Theoretical Economics Letters, 2024, 14, 1193-1216
https://www.scirp.org/journal/tel
ISSN Online: 2162-2086
ISSN Print: 2162-2078

Firm-Specific Capital, Output-Demand Beliefs, and Involuntary Unemployment in a Stock Market Overlapping Generations Model: A Theoretical Investigation

Karl Farmer

Department of Economics, University of Graz, Graz, Austria
Email: karl.farmer@uni-graz.at

How to cite this paper: Farmer, K. (2024). Firm-Specific Capital, Output-Demand Beliefs, and Involuntary Unemployment in a Stock Market Overlapping Generations Model: A Theoretical Investigation. *Theoretical Economics Letters, 14,* 1193-1216.
https://doi.org/10.4236/tel.2024.143061

Received: February 24, 2024
Accepted: June 25, 2024
Published: June 28, 2024

Abstract

It is the aim of this paper to model involuntary unemployment in a perfectly competitive stock market overlapping generations model with firm-specific capital and affine equity-price expectations. In contrast to New-Keynesian macro-models, unemployment is not traced back to inflexible prices and wage rates, but to inflexible aggregate investment based on firm managers' "beliefs" about the expected future demand for production output. After setting up the stock market model, sufficient conditions for the existence and dynamic stability of a Golden Rule steady state with involuntary unemployment are presented and the comparative dynamics of this steady state is investigated. While an increase in managers' optimism decreases unemployment in the short and long run, a smaller savings rate does this only temporarily.

Keywords

Involuntary Unemployment, Managers' Beliefs, Stock Market OLG Model, Existence, Dynamic Stability and Comparative Dynamics of Steady States

1. Introduction

In a little-noticed but seminal article, Magill and Quinzii (2003) study an overlapping generations (OLG) economy in that installed capital is firm-specific, i.e. it cannot be transferred from one firm to another nor can be transformed back into consumption good as in the original Diamond's (1965) OLG model. "Since firms cannot be dismantled at each generational change without losing their value, their ownership is transmitted from generations to generations through a stock market" (Magill & Quinzii, 2003: p. 239). Within this technological and institutional setting, the question arises of how the financial price of a firm is determined. Magill and Quinzii (2003) show that the financial price of a firm is in general lower than the replacement value of its capital without creating profitable arbitrage or hampering the incentives to invest. The discount on the equity price (= difference between replacement value and equity price) turns out as a second dynamic variable besides the capital intensity with the consequence that for economies with under-accumulation, the equilibrium dynamics converges towards the long-run efficient Golden Rule, not towards the inefficient Diamond steady state.

Despite the invaluable contribution of Magill and Quinzii (2003), involuntary unemployment cannot be examined in their model approach because they assume in accordance with Diamond's (1965) full employment of labor force. While this assumption was appropriate with respect to the tight labor market in the 1960s and before the financial crisis of 2008, it remains problematic with respect to the state of the world economy of rather weak economic growth and a global unemployment rate of about 6% since then. Moreover, despite decades of theoretical debate in macroeconomics about Keynes' (1936) concept of an "under-employment equilibrium" or "involuntary unemployment" due to lacking aggregate demand (aggregate demand failure), there is still no agreement among macro-economists as to why aggregate demand in a perfectly operating market economy continues to be below full-employment output.

In contrast to mainstream New-Keynesian stochastic dynamic general equilibrium (DSGE) models in which involuntary unemployment is referred to sticky prices and wage rates, Morishima (1977) and more recently Magnani (2015) trace involuntary unemployment back to inflexible aggregate investment formed independently of aggregate savings[1]. However, Magnani (2015) in Solow's (1956) neo-classical growth model and Farmer and Kuplen (2018) in Diamond's (1965) OLG model simply assume that aggregate investment is somewhat determined by "animal spirits" of investors without any connections to firms' production technologies. To provide production-theoretic foundations for an independent aggregate investment function, it is necessary that savings and investment decisions are separated and undertaken by different, self-interested agents as argued next.

1.1. Separation of Savers and Investors and Firms' Present Value Maximization of Investment

Savers and investors are separated when households do not hold physical capital by themselves but shares in corporations whose managers decide on production of goods and on investment in productive capital as in Magill and Quinzii's (2003) stock market OLG model. This personal separation of wealth allocation

[1] Smith and Zoega (2009) also stress the close connection between investment and unemployment in Keynes (1936) in contrast to the New-Keynesian approaches which focus on matching problems on the labor markets.

and of production and investment decisions questions in general the alignment of the interests of owners and managers of firms. However, in this paper this problem is not addressed[2], but we follow Magill and Quinzii (2003) who assume that the maximization of the net present value of investment by firm managers is in the unanimous interest of firm's shareholders.

As the first-order conditions for present-value maximization in Magill and Quinzii's (2003) stock market OLG model show, the investment quantity of firms is, however, optimally indeterminate, and therefore unsuitable for the production theoretical foundation of an aggregated investment function. To obtain a determinate aggregate investment function compatible with indeterminate firm investment, Morishima's (1977) "flexible acceleration" principle comes to one's mind. This principle says that "the capitalist increases or decreases the rate of investment in capital… when the demand for capital services… is greater or less than the existing stock of capital" (Morishima, 1977: p. 117). However, attempts by the author to integrate Morishima's flexible acceleration principle into Magill and Quinzii's (2003) stock market model proved to be unsuccessful because the acceleration principle turned out to be inconsistent with the intertemporal equilibrium equations in the stock market OLG model. Moreover, there is another fundamental problem with the flexible accelerator principle and related attempts to integrate an investment function in an intertemporal general equilibrium model with economy-wide unemployment which is addressed next.

1.2. Economy-Wide Unemployment versus Increasing Marginal Adjustment Costs of Investment

The fundamental problem with neo-classical attempts to micro-found inflexible aggregate investment when wide-spread unemployment prevailsal ready Ebel (1978) pointed out: determinate investment quantities along an intertemporal equilibrium path can only be obtained when there are strictly increasing marginal adjustment costs of adapting investment towards the optimal capital stock. However, the assumption of strictly increasing marginal adjustment costs of investment is incompatible with wide-spread underemployment since rising marginal adjustment costs are best motivated by full employment of labor force. Ebel's problem insight applies also to all other attempts in the literature to derive determinate investment quantities from firms' intertemporal optimization models by assuming strictly increasing marginal adjustment costs of investment[3].

Fortunately, Magill and Quinzii (2003) do not assume strictly increasing marginal investment adjustment costs. However, with the consequence that net value-maximizing firm investment quantities are indeterminate or perfectly flexible. This leaves us with the following dilemma: while Magill and Qunizii's (2003) assumption with respect to firm investment choice is compatible with economy-wide unemployment it does not provide us with an investment function that

[2]Farmer (2024) addresses the alignment problem between managers' and shareholders' interests.

[3]For a recent critical survey of neo-classical theories of aggregate investment, see Girardi (2021).

seems to be necessary that aggregate demand falls short of aggregate supply. The optimally indeterminate or perfectly flexible firm investment quantities namely ensure that always aggregate demand equals aggregate supply or in Keynes' (1936) words the validity of Say's law which prevents aggregate demand failures. Thus, the research challenge consists of bringing forth a theory of determinate or inflexible investment determination without assuming full employment of labor resources. To meet this research challenge, we recall Farmer's (2013, 2020) "belief function" of shareholders or firm managers.

1.3. (Non-)degenerate Belief Function and Inflexible Aggregate Investment

Farmer (2020) forcefully argues that a belief function of shareholders or firm managers is to be seen as a primitive in addition to the neo-classical primitives of households' preferences, firms' production technologies and economy's resources. Firm managers' beliefs with respect to next period expected equity prices or aggregate investment demand along stochastic or deterministic intertemporal equilibrium paths turn out to be compatible with rational expectations or perfect foresight, respectively, and provide that equation which makes investment determinate or inflexible without implicitly assuming full employment of labor force. While being promising in this regard, it remains the problem encountered already above as of which sort of belief function turns out to be compatible with the rest of intertemporal equilibrium equations.

Farmer (2023a, 2023b) finds out that a degenerate belief function whereby each investor forms a quantity belief with respect to his/her investment demand turns out to be compatible with the intertemporal equilibrium equations in Magill and Quinzii's (2003) stock market OLG model. However, the degeneracy of the belief function remains problematic insofar as there is no connection to firms' production technologies and the expected future demand for firms' production output because of current investment in physical capital. This arouses our first research question of whether a non-degenerate belief function of firms' managers along these lines exists and is consistent with the intertemporal equilibrium system in Magill and Quinzii's (2003) stock market OLG model. Provided the answer to this question is positive, our second research question concerns the implications of such a non-degenerate belief function for the intertemporal equilibrium dynamics and the steady states in such a modified stock market OLG model in Magill and Quinzii (2003). Another modification of Magill and Quinzii's stock market OLG model is to include involuntary unemployment which is dealt with next.

1.4. Endogenous Unemployment and a Production-Function Based, Non-Degenerate Belief Function

To be able to model involuntary unemployment within Magill and Quinzii's (2003) stock market model, Farmer (2023a, 2023b) cancels their labor market-clearing condition and endogenizes the unemployment rate in line with

Magnani (2015)[4]. This procedure leaves the system of intertemporal-equilibrium equations under-determinate and provokes the need of a closing equation. This need does not arise in Magill and Quinzii (2003) since these authors assume labor market clearing or an unemployment rate identically to zero, and perfectly flexible firm investment resulting from the first-order conditions for firms' net value maximization of investment. While we maintain the perfect flexibility of the optimal investment in the following stock market OLG model with involuntary unemployment, aproduction-function based, non-degenerate belief function ensures that the system of intertemporal equilibrium equations is determinate.

Against this research background, the main objective of the present paper is to modify Magill and Quinzii's (2003) stock market OLG model such that the unemployment rate becomes endogenous and the beliefs of firm managers regarding the expected demand for firm production output govern firms' optimally indeterminate investment quantities.

Our first contribution to the literature is thus to show how the structure of the intertemporal equilibrium dynamics derived from households' and firms' optimization conditions, from government's budget constraint and the intertemporal market clearing conditions changes when firms' investment quantities are both optimally indeterminate and determined by firm managers' beliefs regarding the future expected demand for production output.

Our second contribution to the literature consists of proving the existence of a steady state and investigating the dynamic stability of the steady state of the intertemporal equilibrium dynamics in our novel stock-market OLG model with involuntary unemployment.

Our third contribution to the literature is to derive analytically the steady-state effects of main parameter changes on endogenous variables as the capital-output ratio, the equity price, and the unemployment rate. This is completed by a numerical calculation of the intertemporal equilibrium paths of these endogenous variables and the expected sales of production output from corporations' investments in productive capital in response to small parameter changes.

In contradistinction to Farmer (2023a, 2023b), there are two important novelties in the present paper: This is, firstly, the non-degenerate belief function of firms' managers which depicts how the expected demand for production output impacts via the production function current investment. Secondly, and novel, it is shown how the production-function-based belief function changes substantially the intertemporal equilibrium dynamics and the steady state. In contradistinction to Farmer (2024), physical capital is durable and firm-specific, and there is a discount on the equity price of firms in line with Magill and Qunizii (2003).

The structure of the paper is as follows. The next section presents the model set-up. This is followed by derivation of the intertemporal-equilibrium dynamics

[4]Tanaka (2020) also models involuntary unemployment in a three-period OLG model, however without capital accumulation and investment.

and demonstration of sufficient conditions for the existence and dynamic stability of steady states. We then investigate the comparative dynamics of the steady-state responses of the capital-output ratio, the equity-price discount, and the unemployment rate to the main parameter changes. A numerical specification of all model parameters is then used to calculate numerically the intertemporal-equilibrium paths of these dynamic variables in response to small parameter changes. The main conclusions are drawn in the final section of the paper.

2. The Stock Market OLG Model with Involuntary Unemployment

2.1. Informal Model Description

Before embarking on a description of the model economy in technical terms we shortly brush over the assumptions regarding agents' behavior and the market mechanisms coordinating their actions. The economy operates over an infinite number of periods and each period is 25 - 30 calendar years long.

Growth of gross domestic product is governed by exogenous growth of labor productivity and labor force. The economy is composed of infinitely lived firms, an infinitely lived government and two-periods lived households. In each period, a young generation enters the economy which overlaps for one period with the generation that entered the economy one period before (= old generation). Each generation is represented by a continuity of identical households where each consists of one agent. Agents are either employed or unemployed. Each employed agent sells one unit of labor service inelastically to firms in exchange for a real wage rate which is determined by perfectly competitive firms demanding labor services. Unemployed households are supported by unemployment benefits from the government. To balance government's budget, the government collects flat wage taxes on employed households' wage incomes. The unemployed agents do not pay any taxes. Both the employed and the unemployed agent maximize an intertemporal utility function comprising active and retirement consumption subject to budget constraints for the active and retirement period. Due to the former constraint, the active period consumption expenditures plus the expenses for buying corporate bonds and firms' equity shares are to be covered by the net wage rate. Due to the latter constraint retirement consumption expenditures are financed by the revenues from asset sales and the returns on holding assets one period long.

All firms are endowed with identical Cobb-Douglas production functions whereby the services of employed labor and physical capital invested in the previous period generate the production output for households' consumption and firms' investment in physical capital. Physical capital is durable, depreciates at the finite rate each period, and is non-shiftable and firm-specific. It has no resale value. Firms are owned by the young equity holders and are managed such that the net present value of investment is maximized. Firm investment is financed by the sale of corporation bonds but not through equity sales.

There are four markets operating under perfect competition in each period: the market for labor services, the market for corporation bonds, the equity market, and the market for production output. Except for the labor market, all markets are clear in each period. On the equity market, older agents sell their equity shares to the younger households who become firms' owners. On the bond market, firms sell bonds that younger agents buy to invest their savings. On the market for production output, firms supply their output and households demand the output for consumption and firms for investment.

The assumption of perfect competition on all markets does clearly not accord well with nowadays empirical reality. It is made to examine a major claim of the history of economic thought brought forth by Keynes (1936): the existence of an "underemployment equilibrium" or involuntary unemployment in a perfectly competitive market economy.

2.2. Model Description in Technical Terms

In each period $t = 0,1,2,\cdots$, a new generation, called generation *t*, enters the economy. A continuum of $L_t > 0$ units of identical agents comprises generation *t*.

The exogenous growth rate of the population is $g^L > -1$ which implies the following dynamics of population L_t: $L_{t+1} = G^L L_t$, $G^L \equiv 1 + g^L$, $L_0 = \underline{L} > 0$. The exogenous growth rate of labor productivity is $g^a > -1$ which implies the following dynamics of labor productivity a_t: $a_{t+1} = G^a a_t$, $G^a \equiv 1 + g^a$, $a_0 = \underline{a} > 0$.

In exchange for the labor supply each employed household of generation *t* obtains the real wage rate w_t, which denotes the units of the produced good per unit of labor. Thus, the labor supply in period *t* is not equal to L_t, but only to $(1-u_t)L_t$, where $0 \le u_t < 1$ denotes the unemployment rate. The number of unemployed households (= people) is thus $u_t L_t$. Since the unemployed are unable to obtain any labor income from the market, they are supported by the government through the unemployment benefit ς_t (per household) in each period.

To finance the unemployment benefit, the government collects taxes on wages, quoted as a fixed proportion of wage income, $\tau_t w_t h_t, 0 < \tau_t < 1$. The unemployed do not pay any taxes. Young, employed agents, denoted by superscript *E*, split the net wage income $(1-\tau_t)w_t$ each period between current consumption $c_t^{1,E}$ and savings s_t^E. Savings of the employed are invested in the shares of firms, where a share $\theta_t^{j,E}$ of firm $j = 1,\cdots,J$ in period *t* is bought in the stock market at price Q_t^j by the younger households from the older households. Moreover, the younger households also invest their savings in bonds emitted by firms $j(=1,\cdots,J)$, denoted by $b_{t+1}^{j,E}$, with a rate of return i_{t+1}.

In old age, the employed household sells the shares at the price $Q_{t+1}^{j,E}$ to the then younger household in period *t* + 1. The revenues from asset sales and the returns from holding assets one period long, $(1+i_{t+1})\sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J} \theta_t^{j,E}\left(D_{t+1}^j + Q_{t+1}^j\right)$,

are used to finance retirement consumption $c_{t+1}^{2,E}$, where D_t^j denotes the dividend paid by firm *j* in period *t*. In old age, the previously young employed households consume their gross return on assets:
$c_{t+1}^{2,E} = (1+i_{t+1})\sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J}\theta_t^{j,E}\left(D_{t+1}^j + Q_{t+1}^j\right)$. This is also true for the unemployed households who finance their retirement consumption through the returns on equity purchases and firm bonds in youth financed by unemployment benefits: $c_{t+1}^{2,U} = (1+i_{t+1})\sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J}\theta_t^{j,U}\left(D_{t+1}^j + Q_{t+1}^j\right)$, where $c_{t+1}^{2,U}$, represents consumption of the unemployed in old age. To keep it all as simple as possible, we assume that the revenues from equity sales and dividends are not taxed.

2.3. Households' Optimization Problems and Demand Functions

The typical younger, employed household maximizes the following intertemporal utility function subject to the budget constraints of the active period 1) and of the retirement period 2):

$$\text{Max} \rightarrow \varepsilon \ln c_t^{1,E} + \beta \ln c_{t+1}^{2,E}$$

subject to:

1) $c_t^{1,E} + \sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J} Q_t^j \theta_t^{j,E} = w_t(1-\tau_t)$,

2) $c_{t+1}^{2,E} = (1+i_{t+1})\sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J}\theta_t^{j,U}\left(D_{t+1}^j + Q_{t+1}^j\right)$.

Here, $0<\varepsilon\le 1$ depicts the utility elasticity of employed household's consumption in youth and $0<\beta<1$ denotes the subjective future utility discount factor. The intertemporally additive utility function involves the natural logarithm of employed household's consumption in youth weighted by ε, and the natural logarithm of employed household's consumption in old age weighted by $0<\beta<1$.

To obtain the first-order conditions for a maximum of the intertemporal utility function subject to the constraints 1) and 2), we form the following Lagrangian:

$$L_t^E \equiv \varepsilon \ln c_t^{1,E} + \beta \ln c_{t+1}^{2,E} - \lambda_t^E\left(c_t^{1,E} + \sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J} Q_t^j\theta_t^{j,E} - w_t(1-\tau_t)\right)$$
$$-\lambda_{t+1}^E\left(c_{t+1}^{2,E} - (1+i_{t+1})\sum_{j=1}^{J} b_{t+1}^{j,E} - \sum_{j=1}^{J}\theta_t^{j,E}\left(D_{t+1}^j + Q_{t+1}^j\right)\right).$$

Differentiating the Lagrangian with respect to $c_t^{1,E}, c_{t+1}^{2,E}, b_{t+1}^{j,E}, \theta_t^{j,E},\ j=1,\cdots,J$ yields the following first-order conditions for an intertemporal utility maximum:

$$c_t^{1,E} = \frac{\varepsilon}{\varepsilon+\beta}(1-\tau_t)w_t, \tag{1}$$

$$\frac{D_{t+1}^j + Q_{t+1}^j}{Q_t^j} = 1 + i_{t+1},\ j=1,\cdots,J, \tag{2}$$

$$c_{t+1}^{2,E} = \frac{\beta}{\varepsilon+\beta}(1+i_{t+1})(1-\tau_t)w_t, \tag{3}$$

$$s_t^E = \frac{\beta}{\varepsilon+\beta}w_t(1-\tau_t),\ s_t^E \equiv \sum_{j=1}^{J} b_{t+1}^{j,E} + \sum_{j=1}^{J}\theta_t^{j,E}Q_t^j. \tag{4}$$

The typical younger, unemployed household maximizes the following intertemporal utility function subject to the budget constraints of the active period 1) and the retirement period 2):

$$\text{Max} \rightarrow \varepsilon \ln c_t^{1,U} + \beta \ln_t c_{t+1}^{2,U}$$

subject to:

1) $c_t^{1,U} + \sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J} Q_t^j \theta_t^{j,U} = \varsigma_t,$

2) $c_{t+1}^{2,U} = (1+i_{t+1})\sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J}\theta_t^{j,U}\left(D_{t+1}^j + Q_{t+1}^j\right).$

Again, $0<\varepsilon\le 1$ denotes the utility elasticity of consumption in unemployed youth, while $0<\beta<1$ depicts the subjective future utility discount factor and ς_t denotes the unemployment benefit percapita unemployed.

Performing similar intermediate steps as above with respect to the younger, employed household yields the following first-order conditions for a constrained intertemporal utility maximum:

$$c_t^{1,U} = \frac{\varepsilon}{\varepsilon+\beta}\varsigma_t, \tag{5}$$

$$\frac{D_{t+1}^j + Q_{t+1}^j}{Q_t^j} = 1+i_{t+1},\ j=1,\cdots,J, \tag{6}$$

$$c_{t+1}^{2,U} = \frac{\beta}{\varepsilon+\beta}(1+i_{t+1})\varsigma_t, \tag{7}$$

$$s_t^U = \frac{\beta}{\varepsilon+\beta}\varsigma_t,\ s_t^U \equiv \sum_{j=1}^{J} b_{t+1}^{j,U} + \sum_{j=1}^{J}\theta_t^{j,U}Q_t^j. \tag{8}$$

2.4. Firms' Optimization Problem and Its First-Order Conditions

All firms are endowed with an identical (linear-homogeneous) Cobb-Douglas production function which reads as follows:

$$Y_t^j = M\left(a_t N_t^j\right)^{1-\alpha}\left(K_t^j\right)^{\alpha},\ j=1,\cdots,J,\ 0<\alpha<1,\ M>0. \tag{9}$$

here, Y_t^j denotes production output of firm $j=1,\cdots,J$, $M>0$ stands for total factor productivity (equal for all firms), N_t^j represents the number of employed laborers with firm *j*, with productivity of a_t each, while K_t^j denotes the input of capital services of firm *j*, all in period *t*, and $1-\alpha$ (α) depicts the production elasticity (= production share) of labor (capital) services, also equal for all firms. In line with the seminal paper of Magill and Quinzii (2003), we assume that (physical) capital is durable, depreciates at the rate $0<\delta<1$, and needs to be installed one period before it is used. Thus capital K_t^j used by firm *j* is the capital stock that has been carried over from the period before, i.e. period $t-1$. Moreover, we assume "that capital once installed in a firm cannot be 'unbolted' and transformed back into the homogeneous current output or transferred to another firm, without incurring significant adjustment costs—which for simplicity we take to be infinite" (Magill & Quinzii, 2003: p. 242). Consequently, such firm-specific capital has limited value in a resale market. In the extreme, it is completely firm-specific, so that no part of it has a positive value in the second-hand market.

"In such an economy capital accumulation will only take place if the market structure permits firms to be infinitely lived. Invested capital has… value only if the firm retains its identity as income generating unit in the economy. The natural market structure which permits short-lived agents to transfer ownership of long-lived firms from one generation to the next is an equity market for ownership shares of firms" (Magill & Quinzii, 2003: p. 243). Consistent with the firm specificity of capital is that each firm is a corporation with an infinite life where ownership shares are transmitted from one generation to the next through the stock market. As already introduced above, Q_t^j denotes the equity price of firm *j* at date *t*.

Firms are owned by the equity holders[5] and are managed to maximize the payoff of their current owners. These are the younger households who buy the shares of firm *j* endowed with a capital of $(1-\delta)K_t^j$, from the older households for the price Q_t^j. Firm managers decide on the investment $I_t^j \geq 0$ to be made. Magill and Quinzii (2003: pp. 244-245) show that an investment quantity larger than zero is chosen such that the net present value of the investment is maximized:

$$\begin{aligned} &\underbrace{\max}_{\{I_t^j, N_{t+1}^j\}} \left\{-I_t^j + \frac{1}{1+i_{t+1}}\left[\left(K_{t+1}^j\right)^\alpha \left(a_{t+1}N_{t+1}^j\right)^{1-\alpha} - w_{t+1}N_{t+1}^j + Q_{t+1}^j\left((1-\delta)K_{t+1}^j\right)\right]\right\} \\ \Leftrightarrow &\underbrace{\max}_{\{I_t^j, N_{t+1}^j\}} \left\{-I_t^j + \frac{1}{1+i_{t+1}}\left[\left((1-\delta)K_t^j + I_t^j\right)^\alpha \left(a_{t+1}N_{t+1}^j\right)^{1-\alpha} - w_{t+1}N_{t+1}^j \right.\right. \\ &\left.\left. + (1-\delta)^2 K_t^j + (1-\delta)I_t^j - V_{t+1}^j\right]\right\}. \end{aligned} \tag{10}$$

Here, the equivalence between the first and the second line in (10) comes from Magill and Quinzii's (2003: p. 244) insight that in an intertemporal equilibrium shareholders expect an affine (linear) relationship between the expected equity price of non-depreciated capital in period $t+1$, $Q_{t+1}^j\left((1-\delta)K_{t+1}^j\right)$, and non-depreciated capital stock at that time, i.e.:

$$Q_{t+1}^j\left((1-\delta)K_{t+1}^j\right) = (1-\delta)^2 K_t^j + (1-\delta)I_t^j - V_{t+1}^j,\ j = 1,\cdots,J,\ V_{t+1}^j \geq 0, \tag{11}$$

[5]As the optimization calculi of younger households as shareholders suggest, there is no heterogeneity among shareholders. In line with the basic structure of stylized overlapping generations models, there is only heterogeneity between younger and older shareholders.

where $V_{t+1}^{j},\ j=1,\cdots,J$ denotes the discount on the equity price of firm *j* at time $t+1$ due to the non-shift ability of firm *j*'s capital stock.

Maximization of the net present value in the second line of Equation (10) implies the following first-order conditions:

$$\alpha M\left[(1-\delta)K_t^j+I_t^j\right]^{\alpha-1}\left(a_{t+1}N_{t+1}^j\right)^{1-\alpha}=\delta+i_{t+1}, \tag{12}$$

$$M\left[(1-\delta)K_t^j+I_t^j\right]^{\alpha}\left(a_{t+1}N_{t+1}^j\right)^{-\alpha}a_{t+1}=w_{t+1}. \tag{13}$$

Since all firms have the same production function (see Equation (9)) and the capital depreciation rate is the same with all firms, the optimal capital labor ratio will be the same for all firms: $\frac{K_t^j}{a_tN_t^j}=\frac{K_t^{j'}}{a_tN_t^{j'}}=\frac{K_t}{a_tN_t},\quad j\neq j'=1,\cdots,J$. Moreover, since the number of employed workers is $N_t\equiv\sum_{j=1}^{J}N_t^j=L_t\left(1-u_t\right)$, we can rewrite the profit maximization Conditions (12) and (13) as follows:

$$\alpha M\left[K_{t+1}\big/\left(a_{t+1}L_{t+1}\left(1-u_{t+1}\right)\right)\right]^{\alpha-1}=\delta+i_{t+1}, \tag{14}$$

$$(1-\alpha)M\left[K_{t+1}\big/\left(a_{t+1}L_{t+1}\left(1-u_{t+1}\right)\right)\right]^{\alpha}a_{t+1}=w_{t+1}. \tag{15}$$

Finally, the GDP function can be rewritten as follows:

$$Y_t\equiv\sum_{j=1}^{J}Y_t^j=M\left(a_tL_t\left(1-u_t\right)\right)^{1-\alpha}\left(K_t\right)^{\alpha}. \tag{16}$$

2.5. Governments' Budget Constraint

As in Diamond (1965), the government does not optimize, but is subject to the following budget constraint period by period:

$$L_tu_t\varsigma_t=\tau_t\left(1-u_t\right)w_tL_t, \tag{17}$$

where, for the sake of simplicity, it is assumed that the government does not have any other expenditures than the unemployment benefits and that there is no government debt.

2.6. Indeterminate Optimal Investment and the Non-Degenerate Belief Function of Firm Managers

As Magnani (2015: pp. 13-14) rightly states, aggregate investment in Solow's (1956) neo-classical growth model is not micro-, but macro-founded since it is determined by aggregate savings. The same holds true in Diamond's (1965) OLG model of neo-classical growth where perfectly flexible aggregate investment is also determined by aggregate savings of households. As already mentioned in the Introduction above, and as the first-order conditions for optimal investment (14) and (15) show, optimal investment is indeterminate and thus also perfectly flexible in the stock market model of Magill and Quinzii (2003). This is most easily seen if we rewrite Equations (14) and (15) as follows:

$$\alpha M\left(\frac{K_{t+1}}{a_{t+1}L_{t+1}}\right)^{\alpha-1}\left(1-u_{t+1}\right)^{1-\alpha}=\delta+i_{t+1}, \tag{18}$$

$$\left(1-\alpha\right)M\left(\frac{K_{t+1}}{a_{t+1}L_{t+1}}\right)^{\alpha}\left(1-u_{t+1}\right)^{-\alpha}=w_{t+1}. \tag{19}$$

Equations (18) and (19) do not allow for determination of the optimal firm investment quantity.

Morishima (1977), and more recently Magnani (2015), both deviate from neo-classical growth models in maintaining that an independent investment function is needed to determine the level of investment in intertemporal equilibrium models of involuntary unemployment. The big question, however, is where does this function come from in a general equilibrium model with an active stock market and explicit firm maximization calculus to determine investment quantities?

As a first step to provide an answer to this question we recall the no-arbitrage condition between shares and corporation bonds (2) $\left(D_{t+1}^{j}+Q_{t+1}^{j}\right)/Q_{t}^{j}=1+i_{t+1}$, $j=1,\cdots,J$, with $D_{t+1}^{j}=M\left(K_{t+1}^{j}\right)^{\alpha}\left(a_{t+1}N_{t+1}^{j}\right)^{1-\alpha}-w_{t+1}N_{t+1}^{j}-\left(1+i_{t+1}\right)I_{t}^{j}$, $j=1,\cdots,J$. Respecting the first-order conditions for net present value Maximization (18) and (19), and assuming that affine equity price expectations are rational, i.e. Equation (11) holds, then we can show, following Magill and Quinzii (2003: p. 247), that:

$$\frac{D_{t+1}^{j}+Q_{t+1}^{j}}{Q_{t}^{j}}=\frac{K_{t+1}^{j}\left(\delta+i_{t+1}\right)-\left(1+i_{t+1}\right)I_{t}^{j}+\left(1-\delta\right)K_{t+1}^{j}-V_{t+1}^{j}}{\left(1-\delta\right)K_{t}^{j}-V_{t}^{j}}$$, if and only if

$$V_{t+1}^{j}=\left(1+i_{t+1}\right)V_{t}^{j},\ j=1,\cdots,J,\ \forall t\geq 0. \tag{20}$$

As a second step in providing an answer to the seminal question above, we recall Farmer's (2013, 2020) concept of "belief" function which he sees as synonymous with the neo-classical fundamentals like consumer preferences, corporation technologies and the resource endowment of an economy. Farmer and collaborators suggest different expected price or income variables about which investors or firm managers form beliefs (see for an overview Farmer, 2023a).

Here, we assume that corporation managers form beliefs regarding future expected demand for production output, $Y_{t+1}^{ex,j}$, resulting from current period investment I_{t}^{j} and future input of labor services N_{t+1}^{j}. Hereby, the managers consider corporation's production technology $M\left(I_{t}^{j}+\left(1-\delta\right)K_{t}^{j}\right)^{\alpha}\left(N_{t+1}^{j}\right)^{1-\alpha}$ to find out that corporation's investment is determined as follows:

$$I_{t}^{j}=M^{-1/\alpha}\left(Y_{t+1}^{j,ex}\right)^{1/\alpha}\left(N_{t+1}^{j}\right)^{-(1-\alpha)/\alpha}-\left(1-\delta\right)K_{t}^{j},\ j=1,\cdots,J,\ \forall t. \tag{21}$$

Equation (21) does not appear in Magill and Quinzii's (2003) stock market OLG model, since they assume full employment of the labor force, which is equivalent to $u_{t}=0,\forall t$ in our model. For $u_{t}>0$ and u_{t} being endogenous, Equation (21) features as the intertemporal equilibrium condition which makes the whole set of intertemporal equilibrium equations determinate. In contrast to

Morishima (1977: pp. 117-119) and Magnani (2015: p.14), inflexible aggregate investment is not simply assumed to be macro-founded but turns out to be consistent with an indeterminate, market-value maximizing investment quantity of firm *j*. In this restricted sense, we are entitled to claim that inflexible investment results from a production-technology-based firm-managers' belief regarding future expected output demand in our modified stock-market OLG model of involuntary unemployment.

2.7. Market Clearing Conditions

In addition to the restrictions imposed by household and firm optimizations, by the government budget constraint and the belief function, markets for labor, firm bonds, and equity, ought to clear in all periods (the market for the output of production is cleared by means of Walras' law[6]).

$$L_t(1-u_t)=\sum_{j=1}^{J}N_t^j=N_t,\forall t \quad (22)$$

$$L(1-u_t)b_{t+1}^E+Lu_tb_{t+1}^U=\sum_{j=1}^{J}b_{t+1}^j,\ \forall t \quad (23)$$

The demand of the younger employed and the unemployed households for firm bonds (left-hand side of Equation (23)) balance with their supply (right-hand side of Equation (23)). Firms finance their investments by the sales of bonds:

$$\sum_{j=1}^{J}I_t^j=\sum_{j=1}^{J}b_{t+1}^j,\ \forall t. \quad (24)$$

The shares of employed and unemployed younger households sum to unity:

$$L(1-u_t)\theta_t^{j,E}+Lu_t\theta_t^{j,U}=1,\ j=1,\cdots,J,\ \forall t. \quad (25)$$

The sales of equity shares by employed and unemployed older households are equal to the share purchases of employed and unemployed younger households:

$$L_{t-1}(1-u_{t-1})\theta_{t-1}^{j,E}=L_t(1-u_t)\theta_t^{j,E},\ j=1,\cdots,J,\ \forall t, \quad (26)$$

$$L_{t-1}(1-u_{t-1})\theta_{t-1}^{j,U}=L_t(1-u_t)\theta_t^{j,U},\ j=1,\cdots,J,\ \forall t. \quad (27)$$

Using the definition of savings for younger employed households in (4) and younger unemployed households in (8), together with the bond market clearing Condition (23), the investment financing Constraint (24) and Conditions (25)-(27) lead us to the following aggregate savings/investment equality:

$$L_t(1-u_t)s_t^E+L_tu_ts_t^U=\sum_{j=1}^{J}I_t^j+\sum_{j=1}^{J}Q_t^j. \quad (28)$$

On respecting Equation (21) and firm-specific accumulation equation:

$$K_{t+1}^j=(1-\delta)K_t^j+I_t^j,\ j=1,\cdots,J, \quad (29)$$

the following intertemporal equilibrium equation results:

[6]The proof of Walras' law can be obtained upon request from the author.

$$I_t^j = M^{-1/\alpha}\left(Y_{t+1}^{j,ex}\right)^{1/\alpha}\left[a_{t+1}L_{t+1}\left(1-u_{t+1}\right)\right]^{-(1-\alpha)/\alpha} - \left(1-\delta\right)K_t^j,\ j=1,\cdots,J,\ \forall t. \quad (30)$$

Having presented households' and firms' first-order conditions for intertemporal utility and net value maxima of firms, government's budget constraint, managers' belief functions and market clearing conditions, we consider in the next section all that along an intertemporal equilibrium path.

3. Intertemporal Equilibrium

To start with, assume in line with Magill and Quinzii (2003: p. 249) a balanced-growth intertemporal equilibrium in which firms always exhibit the same relative sizes and stock market values. Then, consider initial conditions $\left(K_0^j, V_0^j\right) = \nu_j\left(K_0, V_0\right)$ with $\nu_j > 0$ and $\sum_{j=1}^{J}\nu_j = 1$. If, for the sequence of (real) wage and interest rates $\left(w_t, i_{t+1}\right)_{t\geq 0}$, aggregate equity-price discounts $V_t \geq 0$ and employment-investment decisions $\left(N_t, I_t\right)_{t\geq 0}$ satisfy the Equations (11), (12), (13), (20), and (29)-(30), then $\left(V_t^j, N_t^j, I_t^j\right) = \nu_j\left(V_t, N_t, I_t\right)$ also satisfy Equations (11), (12), (13), (20), and (29)-(30), such that for each firm $\left(N_t^j, I_t^j\right)$ is market-value maximizing, its market value is larger than zero, and the return on equity equals i_{t+1}. Hence, the optimal choices of individual firms can be depicted by the market-value maximizing choice of aggregate employment, investment, and equity price discount.

Acknowledging the linear-homogeneity of firm production functions (16) and the underemployment equilibrium Condition (22), we can switch to aggregate capital per efficient labor $k_t \equiv K_t/\left(a_t L_t\right)$ quantities, and rewrite the first-order Conditions (18) and (19) as follows:

$$\alpha M\left(k_{t+1}\right)^{\alpha-1}\left(1-u_{t+1}\right)^{1-\alpha} = \delta + i_{t+1}, \quad (31)$$

$$\left(1-\alpha\right)M a_{t+1}\left(k_{t+1}\right)^{\alpha}\left(1-u_{t+1}\right)^{-\alpha} = w_{t+1}. \quad (32)$$

As a next step, the aggregate version of Equation (30) is solved for I_t and inserted into the savings/investment Equality (28). If affine equity price Expectations (11) also prevailed in period t, the savings/investment equality can be rewritten as follows:

$$\begin{aligned} & L_t\left(1-u_t\right)s_t^E + L_t u_t s_t^U \\ & = M^{-1/\alpha}\left(Y_{t+1}^{j,ex}\right)^{1/\alpha}\left[a_{t+1}L_{t+1}\left(1-u_{t+1}\right)\right]^{(\alpha-1)/\alpha} - \left(1-\delta\right)K_t + \left(1-\delta\right)K_t - V_t \\ & = M^{-1/\alpha}\left(Y_{t+1}^{j,ex}\right)^{1/\alpha}\left[a_{t+1}L_{t+1}\left(1-u_{t+1}\right)\right]^{(\alpha-1)/\alpha} - V_t. \end{aligned} \quad (33)$$

Next, insert into Equation (33), the optimal savings functions (4) and (8) and the government balanced budget Condition (17):

$$\begin{aligned} & L_t\left(1-u_t\right)\sigma w_t\left(1-\tau_t\right) + L_t u_t \sigma \varsigma_t = L_t\left(1-u_t\right)\sigma w_t\left(1-\tau_t\right) + L_t\left(1-u_t\right)\sigma w_t \tau_t \\ & = L_t\left(1-u_t\right)\sigma w_t = M^{-1/\alpha}\left(Y_{t+1}^{j,ex}\right)^{1/\alpha}\left[a_{t+1}L_{t+1}\left(1-u_{t+1}\right)\right]^{(\alpha-1)/\alpha} - V_t,\ \sigma \equiv \beta/\left(\varepsilon+\beta\right). \end{aligned} \quad (34)$$

Inserting into Equation (34), the first-order Condition (32) for t, and dividing the resulting equation on both sides by $a_t L_t$, we obtain:

$$\left(L_t\left(1-u_t\right)\sigma a_t\left(1-\alpha\right)M\left(k_t\right)^{\alpha}\left(1-u_t\right)^{-\alpha}=M^{-1/\alpha}\left[a_{t+1}L_{t+1}\left(1-u_{t+1}\right)\right]^{(\alpha-1)/\alpha}\left(Y_{t+1}^{ex}\right)^{1/\alpha}-V_t\right)\frac{1}{a_tL_t}$$

$$\Leftrightarrow\left(1-\alpha\right)\sigma M\left(k_t\right)^{\alpha}\left(1-u_t\right)^{1-\alpha}=M^{-1/\alpha}\left[\frac{a_{t+1}L_{t+1}}{a_tL_t}\left(1-u_{t+1}\right)\right]^{(\alpha-1)/\alpha}\left(\frac{Y_{t+1}^{ex}}{a_{t+1}L_{t+1}}\right)^{1/\alpha}-\frac{V_t}{a_tL_t}$$
$$=M^{-1/\alpha}\left[G^n\left(1-u_{t+1}\right)\right]^{(\alpha-1)/\alpha}\left(y_{t+1}^{ex}\right)^{1/\alpha}-v_t, \tag{35}$$

$$G^n\equiv\frac{a_{t+1}L_{t+1}}{a_tL_t}\equiv G^aG^L,\ y_{t+1}^{ex}\equiv\frac{Y_{t+1}^{ex}}{a_{t+1}L_{t+1}},\ v_t\equiv\frac{V_t}{a_tL_t}.$$

By using the capital-output ratio $\kappa_t\equiv K_t/Y_t=K_t\Big/\left[Ma_tL_t\left(1-u_t\right)^{1-\alpha}\left(k_t\right)^{\alpha}\right]=\left(k_t\right)^{1-\alpha}\Big/\left[M\left(1-u_t\right)^{1-\alpha}\right]$ or $k_t=M^{1/(1-\alpha)}\left(\kappa_t\right)^{1/(1-\alpha)}\left(1-u_t\right)$, Equation (35) can be transformed into Equation (36):

$$\left(1-\alpha\right)\sigma M^{1/(1-\alpha)}\left(\kappa_t\right)^{\alpha/(1-\alpha)}\omega_t=M^{-1/\alpha}\left[G^n\left(\omega_{t+1}\right)\right]^{(\alpha-1)/\alpha}\left(y_{t+1}^{ex}\right)^{1/\alpha}-v_t,\ \omega_t\equiv1-u_t,\ \forall t. \tag{36}$$

Equation (36) represents the first difference equation of the intertemporal equilibrium in our stock-market OLG model of involuntary unemployment provided we know how firms' managers form their expectations with respect to future demand for production output.

Like Freitas and Serrano (2015) for the desired capital-output ratio and Plotnikov (2018) for the expected real permanent income, we assume a partial adjustment of the belief of firm j managers about the expected output sales per efficiency capita in period $t+1$ towards its long-run expected level $\hat{y}^{j,ex},\ j=1,\cdots,J$:

$$y_{t+1}^{j,ex}=\left(1-\varphi\right)y_t^j+\varphi\hat{y}^{j,ex},\ 0<\varphi\le1,\ j=1,\cdots,J,\ y_t^j=M^{1/(1-\alpha)}\left(\kappa_t^j\right)^{\alpha/(1-\alpha)}\omega_t,\forall t. \tag{37}$$

Equation (37) represents the second dynamic equation of intertemporal equilibrium.

The third dynamic equation results from the aggregate versions of Equations (29)-(30) and dividing the resulting equation on both sides by a_tL_t:

$$M^{1/(1-\alpha)}\left(\kappa_{t+1}\right)^{1/(1-\alpha)}\omega_{t+1}=M^{-1/\alpha}\left(y_{t+1}^{ex}\right)^{1/\alpha}\left(\omega_{t+1}\right)^{(\alpha-1)/\alpha}. \tag{38}$$

The fourth equilibrium-dynamics equation pops up when Equation (20) is divided on both sides by a_tL_t, and when the definition of v_t and the first-order Condition (31) are used:

$$G^nv_{t+1}=\left[1-\delta+\alpha M\left(k_t\right)^{\alpha-1}\left(1-u_t\right)^{1-\alpha}\right]v_t=\left(1-\delta+\alpha/\kappa_t\right)v_t,$$
$$\text{with } 0\le v_{t+1}\le\left(1-\delta\right)^2M^{1/(1-\alpha)}\left(\kappa_t\right)^{1/(1-\alpha)}\omega_t. \tag{39}$$

The four-dimensional dynamic system (36)-(39) becomes determinate if we assume initial values for the capital-output ratio and the equity-price discount like Magill and Quinzii (2003): $\kappa_0=\underline{\kappa}>0$ and $v_0=\underline{v}>0$, whereby $\underline{\kappa}$ and $\underline{v}$ are exogenously given. The initial value for ω_0 comes from Equation (38) for $t=-1$, and $y_0^{ex}=y_0=M^{1/(1-\alpha)}\underline{\kappa}^{\alpha/(1-\alpha)}\omega_0$.

4. Existence of Steady States and Dynamic Stability of a Positive-Discount Steady State

The steady states of the equilibrium dynamics is depicted by the different Equations (36)-(39) are defined as $\lim_{t\to\infty}\kappa_t=\kappa$, $\lim_{t\to\infty}v_t=v$, $\lim_{t\to\infty}y_t^{ex}=\hat{y}^{ex}$, and $\lim_{t\to\infty}\omega_t=\omega$. Due to the relative simplicity of the dynamic system (36)-(39), explicit steady-state solutions are possible. As in Magill and Quinzii (2003), there are two different steady-state solutions of the equilibrium dynamics (36)-(39): 1) The zero-discount, or so-called Diamond-solution $\lim_{t\to\infty}\kappa_t=\kappa_D$, $\lim_{t\to\infty}v_t=v=0$, $\lim_{t\to\infty}y_t^{ex}=\hat{y}^{ex}$, and $\lim_{t\to\infty}\omega_t=\omega_D$, and 2), the positive-discount steady state $\lim_{t\to\infty}\kappa_t=\kappa$, $\lim_{t\to\infty}v_t=v>0$, $\lim_{t\to\infty}y_t^{ex}=\hat{y}^{ex}$, and $\lim_{t\to\infty}\omega_t=\omega$. Here, we focus on solution (2). This leads us to the following Proposition 1:

Proposition 1. Suppose that $G^n/\left[G^n-(1-\delta)\right]>(1-\alpha)\sigma/\alpha$ and $\hat{y}^{ex}<\dfrac{M^{1/(1-\alpha)}}{\left[G^n-(1-\delta)\right]^{\alpha/(1-\alpha)}}$. Then, the following steady-state solution for $(\kappa,v)>0$, $y^{ex}=\hat{y}^{ex}$, and $0<u<1$ exists:

$$\kappa=\frac{\alpha}{G^n-(1-\delta)}, \tag{40}$$

$$v=\alpha\hat{y}^{ex}\left\{\frac{G^n}{G^n-(1-\delta)}-\frac{(1-\alpha)\sigma}{\alpha}\right\}, \tag{41}$$

$$y^{ex}=\hat{y}^{ex}, \tag{42}$$

$$u=1-\frac{\left[G^n-(1-\delta)\right]^{\alpha/(1-\alpha)}\hat{y}^{ex}}{M^{1/(1-\alpha)}}. \tag{43}$$

Remark: Rearranging the steady-state solution (40) brings forth: $1-\delta+\alpha/\kappa=1+i=G^n$. This means that the positive-discount steady state features the so-called Golden-Rule, intertemporal-consumption allocation which is long-run efficient.

The next step is to investigate the local dynamic stability of the unique steady-state solution (40)-(43). To make the algebraic analysis a little clearer, we assume in the first step $\varphi=1$. The intertemporal equilibrium Equations (36) and (38)-(39) are then totally differentiated with respect to $\kappa_{t+1},\omega_{t+1},v_{t+1},\kappa_t,\omega_t,v_t$. Then, the Jacobian matrix $J(\kappa,\omega,v)$ of all partial differentials with respect to κ_t, ω_t and v_t is formed as follows:

$$J(\kappa,\omega,v)\equiv\begin{bmatrix}\frac{\partial\kappa_{t+1}}{\partial\kappa_t}(\kappa,\omega,v) & \frac{\partial\kappa_{t+1}}{\partial\omega_t}(\kappa,\omega,v) & \frac{\partial\kappa_{t+1}}{\partial v_t}(\kappa,\omega,v)\\ \frac{\partial\omega_{t+1}}{\partial\kappa_t}(\kappa,\omega,v) & \frac{\partial\omega_{t+1}}{\partial\omega_t}(\kappa,\omega,v) & \frac{\partial\omega_{t+1}}{\partial v_t}(\kappa,\omega,v)\\ \frac{\partial v_{t+1}}{\partial\kappa_t}(\kappa,\omega,v) & \frac{\partial v_{t+1}}{\partial\omega_t}(\kappa,\omega,v) & \frac{\partial v_{t+1}}{\partial v_t}(\kappa,\omega,v)\end{bmatrix}, \tag{44}$$

with

$$\frac{\partial \kappa_{t+1}}{\partial \kappa_t} \equiv j_{11} = \frac{\left[G^n - (1-\delta)\right]\sigma}{G^n},$$

$$\frac{\partial \kappa_{t+1}}{\partial \omega_t} \equiv j_{12} = \frac{(1-\alpha)\sigma\left[\frac{\alpha}{G^n - (1-\delta)}\right]^{\frac{\alpha}{1-\alpha}} M^{\frac{1}{1-\alpha}}}{G^n \hat{y}^{ex}},$$

$$\frac{\partial \kappa_{t+1}}{\partial v_t} \equiv j_{13} = \frac{1}{G^n \hat{y}^{ex}},$$

$$\frac{\partial \omega_{t+1}}{\partial \kappa_t} \equiv j_{21} = -\frac{\alpha\sigma\hat{y}^{ex}\left[G^n - (1-\delta)\right]\left[\frac{G^n - (1-\delta)}{\alpha}\right]^{1/(1-\alpha)}}{(1-\alpha)G^n M^{\frac{1}{1-\alpha}}},$$

$$\frac{\partial \omega_{t+1}}{\partial \omega_t} \equiv j_{22} = -\frac{\left[G^n - (1-\delta)\right]\sigma}{G^n},$$

$$\frac{\partial \omega_{t+1}}{\partial v_t} \equiv j_{23} = -\frac{\alpha\left[\frac{G^n - (1-\delta)}{\alpha}\right]^{1/(1-\alpha)}}{(1-\alpha)G^n M^{\frac{1}{1-\alpha}}},$$

$$\frac{\partial v_{t+1}}{\partial \kappa_t} \equiv j_{31} = -\frac{\hat{y}^{ex}\left[G^n - (1-\delta)\right]^2\left[\frac{G^n}{G^n - (1-\delta)} - \frac{(1-\alpha)\sigma}{\alpha}\right]}{G^n},$$

$$\frac{\partial v_{t+1}}{\partial \omega_t} \equiv j_{32} = 0, \quad \frac{\partial v_{t+1}}{\partial v_t} \equiv j_{33} = 1.$$

A glance on the entries of Jacobian (44) reveals that the first row is a $-(1-\alpha)\kappa^{1/(1-\alpha)} M^{1/(1-\alpha)} \big/ \left(\alpha\hat{y}^{ex}\right)$ multiple of the second row. Thus, we know that the determinant of Jacobian (44) is zero. This implies that one of the eigenvalues of the Jacobian, $\lambda_i, i = 1,2,3$, is zero, too: say $\lambda_1 = 0$. The remaining eigenvalues λ_2 and λ_3 read as follows:

$$\lambda_2 = \frac{1}{2} + \frac{\kappa^{\frac{-1}{1-\alpha}} M^{\frac{-1}{1-\alpha}} \sqrt{(1-\alpha)^2 \alpha\kappa^{\frac{-2}{1-\alpha}} M^{\frac{-2}{1-\alpha}} \left(\alpha G^n\left(-4(1-\delta) + 3G^n\right) - 4(1-\alpha)\left[G^n - (1-\delta)\right]^2 \sigma\right)\hat{y}^{ex}}}{2(1-\alpha)\alpha G^n}, \quad (45)$$

$$\lambda_3 = \frac{1}{2} - \frac{\kappa^{\frac{-1}{1-\alpha}} M^{\frac{-1}{1-\alpha}} \sqrt{(1-\alpha)^2 \alpha\kappa^{\frac{-2}{1-\alpha}} M^{\frac{-2}{1-\alpha}} \left(\alpha G^n\left(-4(1-\delta) + 3G^n\right) - 4(1-\alpha)\left[G^n - (1-\delta)\right]^2 \sigma\right)\hat{y}^{ex}}}{2(1-\alpha)\alpha G^n}. \quad (46)$$

As a glance on the eigenvalues in (45) and (46) immediately reveals, $\lambda_2 + \lambda_3 = 1$. Moreover, acknowledging the assumptions in Proposition 1 it follows that the term under the square root in (45) and (46) is strictly larger than zero. Consequently, the eigenvalues λ_2 and λ_3 are real. In addition, they are strictly larger than zero and smaller than one.

Proposition 2. Suppose the assumptions of Proposition 1 hold. Then, the eigenvalues of the Jacobian (44) $\lambda_i, i = 1,2,3$ are real, $\lambda_1 = 0$, $0 < \lambda_2 < 1$ and $0 < \lambda_3 < 1$.

In other words: the equilibrium dynamics with initial values $\kappa_0 = \underline{\kappa} > 0$ and $v_0 = \underline{v} > 0$ in the neighborhood of the positive-discount, steady-state solution in our stock-market OLG model with involuntary unemployment is non-oscillating and converges asymptotically stable towards the steady state (40)-(41) and (43) as time approaches infinity.

As a second step, we investigate the case $0 < \varphi < 1$. Clearly, the same steady-state solution as in (40)-(43) applies in this case. By inserting Equation (37) into Equations (36) and (38), the equilibrium dynamics remains three- dimensional although the entries of the Jacobian (44) change. Without showing the algebraic details of the entries, it turns out that under $0 < \varphi < 1$ the determinant of $J(\kappa, \omega, v)$ equals $(1-\varphi^2)\left[G^n - (1-\delta)\right]/G^n \left\{1-(1-\alpha)\sigma\left[G^n-(1-\delta)\right]/(\alpha G^n)\right\}$, $TrJ(\kappa,\omega,v) = 2-\varphi^2$ and the eigenvalues of $J(\kappa,\omega,v)$ are: $\lambda_1 = 1-\varphi^2$ while λ_2 and λ_3 are exactly equal to (45) and (46).

5. Comparative Dynamics of the Steady-State Solution and the Intertemporal Equilibrium Dynamics

As a next step, it is apt to investigate firstly the comparative dynamics of the positive-discount steady state. The effects of infinitesimal, isolated parameter changes on the positive-discount steady-state solution (40)-(41) and (43) are summarized in the following Proposition 3.

Proposition 3. Suppose that the assumptions of Proposition 1 hold. Then, the effects of infinitesimal, isolated changes of main model parameters on the positive-discount steady-state solution (40)-(41) and (43) read as follows:

$$\frac{\partial\kappa}{\partial\alpha} = \frac{1}{G^n-(1-\delta)} > 0,\ \frac{\partial\kappa}{\partial G^n} = -\frac{\alpha}{\left[G^n-(1-\delta)\right]^2} < 0,\ \frac{\partial\kappa}{\partial\delta} = -\frac{\alpha}{\left[G^n-(1-\delta)\right]^2} < 0, \quad (47)$$

$$\begin{gathered}\frac{\partial v}{\partial\hat{y}^{ex}} = \alpha\left\{\frac{G^n}{G^n-(1-\delta)} - \frac{(1-\alpha)\sigma}{\alpha}\right\} > 0,\ \frac{\partial v}{\partial G^n} = \frac{-(1-\delta)\alpha\hat{y}^{ex}}{\left[G^n-(1-\delta)\right]^2} < 0, \\ \frac{\partial v}{\partial\delta} = \frac{-G^n\alpha\hat{y}^{ex}}{\left[G^n-(1-\delta)\right]^2} < 0,\ \frac{\partial v}{\partial\alpha} = \frac{\alpha\hat{y}^{ex}\sigma}{\alpha^2} > 0,\ \frac{\partial v}{\partial\sigma} = \frac{-(1-\alpha)\alpha\hat{y}^{ex}}{\alpha} < 0.\end{gathered} \quad (48)$$

$$\begin{gathered}\frac{\partial u}{\partial\hat{y}^{ex}} = \frac{-\left[G^n-(1-\delta)\right]^{\alpha/(1-\alpha)}}{M^{1/(1-\alpha)}} < 0, \\ \frac{\partial u}{\partial G^n} = \frac{-\alpha\hat{y}^{ex}\left[G^n-(1-\delta)\right]^{\alpha/(1-\alpha)-1}}{(1-\alpha)M^{1/(1-\alpha)}} < 0, \\ \frac{\partial u}{\partial\delta} = \frac{-\alpha\hat{y}^{ex}\left[G^n-(1-\delta)\right]^{\alpha/(1-\alpha)-1}}{(1-\alpha)M^{1/(1-\alpha)}} < 0, \\ \frac{\partial u}{\partial\alpha} = \frac{\kappa^{\frac{-\alpha}{1-\alpha}}M^{\frac{-1}{1-\alpha}}\hat{y}^{ex}\left(1-\alpha+\log\kappa+\log[M]\right)}{(1-\alpha)^2} > 0.\end{gathered} \quad (49)$$

Considering the results of the comparative-dynamics experiment in (47)-(49) one encounters well-known and not-so-familiar findings. It is well-known from the theory of exogenous growth that a higher capital income share ($d\alpha > 0$), a lower natural growth rate $d\left(G^n - 1\right) < 0$ and a lower capital depreciation rate ($d\delta < 0$) increase the capital-output ratio ($d\kappa > 0$). Moreover, marginal changes of the savings rate (σ) do not impact the steady-state capital-output ratio. New are the findings with respect to the effects of all parameter changes on the steady-state equity-price discount (see the partial derivatives in (48)). More investors' long-run optimism ($d\hat{y}^{ex} > 0$) and a higher capital income share ($d\alpha > 0$) increase the steady-state discount ($dv > 0$), while a higher natural growth rate $d\left(G^n - 1\right) > 0$, a larger capital depreciation rate ($d\delta > 0$) and a higher saving rate ($d\sigma > 0$) decrease the equity-price discount ($dv < 0$). That a higher expected demand for future output raises the equity-price discount, can be intuitively explained by a look on the intertemporal equilibrium condition (28) respective (36): higher expected output demand raises aggregate investment on the right-hand side of condition (28) which for the historically given left-hand side of Equation (28) is only consistent with a lower equity price Q_t implying a larger equity-price discount. Also new, and most important for the main topic of this paper, are the effects of marginal parameter changes on the unemployment rate. Here, the partial derivatives in (49) show that only a larger capital income share ($d\alpha > 0$) increases the unemployment rate ($du > 0$), while more investor's optimism ($d\hat{y}^{ex} > 0$), a larger natural growth rate ($d\left(G^n - 1\right) > 0$) and a larger depreciation rate ($d\delta > 0$) reduce the steady-state unemployment rate ($du < 0$). Notice these typical "Keynesian" results in our neo-classical stock-market OLG model: inter alia, more optimistic investors reduce the steady-state unemployment rate. Notice also that an altered savings rate does not change the steady-state unemployment rate.

Thus, it remains to see whether and if yes how the savings rate impacts the unemployment rate along the intertemporal-equilibrium path towards the new steady state. To be able to answer this question we switch to a numerical specification of our stock-market OLG model of involuntary unemployment. The main model parameters are chosen such that the assumptions of Proposition 1 hold. Moreover, we choose the following parameter set which accords rather well with medium-term stylized facts regarding the growth rate of global real gross domestic product per capita, the real interest rate, the savings ratio, the investment ratio and the unemployment rate of the global economy averaged over the time period between 1960 and 2020 (see IMF, 1990, 2008, 2014, 2023): $G^n = 1.8$ (= annual growth rate approximately 2%), $\beta = 0.31$, $\alpha = 0.22$, $\delta = 0.8$, $\varepsilon = 0.9$, $M = 1.117$, $\hat{y}^{ex} = 0.6191$. Inserting into the steady-state Equations (40)-(43) these parameter values, these equations generate the following steady-state solution: $\kappa = 0.1375$, $v = 0.0294$, $u = 0.06$. The adjustment coefficient will be fixed at: $\varphi = 0.5$.

Consider now a small positive and unexpected shock on ε from 0.9 towards

0.91 implying a small decrease of the savings rate. Then, Table 1 exhibits the intertemporal equilibrium path of main endogenous variables towards the new steady state: $\kappa = 0.1375$, $v = 0.0315$, $u = 0.06$.

Table 1. Intertemporal equilibrium path of $\left(\kappa_t, v_t, u_t, y_t^{ex}\right)_{t>1}$ after a small negative savings-rate shock.

t	0	1	2	3	4	5	6	…	40
κ_t	0.1375	0.1357	0.1356	0.1360	0.1363	0.1366	0.1368	…	0.1375
v_t	0.0294	0.0294	0.0298	0.0302	0.0305	0.0307	0.0309	…	0.0315
u_t	0.0600	0.0563	0.0563	0.0569	0.0576	0.0581	0.0585	…	0.06
y_t^{ex}	0.6191	0.6191	0.6191	0.6191	0.6191	0.6191	0.6191		0.6191

Source: Author's own calculation.

A glance on Table 1 reveals that a small reduction of the saving rate temporarily reduces the capital-output ratio and the unemployment rate, while the equity-price discount steadily increases. After theoretically infinite periods (practically after 40 periods) the capital-output ratio and the unemployment rate return towards the pre-shock values, while the equity-price discount becomes larger. That the unemployment rate temporarily (in the short-term) decreases with a lower saving rate sounds again "Keynesian" in our neo-classical stock-market OLG model with involuntary unemployment.

Starting again from the same steady-state solution as before the savings-rate shock, we assume that firm managers expect a larger long-run demand for their future production output: $\hat{y}^{ex}$ rises from 0.6191 towards 0.62: all other parameters remain on their pre-savings-rate shock values. The effects of this small, positive investment shock on the capital-output ratio, the equity-price discount, the unemployment rate and on the expected demand for future output along the intertemporal-equilibrium path are depicted in Table 2.

Table 2. Intertemporal equilibrium path of $\left(\kappa_t, v_t, u_t, y_t^{ex}\right)_{t>1}$ after a small positive expected output-demand shock.

t	0	1	2	3	4	5	6	…	40
κ_t	0.1375	0.1374	0.1374	0.1375	0.13748	0.13749	0.13750	…	0.1375
v_t	0.0294	0.02950	0.02952	0.02954	0.02955	0.02955	0.02955	…	0.02955
u_t	0.0600	0.0590	0.0586	0.05858	0.05854	0.05851	0.05850	…	0.05848
y_t^{ex}	0.6191	0.6195	0.61980	0.61988	0.61994	0.61997	0.61999		0.6200

Source: Author's own calculation.

As Table 2 reveals, the positive shock on investment temporarily decreases the capital-output ratio and (rather starkly) the unemployment rate, while the equity-price discount only slightly increases in the short- and long-term. While the capital-output ratio increases again along the intertemporal equilibrium path, it returns towards its pre-shock value. The unemployment rate decreases steadily towards its new lower steady-state value: A Keynes-like result even in the long run.

Our last shock experiment concerns the natural growth rate (the impacts of a higher depreciation rate are qualitatively similar). Starting once more from the steady state before the savings-rate shock and the parameters implying it, we increase the natural growth factor from $G^n = 1.8$ to $G^n = 1.85$. The impacts on the capital-output ratio, the equity-price discount, the unemployment rate and on the expected output demand along the intertemporal equilibrium path are depicted in **Table 3**.

Table 3. Intertemporal equilibrium path of $\left(\kappa_t, v_t, u_t, y_t^{ex}\right)_{t>1}$ after a small positive natural-growth shock.

t	0	1	2	3	4	5	6	…	40
κ_t	0.1375	0.1338	0.13318	0.13301	0.13305	0.13310	0.13315	…	0.13333
v_t	0.0294	0.0287	0.02863	0.02867	0.02873	0.02879	0.02883	…	0.02899
u_t	0.0600	0.0526	0.0512	0.0510	0.0511	0.0512	0.0513	…	0.0517
y_t^{ex}	0.6191	0.6191	0.6191	0.6191	0.6191	0.6191	0.6191	…	0.6191

Source: Author's own calculation.

A marginally higher natural growth rate firstly decreases the capital-output ratio, the equity-price discount, and the unemployment rate while from the second respective third period on these dynamic variables increase again towards their nonetheless lower new steady-state values compared to their pre-shock values. The expected output demand does not visibly change along the intertemporal equilibrium path. Similar effects result from a higher depreciation rate.

6. Results and Major Findings

We find in Proposition 1 that a positive-discount steady state exists if the natural growth factor divided by the difference between the natural growth factor and one minus the depreciation rate is larger than the aggregate savings rate (= wage-income share times savings rate of younger households) over the capital income share, and the long-run expected demand for production output from firm investment is not too large, made precise in Proposition 1.

We find in Proposition 2 that local asymptotic stability of the three-dimensional equilibrium dynamics is ensured when the existence condition holds. All three eigenvalues are then equal or larger than zero and smaller than unity.

We also find that a higher capital income share increases the capital-output ratio, while both a higher natural growth rate and a higher depreciation rate decrease the capital-output ratio. In comparison to these well-known responses of the capital-output ratio, the reactions of the equity-price discount are more interesting while new: more manager's optimism and a higher capital-income share increase the equity-price discount since more investment by firms compete for household savings with the share price of companies. A higher natural growth rate, a larger depreciation rate and a higher savings rate decrease the equity-price discount. Most interesting are the responses of the steady-state unemployment rate which increases with a larger capital-income share and decreases with a higher natural growth rate, a larger depreciation rate and more manager's optimism. This last result accords well with short-term Keynesian insights, and it turns out to be valid even in the long run.

We furthermore find that a marginally smaller savings rate temporarily reduces the capital-output ratio and the unemployment rate, while the equity-price discount increases. After about 40 periods (theoretically after an infinite number of time periods) the capital-output ratio and the unemployment rate return to their pre-shock values, while the equity-price discount permanently increases.

We finally find that a positive shock on expected output demand temporarily diminishes the capital-output ratio and rather strongly the unemployment rate, while the equity-price discount increases in the short and long run. Moreover, the positive investment shock reduces the unemployment rate also in the long run. Finally, a marginally higher natural growth rate decreases temporarily and permanently the capital-output ratio, the equity-price discount, and the unemployment rate. A similar effect results from a higher depreciation rate.

7. Conclusion

This paper introduces an endogenous unemployment rate and managers' beliefs regarding the magnitude of the expected demand for production output from capital investment into Magill and Quinzii's (2003) stock-market OLG model with non-shiftable capital and affine equity-price expectations. Firm managers use a production-function-based, non-degenerate belief function in Farmer (2020) to determine their optimally indeterminate investment quantity. The expected demand for future production output per capita is adaptively adjusted towards its long-run expected level given the current production output per capita.

In contradistinction to Magill and Quinzii's (2003) full employment model, in our model, the unemployment rate appears as additional dynamic variable with the consequence that the intertemporal-equilibrium dynamics is three-dimensional instead of two-dimensional as in Magill and Quinzii (2003). The step-by-step derivation of the intertemporal-equilibrium equations from the first-order conditions for intertemporal utility and market value maxima, the government budget constraint, the non-degenerate belief function of firm managers and the market-clearing conditions brings forth that the unemployment rate is a slowly moving dynamic variable in addition to the capital-output ratio and the equity-price discount.

While there are in principle two steady-state solutions, we focus on the positive-discount steady state whereby the capital-output ratio accords to the Golden rule of intertemporal consumption allocation: the interest rate equals the natural growth rate.

At this long-run efficient steady state, we encounter a close connection between investment and the unemployment rate as emphasized by Smith and Zoega (2009) in contrast to New-Keynesian DSGE models, which feature unemployment mainly as a matching problem on labor markets and as consequence of sticky prices and wage rates. The Keynes-like results of our stock market OLG model as the positive impact of more optimistic output-demand expectations and a smaller savings rate along the intertemporal equilibrium path on the unemployment rate are referred to the non-degenerate belief function of firm managers, which makes net value-maximizing firm investment determinate.

Obviously, there is ample space for future research. The highest on the agenda in this respect is the search for another non-degenerate belief function, which is consistent with intertemporal equilibrium in our modified stock-market model of involuntary unemployment. Moreover, to be able to evaluate the policy implications of our model findings, our stock market OLG model should be expanded through endogenizing growth and introducing more policy instruments of the long-lived government.

Acknowledgements

The comments of two anonymous referees to improve the substance and the exposition of the paper are gratefully acknowledged.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

Diamond, P. (1965). National Debt in a Neoclassical Growth Model. *American Economic Review, 55,* 1126-1150.

Ebel, J. (1978). Problems of a Keynesian Investment Function. *Journal of Institutional Economics, 134,* 261-285.

Farmer, K. (2023a). Involuntary Unemployment and Micro-Foundations for Inflexible Aggregate Investment in Diamond-Type Overlapping Generations Models. *Modern Economy, 14,* 455-480. https://doi.org/10.4236/me.2023.144026

Farmer, K. (2023b). Investment Demand Beliefs and Involuntary Unemployment in a Stock Market Overlapping Generations Model. *Theoretical Economics Letters, 13,* 485-503. https://doi.org/10.4236/tel.2023.133031

Farmer, K. (2024). Corporations's Investment, Market Value, and Involuntary Unemployment in a Stock Market Overlapping Generations Model: A Purely Theoretical Exercise. *Modern Economy, 15,* 515-535. https://doi.org/10.4236/me.2024.155026

Farmer, K., & Kuplen, S. (2018). Involuntary Unemployment in a Neoclassical Growth Model with Public Debt and Human Capital. *Studia Universitatis Babes-Bolyai Oeconomica, 63,* 3-34. https://doi.org/10.2478/subboec-2018-0006

Farmer, R. E. A. (2013). Animal Spirits, Financial Crises and Persistent Unemployment. *The Economic Journal, 123,* 317-340. https://doi.org/10.1111/ecoj.12028

Farmer, R. E. A. (2020). The Importance of Beliefs in Shaping Macroeconomic Outcomes. *Oxford Review of Economic Policy, 36,* 675-711. https://doi.org/10.1093/oxrep/graa041

Freitas, F., & Serrano, F. (2015). Growth Rate and Level Effects, the Stability of the Adjustment of Capacity to Demand and the Sraffian Supermultiplier. *Review of Political Economy, 27,* 258-281. https://doi.org/10.1080/09538259.2015.1067360

Girardi, D. (2021). *The Neoclassical Theory of Aggregate Investment and Its Criticisms.* Economics Department Working Paper Series.

IMF (1990). *World Economic Outlook.*

IMF (2008). *World Economic Outlook.*

IMF (2014). *World Economic Outlook.*

IMF (2023). *World Economic Outlook.*

Keynes, J. M. (1936). *The General Theory of Employment, Interest, and Money.* Macmillan.

Magill, M., & Quinzii, M. (2003). Nonshiftable Capital, Affine Price Expectations and Convergence to the Golden Rule. *Journal of Mathematical Economics, 39,* 239-272. https://doi.org/10.1016/s0304-4068(03)00050-8

Magnani, R. (2015). *The Solow Growth Model Revisited. Introducing Keynesian Involuntary Unemployment.* https://hal.archives-ouvertes.fr/hal-01203393

Morishima, M. (1977). *Walras' Economics: A Pure Theory of Capital and Money.* Cambridge University Press.

Plotnikov, D. (2018). Hysteresis in Unemployment: A Confidence Channel. *International Journal of Economic Theory, 15,* 109-127. https://doi.org/10.1111/ijet.12207

Smith, R., & Zoega, G. (2009). Keynes, Investment, Unemployment and Expectations. *International Review of Applied Economics, 23,* 427-444. https://doi.org/10.1080/02692170902954767

Solow, R. M. (1956). A Contribution to the Theory of Economic Growth. *The Quarterly Journal of Economics, 70,* 65-94. https://doi.org/10.2307/1884513

Tanaka, Y. (2020). Involuntary Unemployment and Fiscal Policy for Full-Employment. *Theoretical Economics Letters, 10,* 745-757. https://doi.org/10.4236/tel.2020.104046