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Multiple Contracts with Simple Interest: The Case of the German System of Amortization

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Abstract

The classical systems of amortization, used in house financing all over the world, is based on the compound interest regime, which is characterized by the payment of interest on interest, called anatocism. There have been several questions about its use in Brazil, cf. Jusbrasil (2023), and Italy cf. Annibali et al. (2020). In what appears to be a pioneering contribution De-Losso et al. (2013), it is shown that if a single contract written in terms of the classical system of amortization with constant payments is substituted by multiple contracts, one for each payment of the single contract, the financial institution providing the loan may experience substantial gains in terms of the present value of tax deductions, considering compound capitalization. Other studies using Constant Amortization method, German amortization method have reached the same results. The multiple contracts scheme has been implemented in several amortization methods, such as constant installments, constant amortization and American in simple interest capitalization using simple capitalization, given the problem of anatocism, and has given the same gain in tax reduction, considering simple capitalization. This paper will address the multiple contracts schema to the German amortization method, in simple capitalization, to observe if the same results are obtained. Since this method involves payments of interest at the beginning of each period, some adaptations were required in the proposition of De-Losso (2013). Additionally, a comparison with the French system is also presented.

Keywords

Amortization Systems, Multiple Contracts Scheme, Simple Interest Capitalization, German Amortization System, French Amortization System

1. Introduction

In general, mainly for house financing, the most used system of amortization is the method of constant payments; also known as the French System, cf. Annibali et al. (2016) and de Faro & Lachtermacher (2012). Other methods are also widely spread all over the world, such as Constant Amortization, also known as Italian System, all using compound interest capitalization.

Being worth noticing that, although not very popular, the German system was studied, in Brazil, in Moraes (1967), in Juer (2003) and in de Faro and Lachtermacher (2012), under the hypothesis of compound interest. And, in Italy where a version of it named the Tedesco System, is described in Palestini (2017).

In what appears to be a pioneering contribution De-Losso et al. (2013), it is shown that if a single contract written in terms of the classical system of amortization with constant payments is substituted by multiple contracts, one for each payment of the single contract, the financial institution providing the loan may experience substantial gains in terms of the present value of tax deductions. The amount of tax gains depends on the financial institution's cost of capital.

Similarly, addressing the case of the system of periodic payments of interest only, de Faro (2021), the case of the system of constant amortization, de Faro (2022), and the case of two alternative versions of the SACRE, de Faro & Lachtermacher (2023c), de Faro & Lachtermacher (2023d), and for the German Method, de Faro & Lachtermacher (2024a, 2024b), the same results were observed when the original contracts were substituted by the corresponding multiple contracts.

However, as those systems of amortization are based on the compound interest regime, which are characterized by the payment of interest on interest, called anatocism, there have been several questionings of its use in Brazil, cf. Jusbrasil (2023) and Italy cf. Annibali et al. (2020).

Disregarding the fact that the occurrence or not of anatocism is still a hotly debated subject in Brazil, cf. Puccini (2023) and De-Losso and Santos (2023), and in Italy, cf. Annibali et al. (2020), we are going to address the case of what has been named as the German system of amortization. For the case of using simple interest not yet studied.

Additionally, as the German System, which is characterized by payment of interest in advance, also implies, not counting the first payment, in constant installments, we are also going to make a comparison of a simple interest version of the French System, as in de Faro and Lachtermacher (2023a).

It should be noted that in our knowledge, a version of the Tedesco Method for simple capitalization has not yet been developed.

2. Using Simple Interest Capitalization

Consider the case where a loan in the amount of F units of capital must be amortized by $n+1$ periodic payments. With the k^{th} one identified as P_k, for $k=0,1,\cdots,n$, where n designates the term of the loan.

If the periodic rate of interest, denoted as i, is of compound interest, it is not necessary to specify what is called a focal date; cf. Ayres (1963).

Since, for instance, considering the two most usual focal dates, the first being the beginning of the term, epoch 0, and the second being the end of the term, epoch n, the financial equivalence between F and the sequence of the periodic payments, would imply that, respectively:

$$F = \sum_{k=0}^{n} P_k \times (1+i)^{-k} \tag{1}$$

or

$$F \times (1+i)^{n} = \sum_{k=0}^{n} P_k \times (1+i)^{n-k} \tag{1’}$$

Obviously, expressions (1) and (1’) are equivalent. A result, that expresses the fact that, in the case of the compound interest regime, the period of the interest rate i may be fractionated.

On the other hand, if the rate i is of simple interest we will have:

1) focal date at epoch 0

$$F = \sum_{k=0}^{n} \frac{P_k}{1+i \times k} \tag{2}$$

2) focal date at epoch n

$$F \times (1+i \times n) = \sum_{k=0}^{n} P_k \times \{1+i \times (n-k)\} \tag{3}$$

which implies that we will have different results from Equations (2) and (3), whenever $n > 1$. In what follows, we will focus attention on these two mentioned focal dates.

Being worth noticing that, as pointed out by De-Losso et al. (2020), the focal date at epoch 0 is the one that is prescribed in a Brazilian law of 1964; a legislation that was never revoked, until this moment.

Not withstand, in Brazil, the focal date at epoch n has been considered by Rovina (2009), for the case of the constant amortization system, and by Nogueira (2013), for the case of constant payments.

While in Italy, the focal date at epoch 0 is the one proposed in Mari and Aretusa (2018, 2019). With the focal date at epoch n being described in Annibali et al. (2016). Both address the French Method, using the simple capitalization of interest.

In the next section we will explain the Forger (2009) methodology for implementing system amortization using the simple capitalization of interest.

3. The Concepts of Capitalizable and Non-Capitalizable Components

For the implementation of systems of amortization in the simple interest regime, Forger (2009) introduced the concepts of capitalizable and non-capitalizable components of the amount F.

Denoting by F^C and F^N, respectively, the capitalizable and non-capitalizable components, which are described by:

$$F = F^C + F^N \quad \text{with } F^C = f \times F \text{ and } F^N = (1-f) \times F \tag{4}$$

where *f*, with $0 \le f \le 1$, is defined as a weighting factor, which depends on the focal date chosen, where the superscripts C and N identify the respective capitalizable and non-capitalizable components.

Denoting as S_k the outstanding balance at epoch *k*, immediately after the payment of P_k, and by A_k the respective parcel of amortization, it is supposed that $S_k = S_k^C + S_k^N$, $P_k = P_k^C + P_k^N$, with $A_k = A_k^C + A_k^N$, for $k = 0, 1, \cdots, n$.

Furthermore, for $k = 1, 2, \cdots, n$, it is established that:

$$S_k^C = S_{k-1}^C - A_k^C = S_{k-1}^C - P_k^C \Leftrightarrow A_k^C = P_k^C \tag{5}$$

$$S_k^N = S_{k-1}^N - A_k^N = S_{k-1}^N + J_k - P_k^N \Leftrightarrow A_k^N = P_k^N - J_k \tag{6}$$

with the rate i of simple interest affecting only the capitalizable component S_k^C. Additionally, in the German System, as the interest is paid in advance, we have:

$$J_k = i \times S_k^C, \text{ for } k = 0, 1, \cdots, n \tag{7}$$

Furthermore, extending Forger's original (Forger, 2009) proposal, it is supposed that $A_0^N = A_0^C = P_0^C = 0$.

Regarding epoch 0, which is the date of the financing contract, we have:

$$F = S_0 = S_0^C + S_0^N \text{ with } S_0^C = F \times f \text{ and } S_0^N = F \times (1 - f) \tag{8}$$

Since S_k^C is supposed to decrease linearly from $S_0^C = F \times f$, to $S_n^C = 0$, regardless of the particular system of amortization being considered, it is also established that $P^C = P_k^C = A_k^C = A^C$, for every $k = 1, 2, \cdots, n$, we have:

$$P^C = A^C = F \times f / n \tag{9}$$

From which follows that:

$$S_k^C = F \times f \times (n - k) / n \tag{10}$$

and

$$J_k = F \times f \times i \times (n - k) / n \tag{11}$$

for $k = 0, 1, \cdots, n$.

Considering that we are focusing on the German System, in which the periodic payments are constant, except for the first one, we have that $P_k = P = P^C + P^N$, for $k = 1, 2, \cdots, n$. With P^C as given in Equation (9). It should be noted that $P_0^C = 0$ and $P_0^N = P_0 = J_0$. Therefore, from Equation (6), recursively, as shown in Lachtermacher and de Faro (2023), we have:

$$S_k^N = F \times (1 - f) + \frac{F \times f \times i}{n} \times \left[k \times n - \frac{k \times (k+1)}{2} \right] - k \times P^N \tag{12}$$

From Equations (6) and (12), recursively, it follows that:

$$A_k^N = P^N - F \times f \times i \times (n - k) / n \tag{13}$$

Therefore, Equation (12) can be rewritten as:

$$S_k^N = F \times (1 - f) - \sum_{\ell=1}^{k} A_\ell^N \tag{14}$$

Thus, as we want to have $S_n^N = 0$, it follows that, for $k = n$:

$$S_n^N = F \times (1 - f) - \sum_{\ell=1}^{n} A_\ell^N = 0 \tag{15}$$

Consequently, we have:

$$P^N = \frac{F \times (1 - f)}{n} + \frac{F \times f \times i \times (n-1)}{n} \tag{16}$$

Regarding the initial payment P_0, we should recall that we are, by construction, if the interest rate i affects only the capitalizable component. So that:

$$P_0 = i \times S_0^C = i \times F \times f \tag{17}$$

At this point, it is worth noticing that, in the case of the German System with compound interest, the rate i affects in full the outstanding debt. So that the initial payment is equal to $F \times i$. While, in the case of simple interest, as the rate i, is affected only the capitalizable component, we have $P_0 = i \times F \times f = J_0$. As indicated above, in Equations (11) and (17).

4. The Case of Focal Date at Epoch *n*, with a Single Contract

We will start the analysis by considering the case where the focal date is the end of the term of the loan. Because, in this case, an analytical solution for the weighting factor, *f*, can be provided.

In this case, consider Equation (3), when the first payment is equal to $P_0 = F \times f \times i$, and the remaining periodic payments are constant and equal to *P*, we will have:

$$F \times (1 + i \times n) = F \times f \times i \times (1 + i \times n) + P \times \sum_{k=1}^{n} \left[1 + i \times (n - k)\right] \tag{18}$$

An equation whose analytical solution is:

$$P = \frac{F \times (1 + i \times n) \times (1 - i \times f)}{n + \left[i \times n \times (n-1)/2\right]} \tag{19}$$

Therefore, given that the constant value *P* is partitioned in the components P^C and P^N, it follows from relations (9), (11) and (19), that:

$$\frac{F \times (1 + i \times n) \times (1 - i \times P)}{n + \left[i \times n \times (n-1)/2\right]} = \frac{F \times f}{n} + \frac{F}{n} \times \left[1 - f + \frac{f \times i \times (n-1)}{2}\right] \tag{20}$$

Whose analytical solution is:

$$f = \frac{\dfrac{1 + i \times n}{n + i \times \dfrac{n \times (n-1)}{2}} - \dfrac{1}{n}}{\dfrac{i \times (n-1)}{2 \times n} + \dfrac{(1 + i \times n) \times i}{n + i \times \dfrac{n \times (n-1)}{2}}} \tag{21}$$

Despite the complexity of relation (21), it is easy to implement because it involves only algebraic procedures for real numbers. An alternative is to opt for its numerical resolution as described in Lachtermacher and de Faro (2023).

4.1. Practical Example

Considering a loan F of \$ 100.000, with a term of 12 months at the monthly simple interest rate of 1%, it follows that $f = 0.938967136$, with $P_0 = 938.97$ and $P = 8{,}763.70$. (Table 1)

Table 1. Evolution of the debt in the case of focal date at epoch *n*.

Epoch (k)	J_k	A_k^N	$A^C = P^C$	P^N	S_k^N	S_k^C	S_k
0	938.97	0.00	0.00	938.97	6,103.29	93,896.71	100,000.00
1	860.72	78.25	7,824.73	938.97	6,025.04	86,071.99	92,097.03
2	782.47	156.49	7,824.73	938.97	5,868.54	78,247.26	84,115.81
3	704.23	234.74	7,824.73	938.97	5,633.80	70,422.54	76,056.34
4	625.98	312.99	7,824.73	938.97	5,320.81	62,597.81	67,918.62
5	547.73	391.24	7,824.73	938.97	4,929.58	54,773.08	59,702.66
6	469.48	469.48	7,824.73	938.97	4,460.09	46,948.36	51,408.45
7	391.24	547.73	7,824.73	938.97	3,912.36	39,123.63	43,035.99
8	312.99	625.98	7,824.73	938.97	3,286.38	31,298.90	34,585.29
9	234.74	704.23	7,824.73	938.97	2,582.16	23,474.18	26,056.34
10	156.49	782.47	7,824.73	938.97	1,799.69	15,649.45	17,449.14
11	78.25	860.72	7,824.73	938.97	938.97	7,824.73	8,763.69
12	0.00	938.97	7,824.73	938.97	0.00	0.00	0.00
$\sum$	6,103.29	6,103.29	93,896.71	12,206.57			

4.2. Comparison with the Correspondent Case of Constant Installments (French Method)

Given that, for the German method, except for the initial payment P_0, all the subsequent payments are constant, it appears to be relevant to provide a comparison with the simple interest corresponding version of the classical constant installments' amortization schema, described in de Faro & Lachtermacher (2023a).

Considering a loan F of \$ 100.000, with a term of 12 months, a simple interest rate, i, of 1% per month, it follows that $f = 0.947867299$, with $\hat{P} = 8{,}846.76$, Table 2 presents the correspondent case of our simple numerical example.

Table 2. Evolution of the debt in the corresponding case of constant installments.

Epoch (k)	$\hat{J}_k$	$\hat{A}_k^N$	$\hat{A}^C = \hat{P}^C$	$\hat{P}^N$	$\hat{S}_k^N$	$\hat{S}_k^C$	$\hat{S}_k$
0					5,213.27	94,786.73	100,000.00
1	947.87	0.00	7,898.89	947.87	5,213.27	86,887.84	92,101.11

Continued

2	868.88	78.99	7,898.89	947.87	5,134.28	78,988.94	84,123.22
3	789.89	157.98	7,898.89	947.87	4,976.30	71,090.05	76,066.35
4	710.90	236.97	7,898.89	947.87	4,739.34	63,191.15	67,930.49
5	631.91	315.96	7,898.89	947.87	4,423.38	55,292.26	59,715.64
6	552.92	394.94	7,898.89	947.87	4,028.44	47,393.36	51,421.80
7	473.93	473.93	7,898.89	947.87	3,554.50	39,494.47	43,048.97
8	394.94	552.92	7,898.89	947.87	3,001.58	31,595.58	34,597.16
9	315.96	631.91	7,898.89	947.87	2,369.67	23,696.68	26,066.35
10	236.97	710.90	7,898.89	947.87	1,658.77	15,797.79	17,456.56
11	157.98	789.89	7,898.89	947.87	868.88	7,898.89	8,767.77
12	78.99	868.88	7,898.89	947.87	0.00	0.00	0.00
$\sum$	6,161.14	5,213.27	94,786.73	11,374.41			

As can be seen, the value of the constant installments is increased by $\left[\left(8846.76/8763.70\right)-1\right]\times 100=0.948\%$ and the total payment of interest is increased by $\left[\left(6161.14/6103.29\right)-1\right]\times 100=0.948\%$. This appears to be a general result, as shown in **Table 3**. Where the financing simple interest rate i take the values of 0.5%, 1% and 2% per period, and the number n of periods varies from 60 to 360.

Table 3. Percentual of the total payment of interest over the loan value.

n	German Amortization Focal date at epoch n			French Amortization Focal date at epoch n		
	0.5%	1%	2%	0.5%	1%	2%
60	13.232	23.372	37.888	13.290	23.552	38.365
120	23.225	37.695	54.751	23.314	37.931	55.251
180	31.153	47.507	64.413	31.261	47.757	64.875
240	37.598	54.649	70.674	37.715	54.897	71.091
300	42.939	60.080	75.062	43.062	60.321	75.439
360	47.438	64.349	78.308	47.563	64.580	78.649

Table 3 shows that the financial institution will earn more interest for the same number of units of capital loaned, using the French method than the German method, when using simple interest rate, and focal date at the end of the term (epoch n).

It should be noted that the opposite conclusion was found when comparing the two methods using compound interest capitalization, see de Faro & Lachtermacher (2024a), where the French method charge less interest than the German method.

However, a more relevant comparison must take into consideration the financial institution cost of capital. Which periodic value will be denoted as ρ. That is, considering the rate ρ, we must compare the present values of the corresponding sequences of the parcels of interest payments. Respectively designated as $V_G(\rho)$ and $V_F(\rho)$:

$$V_G(\rho) = \sum_{k=0}^{n} J_k \times (1+\rho)^{-k} \tag{22}$$

and

$$V_F(\rho) = \sum_{k=0}^{n} \hat{J}_k \times (1+\rho)^{-k} \tag{23}$$

where ρ is supposed to be relative to the same period as the financing interest rate *i*.

Considering a loan $F = 100{,}000$ units of capital, term $n = 120$ periods, an interest rate of 1% per period, if ρ_a is the financial institution cost of capital, in annual terms, is equal to 20%, which means that $\rho = 1.531\%$ per month, we have $V_G(\rho) = 22461.13$ units of capital and $V_F(\rho) = 22261.15$, which implies that the financial institution, in terms of the payment of income taxes, should prefer to implement the French system, instead of the German system.

Again, it should be noted that the same conclusion was found when comparing the two methods using compound interest capitalization, as seen by de Faro & Lachtermacher (2024b), where the present value of the French method is smaller than the German method. This finding should prevail over the fact that the German method charges a total interest bigger than the French method, as pointed out.

5. The Case of Focal Date at Epoch 0, with a Single Contract

Now, considering Equation (2), with the first payment being $\bar{P}_0 = F \times f \times i$, and the remaining constant payments being denoted as $\bar{P}$, we have:

$$F = F \times f \times i + \bar{P} \times \sum_{k=1}^{n} \frac{1}{1 + i \times k} \tag{24}$$

In this case, an analytical solution of Equation (24) is not practical. Even for a small number of periods *n*. Therefore, we will use the general procedure suggested in Lachtermacher and de Faro (2023).

5.1. The Case of Our Practical Example

In the case of our numerical example, we will have that the value of the weighing factor is $f = 0.97320714$, with the value of the constant payment being $\bar{P} = \$8779.39$ and $\bar{P}_0 = \$973.21$. **Table 4** summarizes the evolution of the debt.

Table 4. Evolution of the debt in the case of focal date at epoch 0.

Epoch (k)	$\bar{J}_k$	$\bar{A}_k^N$	$\bar{A}^C = \bar{P}^C$	$\bar{P}^N$	$\bar{S}_k^N$	$\bar{S}_k^C$	$\bar{S}_k$
0	973.21	0,00	0,00	973,21	2.679,30	97.320,70	100.000,00

Continued

1	892.11	−222,78	8.110,06	669,33	2.902,08	89.210,64	92.112,72
2	811.01	−141,68	8.110,06	669,33	3.043,75	81.100,58	84.144,34
3	729.91	−60,58	8.110,06	669,33	3.104,33	72.990,53	76.094,86
4	648.80	20,52	8.110,06	669,33	3.083,81	64.880,47	67.964,28
5	567,70	101,62	8.110,06	669,33	2.982,18	56.770,41	59.752,59
6	486,60	182,72	8.110,06	669,33	2.799,46	48.660,35	51.459,81
7	405,50	263,83	8.110,06	669,33	2.535,63	40.550,29	43.085,93
8	324,40	344,93	8.110,06	669,33	2.190,71	32.440,23	34.630,94
9	243,30	426,03	8.110,06	669,33	1.764,68	24.330,18	26.094,86
10	162,20	507,13	8.110,06	669,33	1.257,56	16.220,12	17.477,67
11	81,10	588,23	8.110,06	669,33	669,33	8.110,06	8.779,39
12	0,00	669,33	8.110,06	669,33	0,00	0,00	0,00
$\sum$	6.325,85	2.679,30	97.320,70	9.005,14			

It should be noted that, when comparing the German Method using both focal dates, the installments $\bar{P} > P$ $(8779.39 > 8763.69)$ and the total interest $\sum_{k=1}^{n} \bar{J}_k > \sum_{k=1}^{n} J_k$ $(6325.85 > 6103.29)$ are bigger when using focal date at the beginning of the term. So, for the financial institution, it is better to choose the focal date at the beginning of the term.

This appears to be a general result, as shown in Table 5. Where the financing simple interest i take the values of 0.5%, 1% and 2% per period, and the number n of periods varies from 60 to 360.

Table 5. Percentual of the total payment of interest over the loan value.

n	German Amortization Focal date at epoch n			German Amortization Focal date at epoch 0		
	0.5%	1%	2%	0.5%	1%	2%
60	13.232	23.372	37.888	14,527	27,911	52,337
120	23.225	37.695	54.751	27,785	52,267	95,709
180	31.153	47.507	64.413	40,297	74,746	135,067
240	37.598	54.649	70.674	52,232	95,911	171,860
300	42.939	60.080	75.062	63,701	116,088	206,816
360	47.438	64.349	78.308	74,784	135,483	240,368

5.2. Comparison with the Corresponding Case of Constant Installments

Once more, taking advantage of the presentation in Lachtermacher and de Faro

(2023), **Table 6** presents the corresponding evolution of debt in the case of our numerical example, if the French System would be implemented, with focal date at the beginning of the term.

As can be seen, the value of the constant installments is increased by $\left[\left(8865.67/8779.39\right)-1\right]\times 100 = 0.983\%$ and the total payment of interest is increased by $\left[\left(6388.01/6325.85\right)-1\right]\times 100 = 0.983\%$.

This appears to be a general result, as shown in **Table 7**. Where the financing simple interest i take the values of 0.5%, 1% and 2% per period, and the number n of periods varies from 60 to 360.

Table 6. Evolution of the debt in the case of constant installments—Focal Date $n = 0$.

Epoch (k)	$\hat{J}'_k$	$\hat{A}'^N_k$	$\hat{A}'^C = \hat{P}'^C$	$\hat{P}'^N$	$\hat{S}'^N_k$	$\hat{S}'^C_k$	$\hat{S}'_k$
0					1,722.86	98,277.14	100,000.00
1	982.77	−306.87	8,189.76	675.91	2,029.72	90,087.38	92,117.10
2	900.87	−224.97	8,189.76	675.91	2,254.69	81,897.62	84,152.31
3	818.98	−143.07	8,189.76	675.91	2,397.76	73,707.86	76,105.62
4	737.08	−61.17	8,189.76	675.91	2,458.93	65,518.09	67,977.03
5	655.18	20.73	8,189.76	675.91	2,438.21	57,328.33	59,766.54
6	573.28	102.62	8,189.76	675.91	2,335.59	49,138.57	51,474.16
7	491.39	184.52	8,189.76	675.91	2,151.07	40,948.81	43,099.87
8	409.49	266.42	8,189.76	675.91	1,884.65	32,759.05	34,643.70
9	327.59	348.32	8,189.76	675.91	1,536.33	24,569.29	26,105.62
10	245.69	430.21	8,189.76	675.91	1,106.12	16,379.52	17,485.64
11	163.80	512.11	8,189.76	675.91	594.01	8,189.76	8,783.77
12	81.90	594.01	8,189.76	675.91	0.00	0.00	0,00
$\sum$	6,388.01	1,722.86	98,277.14	8,110.87			

Table 7. Percentual of the total payment of interest over the loan value.

n	German Amortization Focal date at epoch 0			French Amortization Focal date at epoch 0		
	0.5%	1%	2%	0.5%	1%	2%
60	14.527	27.911	52.337	14.596	28.169	53.251
120	27.785	52.267	95.709	27.913	52.723	97.247
180	40.297	74.746	135.067	40.478	75.368	137.113
240	52.232	95.911	171.860	52.459	96.680	174.346
300	63.701	116.088	206.816	63.972	116.990	209.698
360	74.784	135.483	240.368	75.095	136.507	243.612

Table 7 shows that the financial institution will earn more interest for the same number of units of capital loaned, using the French method than the German method, when using simple interest rate, and focal date at the beginning of the term (epoch 0).

Analogously with the case of the focal date at the end of the term of the contract, we also have increases in the value of the payments and the total of interest.

6. The Case of Multiple Contracts

Considering the work of De-Losso et al. (2013), which was formulated under the principles of the compound interest regime, this section will focus on the case where a single contract, written in terms of simple interest, is substituted by multiples contracts. One for each of the $n+1$ payments of the single contract.

The same type of analysis has been made for other amortization methods, see de Faro and Lachtermacher (2023a, 2023b).

6.1. Focal Date at Epoch 0

In this case, with a minor adaptation of the original suggestion of De-Losso et al. (2013), each of the $n+1$ payments of the single contract will be substituted by $n+1$ individual contracts. In such a way, the k^{th} payment of the single contract will be substituted by a single contract, whose principal is equal to the present value, at the same interest *i*, of the single contract.

That is, denoting by $\overline{F}_k$ the principal of the corresponding individual contract, we will have:

$$\overline{F}_k = \overline{P}_k / (1 + i \times k), \quad \text{for } k = 0, 1, \cdots, n. \tag{25}$$

with the k^{th} subcontract stating as the corresponding unique payment. Noticing that, regarding the parcels of amortization, which are not required to be specified in the individual contracts, we have $\overline{A}'_k = \overline{F}_k$, for $k = 0, 1, 2, \cdots, n$.

On the other hand, while also not required to be specified in the individual contracts, it is crucial to observe that, from the strict accounting point of view, the parcel of interest relative to the k^{th} subcontract, denoted as $\overline{J}'_k$, will be:

$$\overline{J}'_0 = \overline{P}_0 \tag{26}$$

and

$$\overline{J}'_k = \overline{P}_k \times \left[1 - 1/(1 + i \times k)\right], \quad \text{for } k = 1, 2, \cdots, n \tag{27}$$

In **Table 8**, considering the consolidation of the $n+1$ subcontracts, for the case of our numerical example, it is presented the corresponding evolution of the consolidated debt.

Even though the total payment of interest is the same both in the case of a single contract and in the case of multiple contracts, there is a crucial distinction regarding the timing of occurrence of their components.

Table 8. Multiples contracts—focal date epoch 0.

Epoch (k)	$\bar{F}_k = \bar{A}'_k$	$\bar{J}'_k$	$\bar{P}_k$	$\bar{J}_k$	$\bar{d}_k = \bar{J}'_k - \bar{J}_k$
0	973.21	0.00	973.21	973.21	973.21
1	8,692.46	86.92	8,779.39	892.11	805.18
2	8,607.24	172.14	8,779.39	811.01	638.86
3	8,523.68	255.71	8,779.39	729.91	474.19
4	8,441.72	337.67	8,779.39	648.80	311.14
5	8,361.32	418.07	8,779.39	567.70	149.64
6	8,282.44	496.95	8,779.39	486.60	−10.34
7	8,205.03	574.35	8,779.39	405.50	−168.85
8	8,129.06	650.32	8,779.39	324.40	−325.92
9	8,054.48	724.90	8,779.39	243.30	−481.60
10	7,981.26	798.13	8,779.39	162.20	−635.92
11	7,909.36	870.03	8,779.39	81.10	−788.93
12	7,838.74	940.65	8,779.39	0.00	−940.65
$\sum$	100,000.00	6,325.85	106,325.85	6,325.85	0.00

A more relevant comparison must take into consideration the financial institution cost of capital. Which periodic value will be denoted as ρ. That is, considering the rate ρ, we must compare the present values of the corresponding sequences of the parcels of interest payments. Respectively designated as $\bar{V}_{\text{single}}(\rho)$ and $\bar{V}_{\text{multiple}}(\rho)$:

$$\bar{V}_{\text{single}}(\rho) = \sum_{k=0}^{n} \bar{J}_k \times (1+\rho)^{-k} \tag{28}$$

$$\bar{V}_{\text{multiple}}(\rho) = \sum_{k=0}^{n} \bar{J}'_k \times (1+\rho)^{-k} \tag{29}$$

where ρ is supposed to be relative to the same period as the financing interest rate i.

For instance, if $\rho_a = 20\%$ per year, which implies 1.531% per month, for a loan term of $n = 12$ and interest rate $i = 1\%$ p.m., we have $\bar{V}_{\text{single}}(\rho) = 5988.23$ and $\bar{V}_{\text{multiple}}(\rho) = 5585.99$, which means that the financial institution, in terms of fiscal gain, should prefer the option of multiple contracts, since it has the smaller present value.

Moreover, this conclusion seems to be always true. Since the sequence of differences $\bar{d}_k = \bar{J}'_k - \bar{J}_k$ has only one change of sign, thus characterizing what is defined a conventional financing project, cf. de Faro (1974), which internal rate of return is known to be unique, and in this case equal to zero. Therefore, $\bar{V}_{\text{single}}(\rho) > \bar{V}_{\text{multiple}}(\rho)$ for $\rho > 0$.

Taking into account that in Brazil the monthly interest rates charged do not exceed 2% per month, in real terms, we are going to analyze the behavior of the percentage increase of the fiscal gain $\bar{\delta} = \left[\bar{V}_{\text{single}}(\rho)/\bar{V}_{\text{multiple}}(\rho) - 1\right] \times 100$, for some values of the corresponding annual opportunity cost ρ_a, with each contract with a term of n_a years. This is depicted in **Tables 9-12**. As can be seen, there is a big advantage for the financial institutions to use, multiple contracts instead of the single ones.

Table 9. Fiscal gain δ—contract with the focal date epoch = 0, $i = 0.5\%$ p.m.

n_a	ρ_a (%)					
	5%	10%	15%	20%	25%	30%
5	8.3678	16.9620	25.7543	34.7170	43.8237	53.0492
10	16.4262	34.2951	53.4320	73.6370	94.6992	116.4104
15	24.4942	52.3522	82.9943	115.7195	149.8170	184.6491
20	32.5819	70.8628	113.3726	158.4381	204.5824	250.7163
25	40.6797	89.5144	143.4894	199.5977	255.6835	310.5405
30	48.7672	107.9948	172.4553	237.7919	301.6162	363.0305

Table 10. Fiscal gain δ—contract with the focal date epoch = 0, $i = 1.0\%$ p.m.

n_a	ρ_a (%)					
	5%	10%	15%	20%	25%	30%
5	8.0647	16.3144	24.7213	33.2588	41.9016	50.6262
10	15.4571	32.0629	49.6308	67.9611	86.8557	106.1284
15	22.6496	47.8241	74.9019	103.2212	132.1739	161.2611
20	29.7113	63.4385	99.6844	137.0046	174.2898	210.8308
25	36.6624	78.6903	123.2118	167.8826	211.3237	252.9508
30	43.5011	93.3705	144.9564	195.2239	242.9630	288.0084

Table 11. Fiscal gain δ—contract with the focal date epoch = 0, $i = 1.5\%$ p.m.

n_a	ρ_a (%)					
	5%	10%	15%	20%	25%	30%
5	7.8208	15.7946	23.8941	32.0937	40.3693	48.6987
10	14.7630	30.4750	46.9449	63.9779	81.3878	99.0062
15	21.4261	44.8562	69.6614	95.2216	121.0040	146.6003
20	27.9113	58.8626	91.3911	124.2219	156.4707	187.6403
25	34.2510	72.3379	111.5637	149.9980	186.6810	221.3350
30	40.4506	85.1281	129.8356	172.2764	211.8165	248.6162

Table 12. Fiscal gain δ—contract with the focal date epoch = 0, i = 2.0% p.m.

n_a	ρ_a (%)					
	5%	10%	15%	20%	25%	30%
5	7.6184	15.3640	23.2102	31.1321	39.1070	47.1136
10	14.2319	29.2658	44.9098	60.9746	77.2849	93.6868
15	20.5344	42.7105	65.9039	89.5314	113.1180	136.3198
20	26.6429	55.6743	85.6767	115.5045	144.4283	172.0882
25	32.5945	68.0350	103.7796	138.1855	170.5608	200.8123
30	38.3965	79.6700	119.9738	157.4933	191.9419	223.6633

6.2. Focal Date at Epoch *n*

Rather than engaging in a single contract, the financial institution has the option of requiring the borrower to adhere to $n + 1$ subcontracts; one for each of the $n + 1$ payments that would be associated with the case of a single contract.

In the case where the interest rate i is of compound interest, we know, cf. De Losso et al. (2013), de Faro (2022), and de Faro and Lachtermacher (2023a, 2023b) that the principal of the k^{th} subcontract is the present value, at the compound interest rate i, of the k^{th} payment of the original single contract.

However, in the present situation, where the interest rate i is of simple interest, and where the focal date is being considered the end term of the contract, an adaptation is thus necessary.

The contractual debt of the k^{th} subcontract, denoted as F'_k, is now defined to be:

$$F'_k = P_k \times \left[1 + i \times (n - k)\right] / (1 + i \times n), \quad \text{for } k = 0, 1, 2, \cdots, n \tag{30}$$

With this proviso, we are assured that the contractual debt F is fully amortized. As for the k^{th} parcel of amortization, similarly to the case of compound interest, we also have:

$$A'_k = F'_k, \quad \text{for } k = 0, 1, 2, \cdots, n \tag{31}$$

On the other hand, regarding the k^{th} parcel of interest, which will be denoted as J'_k, and is equal to the difference $P_k - A'_k$, we will have:

$$J'_k = P \times i \times k / (1 + i \times n) = P - A'_k, \quad \text{for } k = 1, 2, \cdots, n \tag{32}$$

As previously pointed out, it should be noted that the original debt of 100,000 units of capital is fully amortized, since:

$$\sum_{k=0}^{n} F'_k = \sum_{k=0}^{n} A'_k = F \tag{33}$$

and, in this case with $n = 12$.

In Table 13, considering the consolidation of the $n + 1$ subcontracts, for the case of our numerical example, it is presented the corresponding evolution of the debt.

Even though the total payment of interest is the same the case in both of a single contract and in the case of its substitution by multiple contracts, there is a crucial distinction regarding the timing of occurrence.

A more relevant comparison must take into consideration the financial institution cost of capital. Which periodic value will be denoted as ρ.

Table 13. Multiple contracts—focal date epoch *n*.

Epoch (k)	$F'_k = A'_k$	J'_k	P_k	J_k	$d_k = J'_k - J_k$
0	938,97	0,00	938,97	938,97	938,97
1	8.685,45	78,25	8.763,69	860,72	782,47
2	8.607,20	156,49	8.763,69	782,47	625,98
3	8.528,95	234,74	8.763,69	704,23	469,48
4	8.450,70	312,99	8.763,69	625,98	312,99
5	8.372,46	391,24	8.763,69	547,73	156,49
6	8.294,21	469,48	8.763,69	469,48	0,00
7	8.215,96	547,73	8.763,69	391,24	−156,49
8	8.137,72	625,98	8.763,69	312,99	−312,99
9	8.059,47	704,23	8.763,69	234,74	−469,48
10	7.981,22	782,47	8.763,69	156,49	−625,98
11	7.902,97	860,72	8.763,69	78,25	−782,47
12	7.824,73	938,97	8.763,69	0,00	−938,97
$\sum$	100.000,00	6.103,29	106.103,29	6.103,29	0,00

That is, considering the rate ρ we must compare the present values of the corresponding sequences of the parcels of interest payments. Respectively designated as $V_{\text{single}}(\rho)$ and $V_{\text{multiples}}(\rho)$:

$$V_{\text{single}}(\rho) = \sum_{k=0}^{n} J_k \times (1+\rho)^{-k} \tag{34}$$

and

$$V_{\text{multiple}}(\rho) = \sum_{k=0}^{n} J'_k \times (1+\rho)^{-k} \tag{35}$$

where ρ is supposed to be relative to the same period as the financing interest rate *i*.

For instance, if ρ_a = 20% per year, which implies 1.531% per month, for a loan term of n = 12 and interest rate i = 1% p.m., we have $V_{\text{single}}(\rho) = 5778.23$ and $V_{\text{multiple}}(\rho) = 5382.81$, which means that the financial institution, in terms of fiscal gain, should prefer the option of multiple contracts, since it has the smaller present value.

Moreover, this conclusion seems to be always true. Since the sequence of differences $d_k = J'_k - J_k$ has only one change of sign, thus characterizing what is defined a conventional financing project, cf. de Faro (1974), which internal rate of return is known to be unique, and in this case equal to zero. Therefore, $V_{\text{single}}(\rho) > V_{\text{multiple}}(\rho)$ for $\rho > 0$.

Taking into account that in Brazil the monthly interest rates charged do not exceed 2% per month, in real terms, we are going to analyze the behavior of the percentage increase of the fiscal gain $\delta = \left[V_{\text{single}}(\rho)/V_{\text{multiple}}(\rho) - 1\right] \times 100$, for some values of the corresponding annual opportunity cost ρ_a, with each contract with a term of n_a years. This is depicted in **Tables 14-17**.

As can be seen, while the values of δ increase with the opportunity cost of the financing institution, they are the same for all the case where $i = 0.5\%$, 1%, 1.5% and 2% p.p.

Table 14. Fiscal gain δ—contract with the focal date epoch = n, $i = 0.5\%$ p.m.

n_a	ρ_a (%)					
	5%	10%	15%	20%	25%	30%
5	8.7619	17.8064	27.1049	36.6292	46.3515	56.2450
10	17.9458	37.8305	59.5153	82.8142	107.5092	133.3660
15	27.8505	60.7604	98.3346	139.8913	184.5830	231.5234
20	38.4994	86.6763	143.4897	207.0258	275.0592	345.5454
25	49.9109	115.5702	194.5100	282.4785	375.1728	469.3165
30	62.0981	147.3450	250.6167	364.1312	481.3099	598.1394

Table 15. Fiscal gain δ—contract with the focal date epoch = n, $i = 1.0\%$ p.m.

n_a	ρ_a (%)					
	5%	10%	15%	20%	25%	30%
5	8.7619	17.8064	27.1049	36.6292	46.3515	56.2450
10	17.9458	37.8305	59.5153	82.8142	107.5092	133.3660
15	27.8505	60.7604	98.3346	139.8913	184.5830	231.5234
20	38.4994	86.6763	143.4897	207.0258	275.0592	345.5454
25	49.9109	115.5702	194.5100	282.4785	375.1728	469.3165
30	62.0981	147.3450	250.6167	364.1312	481.3099	598.1394

Table 16. Fiscal gain δ—contract with the focal date epoch = n, $i = 1.5\%$ p.m.

n_a	ρ_a (%)					
	5%	10%	15%	20%	25%	30%
5	8.7619	17.8064	27.1049	36.6292	46.3515	56.2450
10	17.9458	37.8305	59.5153	82.8142	107.5092	133.3660
15	27.8505	60.7604	98.3346	139.8913	184.5830	231.5234

Continued

20	38.4994	86.6763	143.4897	207.0258	275.0592	345.5454
25	49.9109	115.5702	194.5100	282.4785	375.1728	469.3165
30	62.0981	147.3450	250.6167	364.1312	481.3099	598.1394

Table 17. Fiscal gain δ—contract with the focal date epoch = n, $i = 2.0\%$ p.m.

n_a	ρ_a (%)					
	5%	10%	15%	20%	25%	30%
5	8.7619	17.8064	27.1049	36.6292	46.3515	56.2450
10	17.9458	37.8305	59.5153	82.8142	107.5092	133.3660
15	27.8505	60.7604	98.3346	139.8913	184.5830	231.5234
20	38.4994	86.6763	143.4897	207.0258	275.0592	345.5454
25	49.9109	115.5702	194.5100	282.4785	375.1728	469.3165
30	62.0981	147.3450	250.6167	364.1312	481.3099	598.1394

This behavior, which is also present for the other values of i, is due to the peculiar way that was used for the formulation of interest. This can look like something incoherent since it was expected that the values should change depending on the interest rate as in the focal date at epoch $k = 0$. It should be noted that a similar result was observed in de Faro & Lachtermacher (2023b), when doing the same type of analysis for the French method of amortization. Given the similarity of both methods, the proof given there is also applicable to this case.

7. Conclusion

This article developed and analyzed the German method of amortization using simple interest capitalization, based on Forger (2009), and compared it with the French method using simple interest.

For the financial institution, it is better to use the French method, as it will yield higher interest earnings than the German method (**Table 3** and **Table 7**) when using simple interest capitalization on both focal dates studied. It should be noted that when using compound interest, in terms of higher interest earning, the conclusion was the opposite, with the German method preferred over the French method, cf. de Faro & Lachtermacher (2024b).

In the comparison of the single contract and the multiple contract scheme, when using the German method using simple interest capitalization, in both focal dates studied, the multiple contracts scheme should be the right choice for the financial institution, since it presents fiscal gains over the single contract.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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