Scientific Research Publishing

Theoretical Economics Letters, 2024, 14, 2552-2555
https://www.scirp.org/journal/tel
ISSN Online: 2162-2086
ISSN Print: 2162-2078

Ergodic Properties of Motion in Quantum Harmonic Oscillator and Finance

Zbigniew S. Kowalski

Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology, Wroclaw, Poland
Email: Zbigniew.Kowalski@pwr.edu.pl

How to cite this paper: Kowalski, Z. S. (2024). Ergodic Properties of Motion in Quantum Harmonic Oscillator and Finance. *Theoretical Economics Letters, 14,* 2552-2555.
https://doi.org/10.4236/tel.2024.146126

Received: August 6, 2024
Accepted: October 15, 2024
Published: December 24, 2024

Abstract

We show the Bernoulli property of the skew product transformation describing particle motion in a one-dimensional quantum harmonic oscillator. Finally, the application of the binomial model for asset prices is presented.

Keywords

Quantum Harmonic Oscillator, Step Skew Product Transformation, Bernoulli Property, Asset Prices

1. Introduction

A model of the motion of particles in a quantum harmonic oscillator on $\mathbb{R}$ was proposed (Kowalski, 2022). This is a skew-product transformation $\hat{S}_p$ where $p \in \left(\frac{1}{2}, \frac{1}{\sqrt[3]{2}}\right)$. The aim of the work is to show that $\hat{S}_p$ has the Bernoulli property. Therefore, we will prove the ergodicity of $\hat{S}_p$. We consider its isomorphic version S_p which is a random walk on $I = [0,1]$. Firstly, we will describe the probability space on which S_p operates. It is a product space $(\Omega \times I, \mathcal{B} \times \mathcal{A}, \mu_p \times \mu_F)$ where Ω is the space $\{0,1\}^{\mathbb{N}}$, $\mathbb{N} = \{0,1,2,\cdots\}$, with the $(p, 1-p)$-Bernoulli measure μ_p on $(\Omega, \mathcal{B})$. Here $\mathcal{B}$ is the Borel product σ-algebra and $\mathcal{A}$ the Borel σ-algebra of subsets of $\mathbb{R}$. The measure μ_F has a probability distribution F which is given by the mathematical description of the oscillator in (Kowalski, 2022) as follows. One-dimensional quantum harmonic oscillator Ψ satisfies the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

where $H = \frac{1}{2}\left(P^2 + Q^2\right)$ is the Hamiltonian. Here $P = -i\frac{\mathrm{d}}{\mathrm{d}x}$ and Q is multiplication by x. Let us consider the n-quantum state solution

$$\Psi_n(x,t)=\frac{1}{\sqrt{n!2^n\sqrt{\pi}}}H_n(x)\exp\left(-\frac{i}{2\hbar}(2n+1)t\right)\exp\left(-\frac{x^2}{2}\right).$$

for $n\geq 0$. Here H_n is nth Hermite polynomial *i.e.*

$$H_n(x)=(-1)^n\mathrm{e}^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2}.$$

The quantum interpretation of

$$\Phi_n(x)=\int_{-\infty}^{x}\left|\Psi_n(y,t)\right|^2\mathrm{d}y=\frac{1}{n!2^n\sqrt{\pi}}\int_{-\infty}^{x}H_n^2(y)\exp\left(-y^2\right)\mathrm{d}y$$

is the probability distribution of occurrence of a particle in $\mathbb{R}$. In Theorem 4.1 (Kowalski, 2017), it has been observed that for $n\geq 1$ there is a unique partition on intervals $\{I_k:k=1,\cdots,2n\}$ of $\mathbb{R}$ such that $\mu_n(I_k)=\mu_{n-1}(I_k)$ where the measure μ_n has distribution Φ_n. The endpoints of intervals come from the equivalence

$$\Phi_n(x)=\Phi_{n-1}(x)\Leftrightarrow H_n(x)H_{n-1}(x)=0.$$

It is convenient to consider the unit interval instead of $\mathbb{R}$, therefore, we use the map $\Phi_{n-1}:\mathbb{R}\to(0,1)$. Φ_{n-1} arise as the distribution function of the Lebesgue measure on the unit interval and Φ_n as $F^{(n)}=\Phi_n\circ\Phi_{n-1}^{-1}$. Here Φ_{n-1}^{-1} is the inverse function of Φ_{n-1}. We will denote $F^{(n)}$ by F as n is fixed. Next, we will describe the construction of the skew-product S_p. Let σ be the one-sided shift on Ω *i.e.* $\sigma(\omega)(i)=\omega(i+1)$. Let us assume that conditions of Remark 2.6 (Kowalski, 2022) hold for some $p\in\left(\frac{1}{2},\frac{1}{\sqrt[3]{2}}\right]$ and for some $1\leq k\leq 2n-1$. Then we get the step skew product transformation in the space $\Omega\times J_{k+1}$ as follows

$$S_p(\omega,u)=\begin{cases}\left(\sigma(\omega),g^{-1}(u)\right) & \text{for } \omega_0=1 \text{ and}\\ \left(\sigma(\omega),\left(2u-g(u)\right)^{-1}\right) & \text{for } \omega_0=0.\end{cases}$$

Here $g(u)$ and $2u-g(u)$ are the increasing self-homeomorphisms of J_{k+1}. The skew product as above preserves the measure $\mu_p\times\mu_F$ on $\Omega\times J_{k+1}$. Here $J_{k+1}=\Phi_{n-1}(I_{k+1})$ for $k=1,\cdots,2n-1$. The formula $\left(2u-g(u)\right)^{-1}$ for the second homeomorphism is equivalent to S_p invariance of the measure $\mu_{\frac{1}{2}}\times\Lambda$ where Λ denotes the Lebesgue measure. Section 3 is dedicated to the binomial model for asset prices.

2. The Bernoulli Property of S_p

Firstly, we normalize the μ_F measure on J_{k+1} and denote it by $\bar{\mu}_F$.

Definition 2.1. $\left(S_p,\mu_p\times\bar{\mu}_F\right)$ *has the Bernoulli property if*

$$\bar{S}_p(\bar{\omega},u)=\begin{cases}\left(\bar{\sigma}(\bar{\omega}),g^{-1}(u)\right) & \text{for } \bar{\omega}_0=1 \text{ and}\\ \left(\bar{\sigma}(\bar{\omega}),\left(2u-g(u)\right)^{-1}\right) & \text{for } \bar{\omega}_0=0.\end{cases}$$

is a Bernoulli automorphism.

Here $\bar{\omega} \in \{0,1\}^{\mathbb{Z}}$ and $\bar{\sigma}$ is the two-sided $(p,1-p)$-Bernoulli shift.

Theorem 2.2. *The skew-product transformation* S_p *has the Bernoulli property.*

Proof. By ergodic decomposition of $\mu_p \times \bar{\mu}_F$ (Kifer, 1986: p. 193, Theorem 1.1) there exists ergodic S_p-invariant measure $\mu_p \times \mu_G$ such that $\mu_G(\{b_k\}) = \mu_G(\{b_{k+1}\}) = 0$. Hence μ_G has the dense support property by Lemma 3 (Kowalski, 2009), *i.e.*, if $\mu_G(A) = \mu_G(J_{k+1})$ then $\bar{A} = J_{k+1}$. Therefore

$$M_p(S_p) = \operatorname{conv}\left\{\mu_p \times \delta_{\{b_k\}}, \mu_p \times \delta_{\{b_{k+1}\}}, \mu_p \times \mu_G\right\}$$

by Theorem 1 (Kowalski, 2003). Here $M_p(S_p)$ denotes the set of S_p-invariant probability measures μ on $\Omega \times J_{k+1}$ such that the left marginal measure of μ is μ_p. So we conclude that $\bar{\mu}_F = \mu_G$ as $\mu_p \times \bar{\mu}_F \in M_p(S_p)$ and $\bar{\mu}_F(\{b_k\}) = \bar{\mu}_F(\{b_{k+1}\}) = 0$. Now we are able to use Corollary 5.2 (Kowalski, 2019) to conclude that $(S_p, \mu_p \times \bar{\mu}_F)$ has the Bernoulli property. ☐

We define the skew product transformation $\hat{S}_p(\omega, x)$ in the space $\Omega \times I_{k+1}$ by putting H_g and $H_{2u-g(u)}$ instead of g and $2u - g(u)$ in the definition of $S_p(\omega, u)$. Here

$$H_g(x) = \Phi_{n-1}^{-1}\left(g\left(\Phi_{n-1}(x)\right)\right) \text{ for } x \in I_{k+1}.$$

This skew product preserves the measure $\mu_p \times \mu_{\Phi_n}$ on $\Omega \times I_{k+1}$ where μ_{Φ_n} has distribution Φ_n. Moreover, its natural extension to automorphism is Bernoulli one. For a detailed description, see (Kowalski, 2022).

3. The Generalized Binomial Model for Asset Prices (Discussion and Conclusions)

A generalized one-asset binomial model, see Bahsoun et al. (2007), Kowalski, (2017), is a random walk on *I*. Here $x \in I$ represents the security price. Physical phenomena at the quantum level, where randomness rules, can well describe the macro world. Therefore, we can take the skew-product $\hat{S}_p$ as a $(n-1)$-asset binomal model. For $n = 3$, see (Kowalski, 2022, Figure 4), we have

$$S_d(x) = h_1(x) \text{ and } S_u(x) = h_0(x) \text{ for } x \in I_4$$

where $I_4 = \left[0, \frac{\sqrt{2}}{2}\right]$. This is the price of one share of common stock of a particular corporation. The subscript u of S_u illustrates that the transformation S_u contains a law that moves the price up, and the subscript d of S_d illustrates that S_d contains a law that moves the price down. The Bernoulli property of $\hat{S}_p(\omega, x)$ implies the mixing of $\hat{S}_p$. Hence

$$\lim_{m \to \infty} \int 1_{\Omega \times A}\left(\hat{S}_p{}^m\right) f \,\mathrm{d}\mu_p \times \mu_{\Phi_n} = \mu_{\Phi_n}(A) \int f \,\mathrm{d}\mu_{\Phi_n}$$

for every $A \in \mathcal{A}$ and $f \in L_\infty\left(\mu_{\Phi_n}\right)$. Therefore

$$\lim_{m \to \infty} \mu_p\left(\left\{\omega : \hat{S}_p^m(\omega, x) \in \Omega \times A\right\}\right) = \mu_{\Phi_n}(A)$$

in weak-$L_1\left(\mu_{\Phi_n}\right)$ convergence.

In a similar way, we consider the second financial asset, price-adjacent to the previous one. Here we take

$$I_5=\left[\frac{\sqrt{2}}{2}.\frac{\sqrt{6}}{2}\right] \text{ and}$$

$$S_d\left(x\right)=h_0\left(x\right) \text{ and } S_u\left(x\right)=h_1\left(x\right) \text{ for } x\in I_5.$$

We can turn the interval $\left[0,\frac{\sqrt{6}}{2}\right]$ into the interval $\left[0,1\right]$ by linearly changing the variables.

For example, the above may be illustrated by the continuous quotations exchange of shares of a company listed on the Warsaw Stock Exchange on May 7, 2024. This is Atende S.A. Here for $n=20$ there are two adjacent zeros of the Hermite polynomials H_n and H_{n-1} such that $x_1\approx 2.79$, $x_2\approx 3{,}16$. If we ignore prices at which the turnover is less than 100, the firm's price distribution strictly increases as does Φ_{20}. Moreover, $p\approx 0.5$, so it is a state of equilibrium.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

Bahsoun, W., Góra, P., Mayoral, S., & Morales, M. (2007). Random Dynamics and Finance: Constructing Implied Binomial Trees from a Predetermined Stationary Density. *Applied Stochastic Models in Business and Industry, 23,* 181-212. https://doi.org/10.1002/asmb.663

Kifer, Y. (1986). Ergodic Theory of Random Transformations. In *Progress in Probability* (Vol. 10). Birkhäuser.

Kowalski, Z. S. (2003). Invariant Measures for Smooth Extensions of Bernoulli Shifts. *Bulletin of the Polish Academy of Sciences. Mathematics, 51,* 261-267.

Kowalski, Z. S. (2009). Iterations of the Frobenius-Perron Operator for Parabolic Random Maps. *Fundamenta Mathematicae, 202,* 241-250. https://doi.org/10.4064/fm202-3-3

Kowalski, Z. S. (2017). Stability of Smooth Extensions of Bernoulli Shifts. *Applicationes Mathematicae, 44,* 85-104. https://doi.org/10.4064/am2298-2-2017

Kowalski, Z. S. (2019). Bernoulli Property of Smooth Extensions of Bernoulli Shifts. *Applicationes Mathematicae, 46,* 275-282. https://doi.org/10.4064/am2376-2-2019

Kowalski, Z. S. (2022). Random Walk as a Model of Motion in a Quantum Harmonic Oscillator. *Applicationes Mathematicae, 49,* 185-195. https://doi.org/10.4064/am2447-12-2022