intervention research into clinical settings has great promise as a means of increasing treatment effectiveness and facilitating greater consistency in practice. Indeed, recognizing what are the root causes of observed, actual health behaviors are more likely to lead to more efficient policy interventions. Riches of models in health behaviors research and epidemiology may serve as a good starting point to retooling an economist's thinking on the subject.

Keywords

Demand for Addictive Goods, Differential Equations, Discontinuous Threshold Phenomenon, Singular-Point Threshold Phenomenon, Quasi-Threshold Phenomenon, Bifurcations

1. Introduction

The existence of a threshold phenomenon within an economic framework imposes constraints on the appropriateness of mathematical models utilized to characterize the system. This paper primarily delves into threshold phenomena within the consumer/demand aspect of an economy. Although instances of thresholds in demand for and consumption of goods or services are plentiful, this paper concentrates on the demand for addictive substances. It will provide a mathematical categorization of threshold phenomena and subsequently suggest a different way to model demand for addictive goods from what has been a standard practice to date.

It is more common in the literature to study indifference curves, appropriately restricted, rather than the demand curves for addictive goods. Primary justification for this approach is that concentrating on mathematical treatment and correlation of addictive goods consumption with prices, income, and other standard variables would move the focus from individual behavior to market. In that process, the connection between environment, behavior and personality, which controls preferences, could be stained or even masked (e.g., Barthold and Hochman, 1988; Miljkovic, 2020). Economics literature on demand for addictive goods typically moves on directly from the discussion of preferences and indifference curves, which clearly point to the presence of thresholds resulting in nonconvexities, to empirical modeling of demand where economic variables, such as prices, have the power to explain and correct market behaviors of addicts. As a result, the economists often make prescriptive statements based on normative narrative developed in theories such as rational or myopic addiction and suggest how changes in prices (or income or other economic variables) may improve or correct individual or group addictive behavior (e.g., Becker and Murphy, 1988; Becker, Grossman, and Murphy, 1994; Miljkovic, Nganje, and de Chastenet, 2008). However, addiction, by definition, implies radically different individual behavior from the mainstream; hence an addict's demand is likely to be characterized by discontinuities and bifurcations rather than a smooth curve where marginal changes in prices or alike variables can magically explain and change that addict's behavior.

We must recognize that continuity and discontinuity may not always be clearly distinguishable. Ultimately, what may be of importance is the precision of the perspective applied to the question at hand (Rosser, 2013). For example, if one is to follow the quantum mechanics premises, then reality is fundamentally discrete at the microscopic level. However, it may be useful to view it as continuous when the analysis is less precise, such as demand analysis in economics. Our mind often draws an invisible line between still points thus creating an illusion of continuity. Mathematically, there are clearly degrees of (dis)continuity. Yet, there are situations, such as one of demand for addictive goods, where discontinuities cannot be ignored.

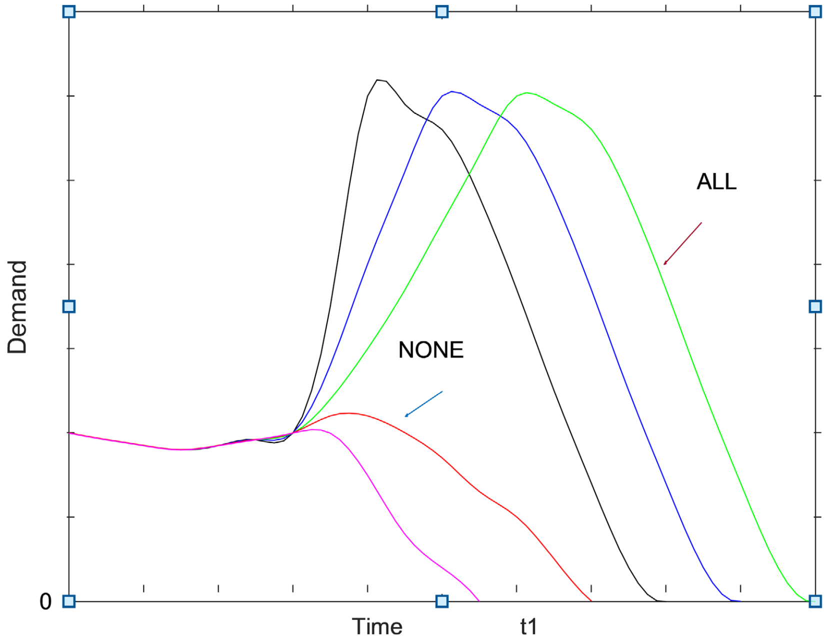
2. Base Model

We can visually represent the fluctuations in demand for an addictive good, denoted as D, by a consumer over the period where the threshold of addictive substance consumption is reached. We assume various levels of exposure to the addictive substance, represented by z, were introduced, concluding at time $t = 0$. Throughout the interval $t \geq 0$, an unchanging non-zero access/exposure to the addictive substance is maintained. The shape of each curve is contingent upon the initial condition, primarily the environment in greater sense, and physiological and mental state of the consumer at $t = 0$. In this context, it is presumed that prices, typically influencing demand decisions, cease to play a role once the threshold is surpassed. The initial condition varies continuously with z. Therefore, the curves depicted in **Figure 1** can be formally described by an equation of the following structure:

$$D = F(z;t) \tag{1}$$

where z is treated as a parameter. As exposure z surpasses a certain threshold value z_θ, the curve's shape undergoes an abrupt change. It can be postulated that if curves corresponding to all z values within a finite interval Z were plotted, the curve shapes would exhibit discontinuous changes as z crosses z_θ. Following the all-or-none principle in physiology, psychiatry or psychology regarding addiction (e.g., Henden, 2013; Henden et al., 2013; Barthold and Hochman, 1988), these curves are categorized into two distinct groups: the "all" and the "none" curves. Within each group, curve shapes vary continuously with z, but there are no intermediate curves between the two classes. Due to random fluctuations in the latency of consumer response to a change in exposure to the addictive substance, it becomes challenging to ascertain, based on a finite number of consumers, whether the latency, as stimulus intensity approaches the threshold from above, is finite or infinite.

Assuming there is a finite maximum delay in the response, at a certain time t_1, when we plot $F(t_1; z)$ against z (see **Figure 2**, labeled "$t = t_1$"), there is a discontinuity at a specific z-value, z_θ. This abrupt change can be interpreted as illustrating the connection between exposure to addictive substances (on the horizontal axis) and the corresponding response (on the vertical axis). However, when we plot the initial state (z; 0) against z (as depicted by the broken line in **Figure 2**), no such abrupt change is observed. Hence, the threshold effect entails a division/separation, occurring somewhere between zero and t_1, in the trajectories of consumer behavior for $z < z_\theta$ and $z > z_\theta$, at least in terms of their potential (and often observed) demand for the addictive product, D. We believe this characteristic should be inherent in any mathematical framework modeling the demand for addictive substances.



$z > z_\theta$ for the upper three "ALL" curves; $z < z_\theta$ for the lower two "NONE" curves. Following a brief shock/exposure at zero time, a constant (unchanging) non-zero access/exposure to the addictive substance is maintained.

Figure 1. Demand for an addictive good, showing the effect of small differences in exposure z near its threshold value z_θ.

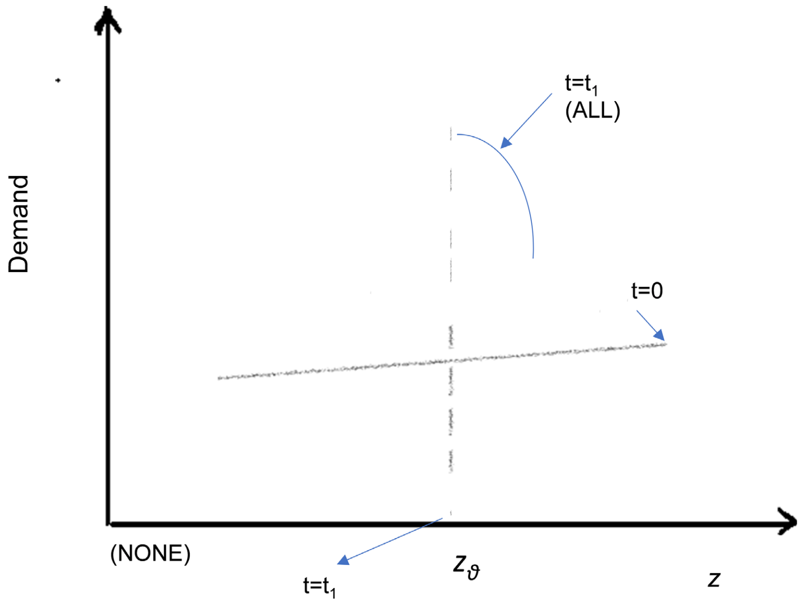


Figure 2. Demand curves measured at two fixed times, zero and t_1, plotted against the exposure z, from the curves in **Figure 1**.

The characteristics being sought pertain not solely to the individual consumer, but to the broader socio-economic system comprising the consumer along with the segment of their environment that enforces an external constraint on addictive substance accessibility. This environment encompasses economic, social, and public policy factors that influence access and exposure to addictive goods and substances. The significance of the environment in shaping threshold behavior is evidenced by the presence of such behavior when the consumer is subjected to excessive exposure to addictive substances, yet absent when there is a sudden change in potential demand (potentially due to a physiological predisposition towards a heightened response to an external stimulus). The threshold phenomena under consideration here encompass only those where the stimulus or the excessive exposure ends at $t = 0$, and the external constraint remains constant throughout the response, i.e., at $t > 0$.

3. Definitions

Let us assume that the total demand system's state at any given time can be described by a finite set of state variables x_n (where $n = 1, 2, \cdots, N$), and the system's behavior can be articulated through a series of differential equations represented as:

$$\mathrm{d}x_n/\mathrm{d}t = f_n\left(x_1, x_2, x_3, \cdots, x_n\right) \tag{2}$$

or, in vector notation,

$$\mathrm{d}x/\mathrm{d}t = f\left(x\right) \tag{3}$$

here, the vectors are denoted in boldface type. The variables x_n can be envisioned as the coordinates within a vector space or phase space of N dimensions. Each point within this space corresponds to a distinct state of the system (Lefschetz, 2016; Nemytskii, 2015). At any given moment, the system's state is represented by a state point, which traces a trajectory within the phase space defined by a solution $\boldsymbol{x}(\boldsymbol{x}^0; t)$ of Equation (3), where $\boldsymbol{x}^0$ denotes the initial point at $t = 0$. During the shock induced by the addictive substance accessibility/exposure constraint, the trajectories within the phase space deviate from those for $t \geq 0$. The point reached at $t = 0$ by the state point following the shock is denoted as $\boldsymbol{x}^0$, which is a function $\boldsymbol{x}^0(z)$ of z. Consequently, the initial point is controlled by the provider of the shock (stimulus), who can manipulate the parameter z as desired before each shock is administered.

All or, more likely, only some of the x_n variables may be observable and thus measurable. The demand for an addictive good, denoted as D, typically quantifiable, can generally be assumed to be a continuous function of the x_n variables. Indeed, one could simply regard the demand for an addictive good as one of the x_n variables.

In the ensuing discussion, definitions of three types of threshold phenomena will be articulated in terms of the characteristics of trajectories within the phase space, rather than solely based on the behavior of D as a function of time.

Nonetheless, it is always presupposed that D is defined as a function of the x_n variables in such a manner that the continuities or discontinuities of shape between adjacent trajectories remain intact when they are transformed into curves of D plotted against t.

Here, we endeavor to delineate a threshold phenomenon mathematically. Figure 3 serves as an illustration of this definition for a phase plane ($N = 2$). The trajectories depicted in the upper right-hand portion of Figure 3 could be delineated in various manners or even left undefined.

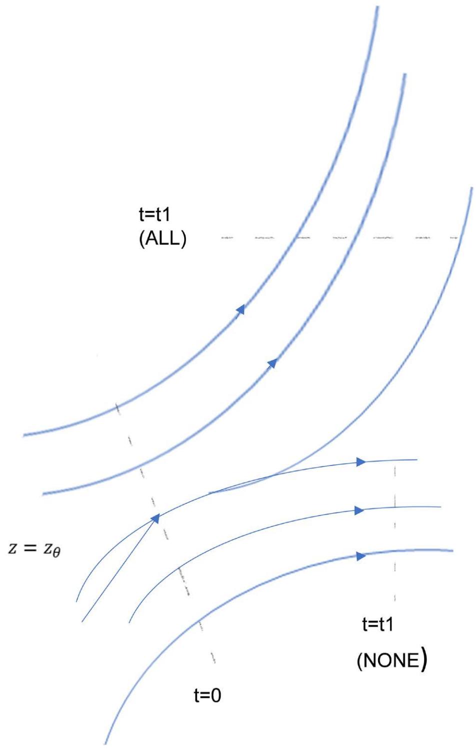


Figure 3. Diagram of a discontinuous threshold phenomenon in a phase plane. Broken line labeled "$t = 0$" in this and subsequent figures is the locus of initial points resulting when the exposure intensity z is varied, $z = z_\theta$ at the point indicated. The two broken lines labeled "$t = t_1$" are loci of state points for time t1 and correspond to "ALL" and "NONE" responses.

3.1. Definition I

If $\boldsymbol{x}^0(z)$ is continuous in z over some interval Z, except possibly for a step discontinuity at $z = z_\theta$, and if, for some time $t_1 \geq 0$, $\boldsymbol{x}(\boldsymbol{x}^0(z); t_1)$ is continuous in z except for a step discontinuity at $z = z_\theta$, then a discontinuous threshold phenomenon (DTP) will be said to exist in the phase space.

This definition is tailored to depict a threshold phenomenon characterized by a bounded latency (t_1). The discontinuity in the system's state at time t_1, as a function of z, can be induced either (a) by a discontinuity in the initial condition at $z = z_0$, without imposing any special conditions on $\boldsymbol{f}(\boldsymbol{x})$, or (b) with the initial condition being continuous in z, in which case constraints must be placed on $\boldsymbol{f}(\boldsymbol{x})$. It may be preferable for both $\boldsymbol{x}^0(z)$ of Definition I and $\boldsymbol{f}(\boldsymbol{x})$ of equation (3) to have as components some of the elementary, differentiable functions of calculus. There are several justifications for such a preference. Firstly, economic processes involving numerous agents are commonly described using such functions. Secondly, empirical studies often reveal significant disparities, suggesting that specifying functions with excessive precision may be unwarranted, hence simpler functions should be favored. Lastly, explaining a discontinuous process by employing discontinuous functions can be viewed as an ad hoc assumption, sidestepping the underlying issue. However, according to the Cauchy-Lipschitz theorem for the existence of solutions of differential equations (Lefschetz, 2016; Nemytskii, 2015), if $\boldsymbol{f}(\boldsymbol{x})$ is differentiable and all partial derivatives are uniformly bounded within a certain region, then the solution $\boldsymbol{x}(\boldsymbol{x}^0; t)$ remains continuous in $(\boldsymbol{x}^0; t)$ for all t. Consequently, if $\boldsymbol{x}^0(z)$ is also continuous in z, a Discontinuous Threshold Phenomenon (DTP) becomes untenable.

It is worth noting that to have a Discontinuous Threshold Phenomenon (DTP) with $\boldsymbol{x}^0(z)$ continuous, it is not mandatory for $\boldsymbol{f}(\boldsymbol{x})$ to be discontinuous; it only needs to violate a Lipschitz condition at certain points. For instance, a DTP does manifest in the following system:

$$\mathrm{d}x/\mathrm{d}t = -x^{1/3}$$

$$\mathrm{d}y/\mathrm{d}t = y^{1/3}$$

If one prefers to utilize differentiable functions, it becomes imperative to establish a revised definition of threshold phenomenon. Definition I can be reworked by relinquishing the requirement of a maximum latency, i.e., finite t_1. A point within a phase space where all $\mathrm{d}x_n/\mathrm{d}t = 0$ is considered a degenerate trajectory and is termed a singular point.

Figure 4 illustrates a phase plane with a saddle point (one type of singular point) situated at the origin of coordinates, as generated by equations of the following form:

$$\begin{aligned} \mathrm{d}x_1/\mathrm{d}t &= p_{11}x_1 + p_{12}x_2 + q_1(x_1, x_2), \\ \mathrm{d}x_2/\mathrm{d}t &= p_{21}x_1 + p_{22}x_2 + q_2(x_1, x_2) \end{aligned} \tag{4}$$

in this case the p's are constants, and the characteristic equation is

$$\begin{vmatrix} p_{11} - \lambda & p_{12} \\ p_{21} & p_{22} - \lambda \end{vmatrix} \tag{5}$$

when λ has one positive and one negative root, and q_1 and q_2 are power series in x_1 and x_2, starting with terms of degree two or higher. If $\boldsymbol{x}^0(z)$ is a continuous function of z and describes a line segment labeled "$t = 0$" in **Figure 4** as z varies over the interval Z, the trajectory with $\boldsymbol{x}^0(z)$ as its initial point undergoes a discontinuous change in shape at a trajectory called the *separatrix*, where z takes on the value z_θ. This discontinuity differs from that in Definition I. In fact, according to the Cauchy-Lipschitz Theorem, $\boldsymbol{x}(\boldsymbol{x}^0(z); t_1)$ is continuous in z for every fixed t_1, and as z varies, $\boldsymbol{x}(\boldsymbol{x}^0(z); t_1)$ continuously traverses a line such as the one labeled "t

= t_1” in **Figure 4**. Let us arbitrarily define the latency as the time taken for x to move from its initial point along an “all” trajectory to a line such as the one labeled “criterion of shock exposure” (one could think of it as a criterion of excitation as well) in **Figure 4**. For a model of the demand for an addictive good, for instance, this line might correspond to the condition that D be halfway between the point of no exposure (and possibly no knowledge about) to an addictive substance and the peak value of the potential impact under unrestricted exposure. With no maximum latency, as z approaches z_θ, the state point remains in the vicinity of the saddle point for an increasingly extended period, where the phase velocity vector $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$ is very small. Consequently, the latency can become arbitrarily large in this manner. This characteristic of the saddle point can be illustrated by plotting latency against z. In **Figure 5**, typical curves of this nature are depicted for a DTP

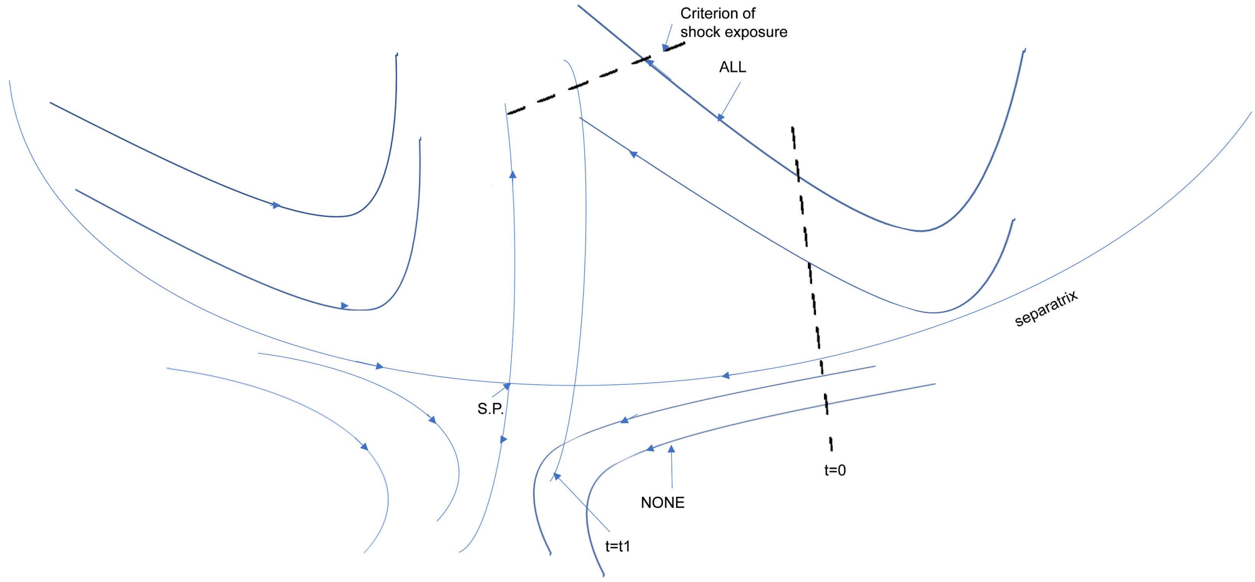


Figure 4. Diagram of an STP in a phase plane. S.P. is a saddle point. Typical trajectories of the “ALL” and “NONE” classes are labeled. More detailed elaboration of the diagram is in the text.

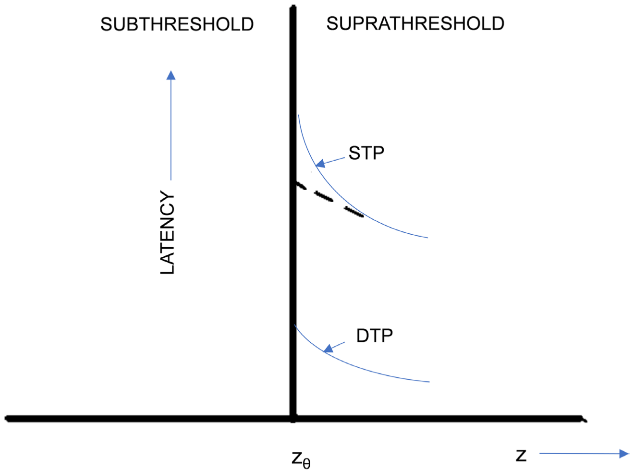


Figure 5. Diagram of latency as a function of stimulus intensity for a DTP and an STP.

and a saddle-point threshold phenomenon (STP). If the latter curve were to deviate from the separatrix only for z very close to z_θ, such a model could be deemed a good approximate representation of the demand for an addictive good. Both the STP and the actual demand for an addictive good may exhibit similar increases in latency near the threshold, except that the latency of the STP approaches infinity as z approaches z_θ, while that of the actual demand remains finite.

In **Figure 4**, the trajectories in the vicinity of the saddle point can be categorized into three classes based on their behavior:

1) Two trajectories are stable, converging towards the saddle point as time progresses.

2) Two other trajectories are unstable, converging towards the saddle point as time decreases.

3) All remaining trajectories are hyperbolic, initially approaching and subsequently departing from the saddle point as time progresses.

Moreover, the hyperbolic trajectories can be further subdivided into four subclasses based on the directions in which they approach and depart from the saddle point. If two trajectories are selected from any two of these distinct classes or subclasses, it is impossible to transform one into the other by traversing through a continuum of intermediate trajectories. This topological property is responsible for the threshold characteristics of the saddle point.

As z traverses through Z and the initial point $\boldsymbol{x}^0(z)$ moves along the line "$t = 0$", it undergoes a discontinuous transition from one subclass of hyperbolic trajectories to another subclass, exhibiting qualitatively different behaviors as t increases. Both of these subclasses occupy contiguous 2-dimensional regions of the phase plane and are separated by a single stable trajectory, the separatrix.

The saddle-point threshold phenomenon can be extended to phase spaces of any finite number of dimensions. The subsequent definition is pertinent to systems with functions $f(\boldsymbol{x})$ that are analytic at a singular point, i.e., they can be expanded in a Taylor series around that point. A more comprehensive treatment is available in Kolmogorov and Piskunov (2019).

3.2. Definition II

A Singular-Point Threshold Phenomenon (STP)[1] is deemed to exist in an N-dimensional phase space ($N \geq 1$) if there exists an isolated singular point characterized by having one positive root and all other roots (if any) with negative real parts. Moreover, if $\boldsymbol{x}^0(z)$ is a continuous function that intersects with, and is not tangent to, the ($N - 1$)-dimensional surface (the separatrix) comprising stable trajectories when $z = z_\theta$.

In two dimensions, the condition on the characteristic roots defines a saddle point. In three dimensions the properties of the singular point can be visualized if we let the trajectories be described by the solutions:

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[1] STP is used as an abbreviation interchangeably to denote a singular-point threshold phenomenon and saddle-point threshold phenomenon as saddle point is a type of singular point.

$$x_n = a_n \mathrm{e}^{\lambda nt} \quad (n = 1, 2, 3) \tag{6}$$

of the differential equations

$$\mathrm{d}x_n / \mathrm{d}t = \lambda_n x_n . \tag{7}$$

The a_n's are constants of integration and are the coordinates of the initial point. The origin of coordinates is a singular point.

If $\lambda_1 < \lambda_2 < 0 < \lambda_3$, then the plane $x_3 = 0$ (the separatrix) encompasses the stable solutions and locally divides the space into two regions, in each of which the trajectories are hyperbolic. However, their behavior for increasing t is qualitatively distinct. Trajectories for which $a_3 < 0$ approach the negative x_3-axis, extending towards negative infinity along that axis, and may signify the "none" response. Conversely, trajectories for which $a_3 > 0$ approach the positive x_3-axis, extending towards positive infinity, may represent the "all" response. The separatrix segregates the "all" trajectories from the "none" trajectories.

Conversely, if $\lambda_1 < 0 < \lambda_2 < \lambda_3$, the hyperbolic trajectories fall into two subclasses that do not differ qualitatively in their behavior for increasing t. Instead, they approach the plane $x_1 = 0$, pointing in all possible directions within that plane. In this latter scenario, there is no threshold phenomenon.

In general, for a Singular-Point Threshold Phenomenon (STP) to manifest in any N-dimensional phase space, the singular point must possess the property that an $(N - 1)$-dimensional surface consisting of stable trajectories (the separatrix) serves as a local boundary between two N-dimensional regions. Both of these regions comprise hyperbolic trajectories, which, for sufficiently large t, diverge from the singular point in two opposite directions. If all the characteristic roots have non-zero real parts, then the aforementioned conditions are met if the roots adhere to the specifications outlined in Definition II (Kolmogorov and Piskunov, 2019; Lefschetz, 2016). However, cases in which some of the roots have zero real parts are more intricate and necessitate separate examination.

A second approach to revising Definition I involves retaining the concept of a maximum latency while relinquishing the discontinuity between the "all" and "none" trajectories. **Figure 6** illustrates an example in a phase plane. For all t_1, $\boldsymbol{x}(\boldsymbol{x}^0(z); t_1)$ remains continuous in z, but for certain values of t_1, it exhibits rapid variation when z is close to z_θ. **Figure 7** depicts how $D = F(t_1; z)$ might appear when plotted against z. The discontinuous curve observed in **Figure 2** has been replaced by a continuous curve with a sharp ascent near $z = z_\theta$.

These properties may be described mathematically as follows.

3.3. Definition III

If $\boldsymbol{x}^0(z)$ is continuous in z over some interval Z, and if there exist a positive time t_1 and two values z_1 and z_2 of z such that the ratio $\left| \boldsymbol{x}\left[\boldsymbol{x}^0(z_2); t_1\right] - \boldsymbol{x}\left[\boldsymbol{x}^0(z_1); t_1\right] \right| / \left| z_2 - z_1 \right|$ is sufficiently large, then a quasi-threshold phenomenon (QTP) will be said to exist in the phase space.

This definition is deliberately imprecise, as the phrase "sufficiently large" is open to subjective interpretation. Consequently, a QTP might even fall into the category of what, in practical terms, is not a threshold phenomenon at all. Nevertheless, one could regard and assess it as such based on statistical considerations.

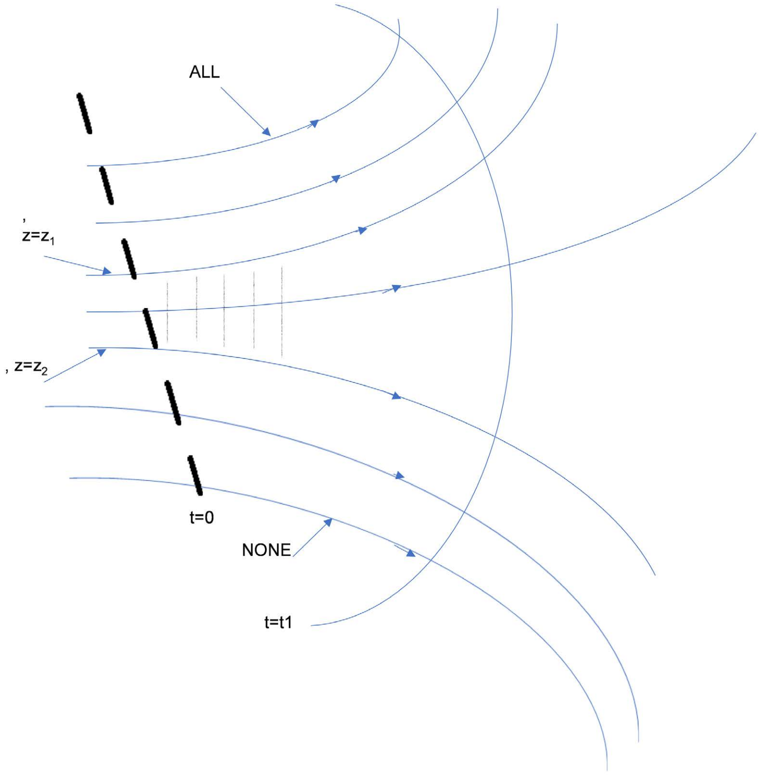

Figure 6. Diagram of a QTP in a phase plane. The shape of the trajectory changes continuously as *z* varied. The two-dimensional separatrix is cross hatched.

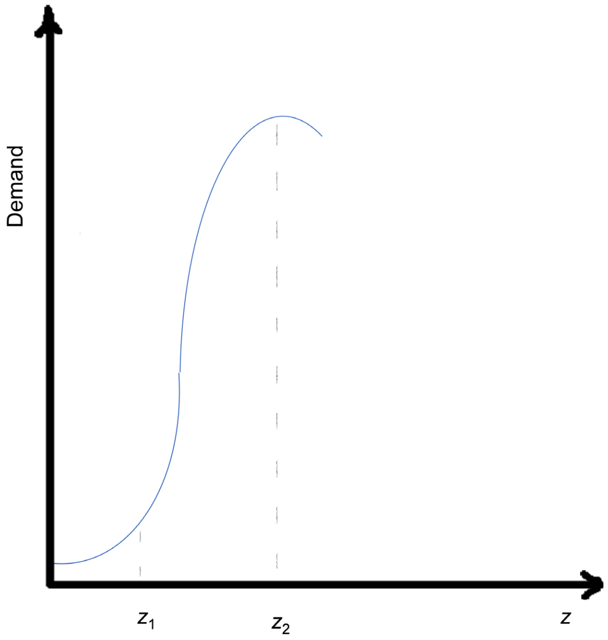

Figure 7. Curve equivalent to that of **Figure 2**, but plotted for the case of **Figure 6**.

For comparison, one could argue that in the QTP, the $(N-1)$-dimensional separatrix of the STP has been substituted with a "thin" N-dimensional vicinity surrounding an $(N-1)$-dimensional surface, as depicted by the cross-hatched region in **Figure 6**. The width of this vicinity, or the extent of its smallest dimension, dictates the sharpness of the QTP. Moreover, the single threshold value z_θ of stimulus (shock) may be replaced by the closed interval $[z_1, z_2]$.

The three types of threshold phenomena mentioned above do not encompass all possibilities. It is conceivable to establish a threshold phenomenon with certain properties akin to the saddle-point type by employing a line or surface comprising singular points instead of an isolated one. Additionally, a limit cycle (a periodic closed trajectory), where some trajectories are stable while others are unstable, can be substituted for the saddle point. In this scenario as well, hyperbolic trajectories can be identified with any specified latency, regardless of how extensive it may be. Most of these trajectories may entail the state point oscillating within the vicinity of the limit cycle.

4. Implications

A lack of mathematical economic demand models for addictive goods that account for nonconvex consumer preferences with thresholds resulting in discontinuous demand curves (frequently with bifurcations) is often attributed to the need that economic variables must be relevant to induce behavioral changes in addicts. An important thing to keep in mind is that demand shows an addict's willingness to purchase a product is not always or mostly driven by its price, but also other environmental, physiological or psychological factors. Sometimes, addicts will try to acquire desired product(s) such as drugs or alcohol using other, even illegal means such as theft, rendering price irrelevant as their need for the product is completely or primarily determined by non-economic factors. This is often true, but especially when it comes to addicts' behavior. Hence, it is not a surprise to note that there are plentiful models in health behaviors and epidemiology literature that are more accurate reflection of actual addicts' behaviors than in economics literature. Below are a few (recent) examples of such models.

Li and Wu (2024) model synthetic drug addiction. They introduce a comprehensive synthetic drug addiction model that considers both psychosocial and environmental factors. Additionally, their model encompasses aspects related to relapse and the drug supply chain. Through a dynamic analysis of this model, they uncover a multitude of intricate dynamical phenomena, including saddle-node bifurcation and Hopf bifurcation. Meanwhile, they have also obtained an elliptic-type nilpotent singularity. Furthermore, they provide bifurcation diagrams and corresponding phase diagrams to offer epidemiological insights into these intricate dynamic phenomena. Liu and Wang (2016) considered a delayed multi-group heroin epidemic model with relapse phenomenon and nonlinear incidence rate. The main contributions of the paper are the proofs of global stability of equilibria. The distributed delays are introduced by the time needed to return to an untreated drug user, which are not constants but vary according to drug users' different temporal, social, and physical contexts. Although including the nonlinear incidence rate combined with the relapse distributed delays into the multigroup model leads to the analysis of the resulting system becoming very complex, they are able to make a rigorous analysis of the model and establish a sharp threshold property. By using the method of constructing Lyapunov functionals based on graph-theoretical approach for coupled systems, sufficient conditions for the global stability of equilibria are given. In similar fashion, Ma, Liu and Li (2017) discuss the dynamical properties of a heroin model with nonlinear contact rate. The authors analyze the types of the equilibria and show that the model exhibits numerous kinds of bifurcation, such as the saddle-node bifurcation, the Hopf bifurcation, Bogdanov-Takens bifurcation of codimension 2 and so on as the parameters' values vary. They establish that these results have certain effect to control the heroin prevalence. In summary, the health behaviors literature on non-linear dynamics, discontinuities and bifurcations in modeling synthetic drugs and other illicit drugs addictions is rather rich (e.g., Ma et al., 2018; Mulone and Straughan, 2009; Nyabadza et al., 2013).

Importantly, other "typical" addictions are studied in this dynamic, non-linear fashion. For example, Ramzan et al. (2024) conduct the bifurcation analysis and numerical investigation of alcohol consumption dynamics. Duncan et al. (2019) study the relapse frequency in addicts. To analyze the underlying mechanisms driving relapse-recovery cycles, they construct a fast-slow dynamical system model of the interplay between the mood and craving level of an addictive disorder patient. The model captures the dynamics of addiction by admitting relaxation oscillations with excited states for relapse and mood crash, and relaxed states for recovery and craving satiation. Exploiting the separation of time scales by neglecting fast states of the relaxation oscillation and linearizing the slow branches of the limit cycle decouples the system and enables an analytic solution of the differential equation governing craving levels. The solution is used to calculate the durations of the craving build-up and satiation phases, the sum of which yields an analytic approximation of the full cycle period, i.e., a prediction of relapse frequency in terms of psychologically relevant model parameters. Comparisons of the period approximation with numerical simulation show good agreement, with relative errors below 5% - 10% over most of the pertinent parameter space. Similarly, tobacco addiction has been studied in this context as well. Sharma and Misra (2015) propose and analyze a nonlinear mathematical model to study the effect of environment, media campaigns specifically in this case, on smoking cessation. They obtain equilibria in their model and discuss their stability. Using center manifold reduction theory, they reduce the proposed model to a system of lower dimension. The reduced system contains all the necessary information regarding the asymptotic behavior of small solutions of the original system. The analysis shows that when changing one parameter of the system (reproduction number, R, which depends on various other parameters), two different manifolds of fixed points cross each other and trans-critical bifurcation occurs. Further, for large value of relapse rate the bifurcation is subcritical (backward). This shows that requirement R < 1 is only necessary, but not sufficient, for smoking cessation. Numerical simulation also supports their analytically obtained results.

Normative economic theories/research on modeling demand for addictive goods focus on economic variables. The very nature of an addict, where prices or income stop being (important) determinants of their behavior at some threshold point, point to inadequacy of theories and resulting prescriptive implications resulting from such modeling efforts. While economic variables may play a preventative role in addictive goods consumption (e.g., Miljkovic, Nganje and de Chastenet, 2008), prescriptive component in addiction research is more realistic and useful when coming from positive, rather than normative, framework and analysis. Once addiction thresholds are reached, and economic variables such as prices or taxes become less efficient or completely inefficient in their corrective role, addressing underlying causal addiction issues becomes maybe the only remedy to the problem. In such cases, the transfer of knowledge acquired through addiction intervention research into clinical settings has great promise as a means of increasing treatment effectiveness and facilitating greater consistency in practice (Glasner-Edwards and Rosson, 2010). Indeed, recognizing what are the root causes of observed, actual health behaviors is more likely to lead to more efficient policy interventions. Riches of models in health behaviors research and epidemiology may serve as a good starting point to retooling an economist's thinking on the subject.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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