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The Valuation of Options on Real Estate Using Laplace Transforms

Rebecca Abraham

Huizenga College of Business, Nova Southeastern University, Fort Lauderdale, Fl, USA
Email: abraham@nova.edu

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Abstract

Options on real estate give the buyer the right to purchase or sell property at a specified price (exercise price) within a specific time period. Investors in real estate options are typically moderate to high risk-takers who seek to gain from rising or falling real estate prices by purchasing at bargain prices, or selling at inflated prices. This paper constructs mathematical models that value call options on real estate. We consider 3 different types of properties including real estate investment trusts (REITs), resorts, hotels, and large shopping malls. Investor sentiment for moderate risk-takers was modeled by gamma distributions, whose step function indicated a gradual increase in risk-taking propensity. Investor sentiment for risk-takers was modeled by exponential distributions with sharp upward increases in risk-taking propensity. Options were valued based on their intrinsic value and time value. The time distribution of option prices was modeled using Laplace transforms. Laplace transforms are presented as approximations of time value due to their ability to accommodate revisions in price expectations during trading. Laplace transforms of different types were considered such as the linear form, periodic summation, and the Kulback-Liebler divergence.

Keywords

Real Estate Options, Real Estate Derivatives, Gamma Distribution, Exponential Distribution, Laplace Transform

1. Introduction

Research on the valuation of derivatives on security prices has a long history where the call option is the right to purchase or sell stock at a specified price within a specified time period. Bachelier (1900) created an early option valuation model, although his model did not capture the intrinsic value and time value of the option. In 1973, Black and Scholes presented their model of the price of a call option on stock that overcame the problem of estimating security returns by substituting security prices for returns (Black & Scholes, 1973). This seminal model has driven much of the research on option pricing in subsequent years. An under-researched area in option pricing is options on real estate as the underlying asset. A call option on real estate is the right to purchase real estate at an exercise price during a specific time period. The call buyer gains if the market price of the property is above the exercise price as he or she can purchase the property at the exercise price and sell at a gain at the higher market price. The call buyer pays the call writer a premium for the call option which confers this price privilege.

The purpose of this study is to construct mathematical models to value call options and put options on real estate. The paper creates models based on varying investor sentiment and varying option prices. Investors in real estate options are either moderate risk-takers or risk-takers. We model the sentiments of moderate risk-taking call buyers and put buyers as gamma distributions, as the step function of the gamma distribution suggests gradual increase in investor sentiment favoring option purchase. The step distribution has no sharp upward swings or downward swings in investor sentiment supporting the notion of moderation in risk-taking. We further model the sentiments of risk-taker investors as an exponential distribution which accommodates sudden changes in investor sentiment. We consider 3 different types of real estate investments. Most of them are in commercial real estate. They include 1) real estate investment trusts (REITs), 2) upscale resorts and hotels, and 3) large shopping malls. Option values have an intrinsic (current) value, and a time value based on the trajectory of price changes during the time period from today until the date of expiration of the option (typically, 1 day-3 months). The intrinsic value is based on current market prices, which are a constant value. The time value is based on the trajectory of changes in prices over a three-month period. This study proposes Laplace transforms to model real estate price movements over the three-month time value period as Laplace transforms are the mathematical formulation that permits the revision of prices over time.

Laplace transforms were selected to model call option prices as they convert differentiation and integration in the time domain into simpler multiplication and division in the Laplace form. The Laplace transform of the Fourier transform decomposes a function into its moments contained in a power series. This is applicable to call option pricing during the exercise period. During this time period, call prices change constantly, and are revised several times. The power series of the Laplace transform captures the call prices and their revisions. The Laplace transform's ability to decompose the Fourier transform into moments is useful in call price valuation, as moments describe volatility, or the distance of call prices from the mean. As property values change, some call prices will be close to mean property values, while other call prices will differ. The Laplace transforms moments will describe these changes in call prices from the mean. A Fourier transform per se cannot do this as it decomposes a function into its components, not its moments.

This study recognizes that different types of property have different predictors of valuation. The chief predictors are 1) stability of rental income, and 2) property values. We maintain that real estate prices will remain high, and therefore, attractive to call buyers. As properties are primarily occupied by businesses, businesses that are financially viable will provide stable rental income. Assuming that these businesses are located in areas with growth and economic prosperity, property values will continue to increase, thereby increasing the attractiveness of call option purchases and put writing. Real estate investment trusts, upscale resorts and hotels, and large shopping malls are financially sound businesses with a steady stream of rental income. REITs are large properties that attract sophisticated property management firms who deliver increasing rental income. Resorts and hotels cater to tourists who provide growing revenue. Large shopping malls traditionally provided stable rents. Recently, online sales have emptied stores so that rental income has become unstable. The stream of rental income is capitalized into a perpetuity with high property values (see Equation (1) below).

$$\text{Property value} = \text{Rental income}/r \tag{1}$$

Where r = rate of return on the real estate investment

$$\text{Rental income} = \text{Rental income 1}\frac{1}{1+r} + \text{Rental income 2}\frac{1}{(1+r)^2} + \text{Rental income 3}\frac{1}{(1+r)^3} + \cdots$$

This paper models the conduct of call buyers who have the optimistic belief in rising property values and stable rental income. We do not consider put buying which as the pessimistic belief in declining property values and declining rental income. These businesses include strip malls, and farmland. Strip malls are of small size with franchises and independent retailers. While franchises have name recognition, and therefore an inflow of customers, small independent pharmacies, clothing stores, and general stores may not survive as they lack an established customer base. Farmland is located in rural areas at a distance from urban centers. As the number of Americans employed in agriculture has diminished from 80% to 3% in the past century (Statistical Abstract of the United States, 2022), property demand in the agricultural sector has declined diminishing the value of farmland.

This study contributes to the sparse literature on mathematical models that value real estate options. A few recent studies are notable. Cirjevskis (2021) presented an option pricing model for commercial real estate development in Latvia. In the high economic uncertainty environment of the post-Covid era, developers reduced risk by holding a series of call options on their development project. Each option had a life of a segment of the project. For example, Option 1 lasted for the duration of foundation-building, with its return being based on the value of the foundation-building phase. Option 2 had a duration of outer wall construction, with a higher return than Option 1 to be realized upon completion of the outer walls, and so on. This time period, i.e. the post Covid-era, may not be applicable to modern times. Zhong and Hui (2021) created a model that valued options on mixed-used properties in Hong Kong, China. Mixed-use properties may not be relevant to this study, as we focus on purely commercial real estate, while mixed-use properties contain both residential properties and commercial properties. Wang et al. (2023) constructed option models based on both tenant and landlord perspectives in the Hong Kong, China, real estate market. They reasoned that the stream of rent payments by tenants would earn a negative return upon lease renewal, assuming that rents would increase. The price value of the option would have a negative relationship with the original lease terms, suggesting declining gain for their call option. Conversely, landlords as call writers experienced gain upon lease renewal, as tenants pay them a premium to keep rents from rising more sharply than the original level.

This study differs from this body of work in that it considers multiple types of properties, while the existing studies are confined to a single type of property. This study includes investor sentiment in the analysis as equilibrium prices depend upon an investor's attitude to risk. Our model is more general in its applicability to different countries and regions while the aforementioned studies describe options on real estate in a single region, such as Latvia or Hong Kong, China. We recognize that real estate investment must include properties of widely different valuations, from real estate investment trusts and upscale resorts and hotels, to mid-range shopping malls. The remainder of this paper is organized as follows. Section 2 is a Review of Literature, Section 3 contains the mathematical formulations of call options, and Section 4 is the consists of Conclusions.

2. Review of Literature

2.1. The Concept of Options on Real Estate

A few seminal papers have explored the concept of options on real estate. Titman (1985) created an early model in which investors purchased call options on property under construction. As there was a delay in completing construction, there was a wait period before investors could realize returns. The uncertainty about construction costs and the risk-free rate used in valuing the call option reduced the value of the call option. Childs et al. (1996) set forth that mixed use of the property and continuous redevelopment contributed to the value of the property increasing call option values. Williams (1991) verified the Childs et al. (1996) model using property values, showing that redevelopment of the property generated higher value than single redevelopment, so that call option values based on multiperiod redevelopment are valued more highly than those based on single period redevelopment. Based on his review of these papers, Lucius (2000) set forth the following characteristics of options on real estate.

Entrepreneurial Flexibility. Real estate investors are entrepreneurs with a range of investment choices. Upon considering the purchase of call options on real estate, they concurrently evaluate other options, such as call options on a strip mall, hotel, or large shopping mall. Based on construction costs and projections of future rent, the investor may abandon the original option in favor of an option with a different underlying asset that offers lower construction costs, shorter time period to completion of construction, and higher, stable rents.

Limited Substitutability. Within a type of property, there may be few real estate choices. An investor who wishes to purchase call options on large shopping malls in Florida, USA, may not have many choices.

Immobility. Real estate cannot be moved. If construction on a piece of property is delayed, the investor is required to wait till construction resumes. In contrast, a stock option will permit the investor to sell the stock and move onto a different investment.

Irreversibility of Real Estate Investment. Construction projects cannot be easily altered and sold off if original plans do not materialize. Likewise, market forces may adversely influence the stream of rental income. Online sales may drive large shopping mall occupants to closure disrupting rental income streams for the investor whose call options will rapidly lose value.

High Transaction Costs. As they are less liquid, the supply of real estate options is more limited than options on securities. Therefore, transaction costs incurred upon purchase or sales of these options are higher than the transaction costs on stock options. Further, the transaction costs on the underlying real estate asset are high, as closing costs, real estate commissions and fees have to be paid on purchased property, while these expenses do not exist for other underlying assets, such as stock, market indexes, oil, soybeans, and agricultural commodities.

2.2. Research on the Valuation of Real Estate Options

Titman's (1985) seminal paper focused on the valuation of vacant land. An investor could purchase a call option on vacant land based on the assumption that a future building would increase land prices significantly adding value to the call option. Conversely, if the increase in price of the developed property was not forthcoming, the owner could let the land remain undeveloped. This is the expiration of the option if the strike price is less that the asset price. Alternatively, the owner could construct a building on land that is depreciating in value. The owner could earn arbitrage profits by short selling the property consisting of land plus building. The proceeds from the short sale could be invested in a risk-free asset. Titman (1985) did not consider the alternative options strategy of buying a put option rather than short selling the underlying asset.

Cirjevskis (2021) presented a case study to value options on Latvian real estate. His case study does not present a mathematical formulation. It assumes that options on Latvian real estate are valued in a multistep process. First, the net present value of the property construction project is determined, then critical variables for project success are measured, followed by best case, normal, and worst case scenarios. Finally, net present values are computed for each case. Black-Scholes and binomial option pricing models are used to value call options. If the call options have positive value (NPV of constructed property is positive), the project is expanded. If the NPV of the constructed property is negative, the project is abandoned, or an alternative project with positive call option value is sought.

While both the Titman (1985) study and the Cirjevskis (2021) model focused on the intrinsic value of the option, Zhong and Hui (2021) provided the path of price movements of call options on mixed-used properties. However, they employed a simple e^X function to describe the value of the property. Their model created a maximization function of option values subject to the option values being above a minimum threshold at a market-driven hurdle rate. The inclusion of the path of price movement resulted in the continuous adjustment of call option values, as property values were revised upwards. The final value of the call option was achieved when the increase in value between two stages was minimized.

Unlike the aforementioned studies which focused on the construction of property, Wang et al. (2023) created a model valuing options based on the value of the stream of rents received by property-owners. If vacancy rates and vacancy periods are high, market rents will decrease call option values. If the opposite is true, i.e. vacancy rates and vacancy periods are low, market rents will increase call option values. This finding suggests that call buyers can renew their contracts at fair market value.

3. Findings and Analysis

3.1. Valuation of Call Options on REITs

Real Estate Investment Trusts are of large size as they are formed by the bundling together of multiple commercial properties. Investors with large real estate portfolios seek REITs which are frequently offered at competitive prices. The investors typically repair the properties and resell them. Call buyers on REITs may purchase a series of call options. Upon expiration of the first option, the call buyer would exercise the option, purchasing the property at the cheap exercise price, and taking a gain. The moderate risk-taker would cease investment with the second or third call option, while the risk-taker would continue to purchase options, exercise them, and take further gains. The risk-taker would continue to purchase calls until no further gain is realized.

3.1.1. The Moderate Risk-Taker

Figure 1 shows the maximization of the probability density function of the moderate risk-taker which follows the path of a gamma distribution. The gamma distribution's gradual upward trajectory matches the investor sentiment of the moderate risk taker who purchases the call option on the REIT, takes the gain, assesses profits, purchases a second option and repeats the process till prices rise to level t, upon which the investor exits the market.

OB is the gamma distribution of investor sentiment. Lines T, U, and V represent Laplace transforms at which revision of call price expectations takes place.

Maximization of gains occurs at the three intersections of the gamma distribution, *OB*, with the Laplace transform of price changes *XY*.

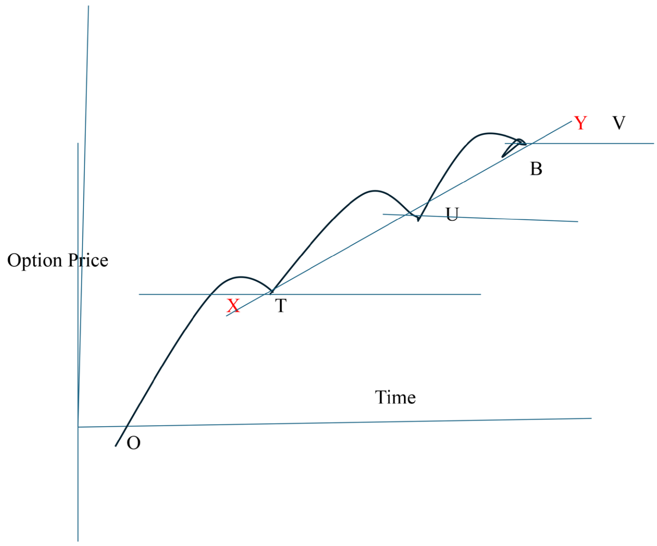

Figure 1. The moderate Risk-Taker's purchase of a call on REIT.

The following formulation describes the moderate risk-taker's gain at the first level of trading volume, *T*.

Max

$$(S-X)+REIT*[1/\Pi(k)\theta^k x^{k-1}e^{-x}]\theta \tag{2}$$

Intrinsic value *S*, current property value, - *X* strike price + time value of REIT*gamma with shape parameter *k*, and scale parameter θ.

Subject to

$$s^2F(s)-s(0^-)-f'(0^-)=0 \tag{3}$$

Second derivative of the power series of Laplace transform function $f(0^-)$.

$$L>Y \tag{4}$$

L is the property value > a minimum threshold, as REITs are of large size.

$$S=R \tag{5}$$

$$S = \text{property value at a maximum of } R, \tag{6}$$

Taking Lagrangians of the objective function and constraints,

Max

$$(S-X)+REIT*[\Pi(k)\theta^k x^{k-1}e^{-x}]\theta-P_1(s^2F(s)-s(0^-) \\ -f'(0^-)-0-P_2(L-Y)-P_3(S-R) \tag{7}$$

Taking first derivatives of Equation (7),

$$REIT*[\frac{1}{\Pi(k)\theta^{k-1}e^{-x}}]\theta-P_1(2sF'(s)-s'(0^-)-f''(0^-)]-P_2dS/dx \tag{8}$$

Taking the derivative of Equation (8) which maximizes the objective function,

$$REIT*[\frac{1}{\Pi'(k)k(k_1)\theta^{k-2}(k-1)(k-2)x^{k-2}e^{-x}}]\theta \\ -P_1(2F''(s)-s''(0^-)-f''(0^-)]-P_2d^2S/dS^2 \quad (9)$$

The following formulation describes the moderate risk-taker's gain at trading volume *U*. At point *T*, the call buyer revises expectations of real estate prices. He or she anticipates that real estate prices will increase, but being cautious, may decide to take the gains from the call purchase and exit the market.

Max

$$(S_1-X)+REIT*[\frac{1}{\Pi(k)\theta^k x^{k-1}e^{-x}}]\theta \quad (10)$$

Intrinsic value S_1, property value at *U*, - *X* strike price + time value of REIT*gamma with shape parameter *k*, and scale parameter *θ*.

Subject to

$$s^2F(s_1)-s_1(0^-)-f'(0^-)=0 \quad (11)$$

Second derivative of the power series of Laplace transform function $f(0^-)$.

$$L>Y \quad (12)$$

L is the property value > a minimum threshold, as REITs are of large size.

Taking Lagrangians of the objective function and constraints,

Max

$$(S_1-X)+REIT*[\frac{1}{\Pi(k)\theta^k x^{k-1}e^{-x}}]\theta-P_1(s^2F(s_1)-s_1(0^-) \\ -f'(0^-)-0-P_2(L-Y)-P_3(S-Z) \quad (13)$$

Taking first derivatives of Equation (13),

$$REIT*[\frac{1}{\Pi(k)\theta^{k-L}(k-L)x^{k-2}e^{-x}}]\theta-P_1(2sF'(s_1)-s_1'(0^-) \\ -f''(0^-)]-P_3dS/dx \quad (14)$$

Taking the derivative of Equation (14) which maximizes the objective function,

$$REIT*1/\Pi'(k)k(k_1)\theta^{k-2}(k-1)(k-2)x^{k-2}e^{-x}]\theta \\ -P_1(2F''(s_1)-s_1''(0^-)-f''(0^-)]-P_3d^2S/dS^2 \quad (15)$$

3.1.2. The Risk Taker

The risk-taker's sentiment follows an upward-sloping exponential distribution, *AB*, in **Figure 2**. The risk-taker is undisturbed by higher REIT prices with decreasing gains in each round of trading. As long as gains are to be made the risk-taker will continue to purchase call options only exiting the market when property prices have reached their maximum so that no further gains are possible.

We maximize the investor sentiment represented by an exponential distribution. The exponential distribution describes the distance between event in a Poisson process of the beginning of purchasing a call option, holding it as *REIT* prices rise, and selling the call at the maximum *REIT* price. **Figure 2** shows that distance *AB* between the maximum price of the call and the price at the beginning of trading. The Laplace transform of the price distribution of the call only occurs at the maximum REIT prices, as minimal revisions take place before the risk-taker exits the market.

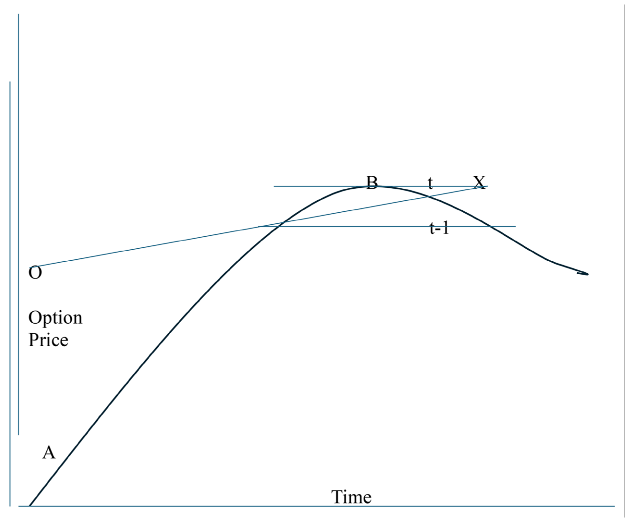

Figure 2. The Risk-Taker's purchase of a call on REITs.

AB is the investor sentiment of the risk-taker. At *t*-1, the risk-taker reassesses the purchase of additional call options, based upon *OX*, the Laplace transform of property prices. The risk-taker may exit the market at *t*-1, or continue to trade until time period, *t*.

Max

$$\lambda e^{-\lambda x} + \log(\lambda_0) \pm \log(\lambda) + \frac{\lambda}{\lambda_0} - 1 \quad (16)$$

where the first term in Equation (16) is the probability density function of an exponential distribution with λ as the rate parameter over the interval (0, *α*). The second term is the Kullback-Leibler divergence showing the divergence from the investor sentiment at time *t*-1 and time *t*, at which the risk-taker evaluates the call option which has such low gain that the investor may reject it, and exit the market.

Subject to

$$s^{tk} F(s) - \sum k = 1\, to\, n\, s^{n-k} f^{(k-1)}(0^-) > 1 \quad (17)$$

Where Equation (17) represents the Laplace transform at period, *t*-1, at which the call buyer decides to trade 1 more round to maximize gain. The Laplace transform is the revision of price expectations to accept low gains in the final round of trading.

Combining the objective function and constraint using Lagrangians,

$$[\lambda e^{-\lambda x} + \log(\lambda_0) \pm \log(\lambda) + \frac{\lambda}{\lambda_0} - 1] - L[s^{tk} F(s) - \sum k = 1\, to\, n\, s^{n-k} f\, k - 10 - 1] \quad (18)$$

Taking the first derivative,

$$[\lambda e^{-\lambda x} + 1/\lambda_0 \pm 1/\lambda] - L[tks^{tk-1}F'(s) - \sum k = 1\, to\, n\,(n-k)s^{n-k-1} f'k - 1] \quad (19)$$

Taking second derivatives,

$$\begin{aligned}&[\lambda e^{-\lambda x} + 1/\lambda_0 \pm 1] - L[tk(tk-1)s^{tk-2}F''(s) \\ &- \sum k = 1\, to\, n\,(n-k)(n-k-1)s^{n-k-2} f''k - 1]\end{aligned} \quad (20)$$

Equation (20) maximizes the objective function.

3.2. Valuation of Call Options on Resorts and Hotels

Upscale resorts and hotels offer vacation packages to high net-worth clients. The packages include a variety of amenities such as sightseeing trips, fine dining, cruises, and professional caregivers. They may hire professional staff with special skills. The price of the vacation packages will continue to rise as long as additional amenities are added. The constraints are increased competition from other resorts and hotels offering similar amenities.

3.2.1. The Moderate Risk-Taker

The value of the resorts and hotels will continue to increase over time. The moderate risk-taker, although supportive of the price increases, is sufficiently cautious to exit the market after a few rounds of trading, reasoning that the value of the resort will decline rapidly if a competitor starts selling vacations with more amenities. The gamma distribution will model investor sentiment. Figure 3 shows the gamma distribution *ABCD*. At point *C*, the moderate risk-taker exits the market. The constraints are the price function of the Laplace transform with no revision as the risk-taker leaves the market without revising expectations for future trading. There will be additional constraints for new attractions to stay competitive.

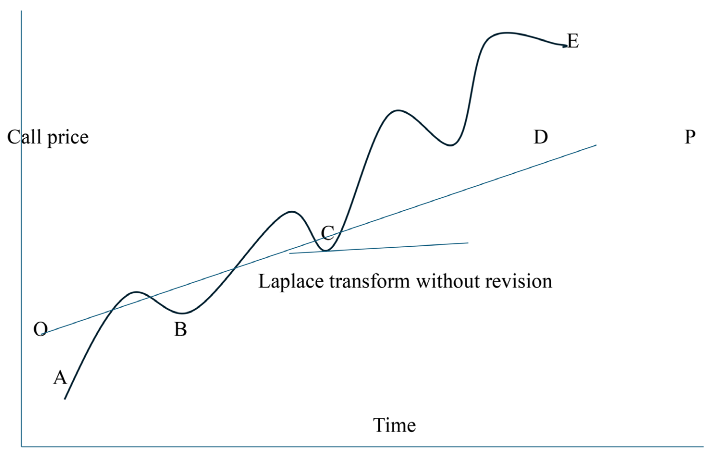

Figure 3. The Moderate Risk-Taker's Purchase of Calls on Upscale Hotels and Resorts.

ABCDE is the gamma distribution describing the sentiment of calls on hotels and resorts for moderate risk-takers. At point *C*, the moderate risk-taker, exercises calls and exits the market upon seeing a rising price distribution represented by *OP*, the Laplace transform without revision.

Max

$$(S_1 - X) + HOTELPRICE * [\frac{1}{\Pi(k)\theta^k x^{k-1} e^{-x}}]\theta \tag{21}$$

Intrinsic value S_1, property value at point $C = -X$ strike price + time value of HOTEL OR RESORT VALUE*gamma with shape parameter k, and scale parameter θ.

Subject to

$$aF(s) + G(s) > 0 \tag{22}$$

Equation (22) is a linear Laplace transform with no reduction in price of the hotel or resort.

$$X_1 + X_2 + \cdots + X_n > 1 \tag{23}$$

Equation (23) shows the addition of multiple attractions $X_1 + X_2 + \cdots + X_n$ to the original vacation package.

$$\int 0 \; to \; \alpha \;\; f(t) e^{-\sigma t} e^{irt} dt \tag{24}$$

Equation (24) is a Fourier transform of the addition of skilled staff. $e^{-\sigma t}$ is the addition of staff with skills $-\sigma$, while e^{irt} is the addition of staff with different skills, ir at time t.

Taking Lagrangians,

$$\begin{aligned}&(S_1 - X) + HOTELPRICE * [\frac{1}{\Pi(k)\theta^k x^{k-1} e^{-x}}]\theta - L[aF(s) + G(s) - 0 \\ &+ X_1 + X_2 + \cdots + X_n - 1 + \int 0 \; to \; \alpha \;\; f(t) e^{-\sigma t} e^{irt} dt]\end{aligned} \tag{25}$$

Taking the first derivative of Equation (25),

$$\begin{aligned}&HOTELPRICE * [\frac{1}{\Pi(k)\theta^k x^{k-1} e^{-x}}]\theta - L[aF'(s) + G'(s) - 0 \\ &+ X_1 + X_2 + \cdots + X_n - 1 + f(t) e^{-\sigma t} e^{irt}]\end{aligned} \tag{26}$$

Taking the second derivative of Equation (26),

$$\begin{aligned}&HOTELPRICE * [\frac{1}{\Pi(k)(k-1) x^{k-2} e^{-x}}] - L[aF''(s) + G''(s) \\ &+ X_1 + X_2 + \cdots + X_n + f'(t) e^{-\sigma t} e^{irt}]\end{aligned} \tag{27}$$

Equation (27) maximizes the objective function.

3.2.2. The Risk-Taker

The risk-taker's sentiment will follow an exponential distribution with call buyers continuing to purchase calls as property prices rise. The continuous upward trajectory of prices is modeled as a Laplace transform with no revision in price expectations. Figure 4 shows the investor sentiment as OA, and the price function as *CD*. The call buyer purchases at *O*, holds the call through several rounds of trading, and finally exercises the options at point *A*, when the full gain is taken, and the buyer exits the market. Constraints emerge from the need to continuously upgrade products to meet competitive threats.

Max

$$\lambda e^{-\lambda x} + \frac{n!}{\lambda^n \sum k} = 0 \; to \; n \, (-1)^k / k! \tag{28}$$

Equation (28) consists of two terms. The first term is the probability density function for an exponential distribution, while the second term is the moment about the mean of the exponential distribution to capture slight variations in investor sentiment.

Subject to

$$1/e^{-ts} F(s) \tag{29}$$

Equation (29) describes the continuous upward movement in property prices as the periodic function of a time shifting Laplace transform. Time shifting accounts for variations from prices over specific time periods.

$$X_1 + X_2 + X_3 + \cdots + X_n > Y_1 + Y_2 + Y_3 + \cdots + Y_n \tag{30}$$

where $X_1, \cdots, X_n$ represents the attractions of the hotel/resort to stay competitive, that exceed similar attractions $Y_1, \cdots, Y_n$ offered by a competitor.

Taking Lagrangians,

$$\begin{aligned} & e^{-\lambda x} + \frac{n!}{\lambda^n \sum k} = 0 \; to \; n \frac{(-1)^k}{k!} \\ & -L[X_1 + X_2 + X_3 + \cdots + X_n - (Y_1 + Y_2 + Y_3 + \cdots + Y_n) \end{aligned} \tag{31}$$

Taking first derivatives of Equation (31),

$$\begin{aligned} & e^{-\lambda x} + \frac{n!}{n\lambda^{n-1} \sum k} = 0 \; to \; n \frac{(-1)k^{k-1}}{k!} \\ & -L'[X_1 + X_2 + X_3 + \cdots + X_n - (Y_1 + Y_2 + Y_3 + \cdots + Y_n) \end{aligned} \tag{32}$$

The function is maximized in Equation (32), the second derivative of Equation (31),

$$\begin{aligned} & e^{-\lambda x} + \frac{n!}{n(n-1)\lambda^{n-2} \sum k} = 0 \; to \; n \frac{(-1)(k-1)k^{k-2}}{k!} \\ & -L''[X_1 + X_2 + X_3 + \cdots + X_n - (Y_1 + Y_2 + Y_3 + \cdots + Y_n) \end{aligned} \tag{33}$$

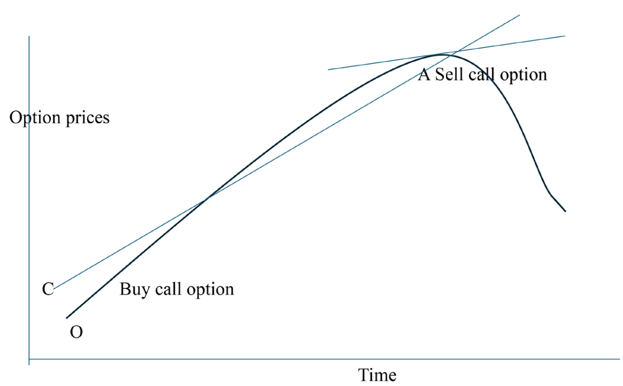

Figure 4. The Risk-Taker's purchases of calls on upscale hotels and resorts.

Figure 4 shows purchase of the call option at *O*, and sale at *A*, the maximum expectation of property price increases. The price function *CD* of the Laplace transform without revised price expectations, intersects the investor sentiment at point *A*.

3.3. Valuation of Call Options on Large Shopping Malls

The large shopping mall has an anchor store that occupies the most space and pays the highest rent. It shares the mall with smaller stores that pay lower rent. The call buyer views gains as being achieved by continuously rising property values, where property values increase based on annual increases in rent.

3.3.1. The Moderate Risk-Taker

The moderate risk-taker's sentiments may be modeled by a gamma distribution in which price expectations rise gradually based on annual increases in rent. At time *t*-2, the call buyer ceases call purchases as online sales may end the domination of the anchor store, and possibly even the smaller stores. The call buyer will sell call options, taking the maximum gain, and exiting the market. No revision of price expectations occurs, as the moderate risk-taker does not wish to experience any losses from declining mall sales due to competition from online vendors. The Laplace transform of the price distribution is a periodic function of a time shifting geometric series as rents change minimally over time.

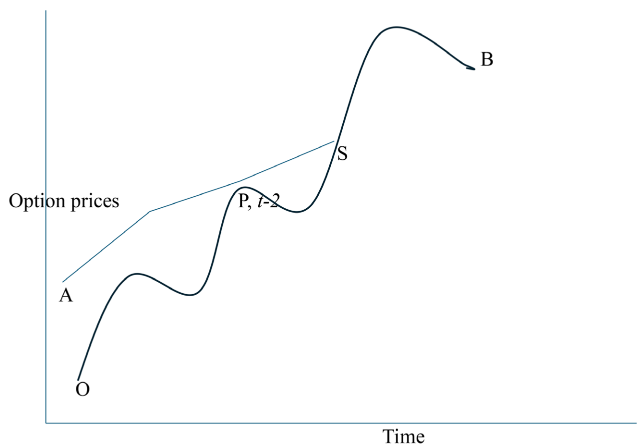

Figure 5. The Moderate Risk-Taker's purchase of calls on large shopping malls.

Figure 5 shows *OB*, the moderate risk-taker's sentiment of call prices. Rising expectations of prices abruptly end at time *t*-2, when rents decline. The Laplace transform of property values *AS*, is a broken line as there is time shifting of prices as online sales gradually replace mall sales, reducing mall rents. At *t*-2, the call buyer exercises the option, taking maximum gain, and exiting the market.

Max

$$\frac{1}{\Pi(k)\theta^k x^{k-1} e^{-x}}\theta_1 + \frac{1}{\Pi(k)\theta^k x^{k-1} e^{-x}}\theta_2 + \frac{1}{\Pi(k)\theta^k x^{k-1} e^{-x}}\theta_3 \quad (34)$$

where Equation (34) is the gamma distribution of the stream of rents. The first term represents the rents from Business (1), the second term represents the rents from Business (2), and so on.

Subject to

$$1/(1e^{-tn}\int 0\ to\ t\ e^{-st} f(t)dt < x \tag{35}$$

Equation (35) is a time shifting Laplace transform of mall rents until time *t*-2, when trading ends.

Taking Lagrangians,

$$\begin{aligned}&\frac{1}{\Pi(k)\theta^k x^{k-1}e^{-x}}\theta_1+\frac{1}{\Pi(k)\theta^k x^{k-1}e^{-x}}\theta_2+\frac{1}{\Pi(k)\theta^k x^{k-1}e^{-x}}\theta_3\\&-L[1/(1e^{-tn}\int 0\ to\ t\ e^{-st} f(t)dt-x\end{aligned} \tag{36}$$

Taking the first derivative of Equation (36),

$$\begin{aligned}&\frac{1}{\Pi(k)\theta^k(k-1)x^{k-2}e^{-x}}\theta_1+\frac{1}{\Pi(k)\theta^k(k-1)x^{k-2}e^{-x}}\theta_2\\&+\frac{1}{\Pi(k)\theta^k(k-1)x^{k-2}e^{-x}}\theta_3-L'[\frac{1}{1e^{-tn}e^{-st}f(t)dt}-1]\end{aligned} \tag{37}$$

Taking the second derivative of Equation (36), which maximizes this function,

$$\begin{aligned}&\frac{1}{\Pi(k)\theta^k(k-1)x^{k-2}e^{-x}}\theta_1+\frac{1}{\Pi(k)\theta^k(k-1)x^{k-2}e^{-x}}\theta_2\\&+\frac{1}{\Pi(k)\theta^k(k-1)x^{k-2}e^{-x}}\theta_3-L'[\frac{1}{1e^{-tn}e^{-st}f(t)dt}-1]\end{aligned} \tag{38}$$

3.3.2. The Risk-Taker

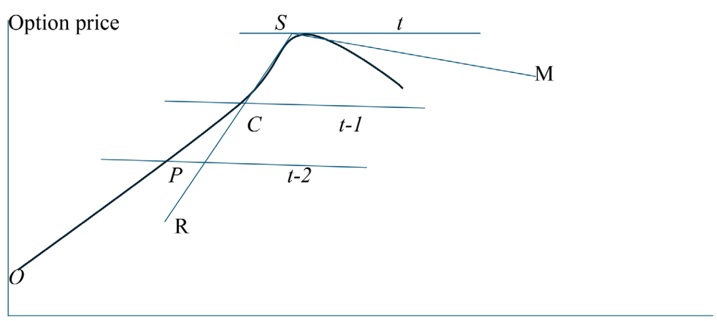

Figure 6. The Risk-Taker's purchase of calls on large shopping malls.

Figure 6 shows the exponential distribution of price expectations *OS*. At time *t*-2, the risk-taker evaluates the call price at point *P*. Being a risk-taker, this individual does not fear price decreases due to online sales replacing mall sales. The risk-taker continues to trade, taking a small gain by exercising the call at point *C* at time period, *t*-1 when property prices are still rising. A few risk-takers will continue to buy calls till time period, *t*, when they experience losses upon the displacement of mall sales by online sales. RCM is the curved Laplace transform that includes the slowing of price increases from *t*-2 to *t*-1, followed by the reduction in prices during time *t*.

As seen in **Figure 6**, the exponential distribution of price expectations *OS* is on a continuous upward trajectory of property price increases. At time *t*-2, the risk-taker evaluates the call price at point *P*. Being a risk-taker, this individual does not fear price decreases due to online sales replacing mall sales. He or she purchases more calls. If the trader judges price changes accurately, he or she will be able to take a small gain by exercising the call at point *C* at time period, *t*-1, when property prices are still rising. However, not all traders will judge price changes correctly. Some traders will expect prices to rise in time period, *t*, when they actually fall. They will continue to buy calls till time period, *t*, when they experience losses upon the displacement of mall sales by online sales. *RCM* is the curved Laplace transform of prices.The curvature of the Laplace transform accounts for the change in intensity of price increases from *t*-2 to *t*-1, and the change in direction of prices in period *t*.

The objective function and constraints of the risk-taker are given below.

Max

$$[(t-2)\lambda e^{-\lambda x}]+[(t-1)\lambda e^{-\lambda x}]+(\frac{t\lambda_0}{\lambda}+\frac{\lambda}{\lambda_0}-1) \tag{39}$$

where the first term in Equation (38) is the exponential distribution of price expectation at time period *t*-2, the second term is the exponential distribution of price expectation at time period, *t*-1. The third term is the Kullback-Leibler divergence of the change in direction of property price expectations at time *t*, upon the displacement of mall sales by online sales. We recognize that this divergence will result in losses, so we give this term a negative sign in order to minimize the losses.

Subject to

$$(t-2)n!/(s+\alpha)^n+(t-1)n!/(s+\alpha)^n+(t)\frac{n!}{(s+\alpha)^n}>0 \tag{40}$$

where Equation (40) consists of *n*th power Laplace transforms with frequency shift to account for the varying shapes of the price function in the three time periods, *t*-2, *t*-1, and *t*.

Taking Lagrangians,

$$\begin{aligned}&[(t-2)\lambda e^{-\lambda x}]+[(t-1)\lambda e^{-\lambda x}]+(\frac{t\lambda_0}{\lambda}+\frac{\lambda}{\lambda_0}-1)\\&-L[(t-2)\frac{n!}{(s+\alpha)^n}+(t-1)\frac{n!}{(s+\alpha)^n}+(t)\frac{n!}{(s+\alpha)^n}-0]\end{aligned} \tag{41}$$

Taking first derivatives of Equation (41),

$$\begin{aligned}&[(t-2)e^{-\lambda x}]+[(t-1)e^{-\lambda x}]+(\frac{t\lambda_0}{\lambda}+\frac{\lambda}{\lambda_0})\\&-L'[(t-2)\frac{n!}{(s+\alpha)^n}+(t-1)\frac{n!}{(s+\alpha)^n}+(t)\frac{n!}{(s+\alpha)^n}]\end{aligned} \tag{42}$$

Taking second derivatives of Equation (40) which maximizes the objective function.

$$[(-1)e^{-\lambda x}]+(\frac{\lambda_0}{\lambda}+\frac{\lambda}{\lambda_0})-L''[(-1)n!/(\alpha)^n+n!/(\alpha)^n] \quad (43)$$

4. Conclusion

This study makes contributions to theory and practice. It is unique in its differentiation of investor sentiment between the moderate risk-taker and risk-taker with different mathematical formulations for each type of investor. The formulations of gamma distributions for the moderate risk-taker's sentiments and exponential distribution for the risk-taker's sentiments are distinctly different from each other.

We employ different types of Laplace transforms, including the linear form, the general derivative, time shifting, and periodic summation. The use of different Laplace transforms enriches the formulations by exploring the fuller scope of price functions than a single type of Laplace transform.

Earlier mathematical formulations have been on financial assets such as foreign currencies (Abraham, 2018a; Abraham, 2018b), commodities and cryptocurrencies (Abraham & El-Chaarani, 2022). In contrast, this study presents formulations on options on real assets, such as real estate. Real assets are illiquid, high-priced, and immobile, thereby presenting unique challenges to investors.

The findings of this study are of interest to practitioners. Real estate investors are concerned about having to make large cash investments in properties. In the event that the properties do not sell, the investors will have to reduce prices significantly which may result in losses. Call options on real estate do not require large cash investment, thereby preserving liquidity. The models in this study show that if the call option investor exits the market before property price reductions can occur, then profits can be earned. Therefore, the challenge lies in timing the investment, rather than being subject to changes in price due to market conditions that are not within the control of the investor.

Study Limitations and Directions for Future Research

Defreitas and Kane (2022) reduced numerical errors in the inversion of Laplace transforms in a multi-precision environment. They present the Talbot and Gaver functionals as the optimal choice to reduce pertubations in the inversion of Laplace transforms. Future research must employ these functionals in the valuation of call options as they will reduce numerical errors in estimating the property values underlying call options on real estate.

The study was confined to properties with rising values which offer gains to call buyers. Future research must consider the ability of investors to make gains on put options on real estate. Strip malls are collections of small stores with uncertain cash flows. These firms may not be able to pay stable rents. Therefore, property values will decline suggesting that put buyers may make gains in multiple rounds of trading.

The study does not consider multifamily residential property with 10-100 units. The prices of these properties depend on their location, i.e. whether they are accessible to offices and stores. Prices of these properties may rise or fall so that either call options or put options may be attractive investments. Future research should present mathematical formulations of both calls and puts on multifamily residential properties.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

(2022). *Statistical Abstract of the United States 2022: The National Data Book.* Bernan Press and ProQuest.

Abraham, R. (2018a). The Valuation of Currency Put Options. *Theoretical Economics Letters, 8,* 2569-2593. https://doi.org/10.4236/tel.2018.811165

Abraham, R. (2018b). The Valuation of Currency Call Options. *Theoretical Economics Letters, 8,* 2271-2289.

Abraham, R., & El-Chaarani, H. (2022). A Mathematical Formulation of the Valuation of Ether and Ether Derivatives as a Function of Investor Sentiment and Price Jumps. *Journal of Risk and Financial Management, 15,* Article 591. https://doi.org/10.3390/jrfm15120591

Bachelier, L. (1900). *The Theory of Speculation.* Ph.D. Thesis, Sorbonne University.

Black, F., & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy, 81,* 637-654. https://doi.org/10.1086/260062

Childs, P. D., Riddiough, T. J., & Triantis, A. J. (1996). Mixed Uses and the Redevelopment Option. *Real Estate Economics, 24,* 317-339. https://doi.org/10.1111/1540-6229.00693

Cirjevskis, A. (2021). Value Maximizing Decisions in the Real Estate Market: Real Options Valuation Approach. *Journal of Risk and Financial Management, 14,* Article 278. https://doi.org/10.3390/jrfm14060278

Defreitas, C. L., & Kane, S. J. (2022). The Numerical Inversion of the Laplace Transform in a Multi-Precision Environment. *Applied Mathematics, 13,* 401-418. https://doi.org/10.4236/am.2022.135027

Lucius, D. I. (2000). Real Options in Real Estate Development. *Journal of Property Investment & Finance, 19,* 73-78. https://doi.org/10.1108/14635780110365370

Titman, S. (1985). Urban Land Prices under Uncertainty. *The American Economic Review, 75,* 505-514.

Wang, J. J., Shen, J., & Pretorius, F. (2023). Valuing Options to Renew at Future Market Value: The Case of Commercial Property Leases. *Financial Innovation, 9,* Article No. 70. https://doi.org/10.1186/s40854-023-00479-1

Williams, J. (1991). Real Estate Development as an Option. *The Journal of Real Estate Finance and Economics, 4,* 191-208. https://doi.org/10.1007/bf00173124

Zhong, J., & Hui, E. C. M. (2021). Real Option and Vertical Mixed-Use Development. *International Journal of Strategic Property Management, 25,* 382-395. https://doi.org/10.3846/ijspm.2021.15059