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Dynamic Poverty Measures

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Abstract

In this paper we propose methods for detecting the number of pores based on dynamic optimization techniques. An illustration is provided and the results are discussed based on Government's objectives and control variables.

Keywords: Optimization Techniques, Poverty Measures

1. Introduction

How poverty is measured is a central topic in economic and policy analyses. However, recently, it has clearly appeared that it is not only the determination of particular poverty levels at particular instants (based on several indices available in the literature) that matter the most. The paths of poverty levels over time are also critical and crucial indicators in assessing the efficiency of poverty measures (Ciarlet, 2006 [1]).

This paper adds to the literature on this topic by providing methods for measuring poverty in dynamic environment (Dia and Popescu, 1996 [2]). The proposed approach answers the following question: How important are dynamic optimization techniques for poverty measure and analysis?

The remainder of the paper is organized as follows: In Section 2, we address the problem of poverty measures in a dynamic context based on optimal control. An illustration is provided and discussed in Section 3. Finally, some concluding remarks appear in Section 4.

2. Dynamic Poverty Measures

In the context of dynamic optimization, time does matter and the paths of poverty indexes are important.

2.1. The Problem in Discrete Time

Consider t_0 an initial instant. We assume that a static model has been used to determine the number of pores in a given population based on a given poverty line. Let t be the time index, $Y_t \in \mathbb{R}_+^{*Q}$, the vector of revenues of the Q pores who have been identified at time t. Let $u_t \in \mathbb{R}^p$, be the vector of the p control variables which represent the set of commands. The objective is to measure the performance of the system. To this end, we consider an objective function to optimize subject to some constraints. The above problem can be formalized as follows (Rustagi, 1997 [3])

$$\text{Opt } J, \quad J := \sum_{t=t_0}^{T-1} k\left(Y_t, a_t, u_t, t\right) \tag{1}$$

$$Y_{t+1} = g_t\left(Y_t, u_t\right), \; t = t_0, t_1, \cdots, T-1 \tag{2}$$

where the second equality represents the constraints on the state vector which is the vector of revenues Y_t, a_t the objective at each period of time t and u_t the control variables. The problem now is to choose the best control vector u_t, at each period of time, according to available resources, such that the above system is satisfied (Rustagi, 1997 [3] and Mart, 1997 [4]). In this type of problem, it is the final stage which is the most important since the objective is to reduce or eliminate poverty. This of course depends on intermediary objectives. More specifically, the dynamic optimization problem can be set up as a minimization problem of J (Troutman, 1980 [5]),

$$\min J, \; J := \sum_{t=t_0}^{T-1} k\left(Y_t, a_t, u_t, t\right)$$
$$= \sum_{t=t_0}^{T-1} \left(Y_t - a_t\right)\left(Y_t - a_t\right)^t + u_t u_t^t \tag{3}$$

How to justify the choice of the function $k(Y_t, a_t, u_t, t)$? In the above problem, it is the final revenue

which is the most important, *i.e.*, Y_{T-1}. In fact, the objective is to get the individuals in the vector of revenues Y_{T-1} out of poverty. Therefore, intermediary objectives to be reached and any set of decisions at time $T-1$ should be such that, $Y_i^t \geq Z, \forall i$, where Z is the poverty line. From a mathematical point of view, at $T-1$, the Euclidian norm of a_{T-1}, *i.e.*, $\|a_{T-1}\|$ must be at least of the same order as NZ^2, the poverty line of the population under consideration. To this end, it suffices to prove that,

$$\left\| \sum_{i=1}^{Q} \left(a_i^{T-1} \right)^2 - NZ^2 \right\| \geq 1 \tag{4}$$

or

$$\left\| \sum_{i=1}^{Q} \left(a_i^{T-1} \right)^2 - NZ^2 \right\| = o(1) \tag{5}$$

where o is the Landau notation.

Equation (4) has an immediate solution

$$a_i^t \geq \left(\frac{t}{T-1} \right) \sqrt{\frac{1+NZ^2}{Q}} , \tag{6}$$

$$i \in \{1, \cdots, Q\}; t \in \{t_0, \cdots, T-1\}$$

In addition, another constraint is that when $r \to NZ^2$, the level of richness of the individual i , r , must be in adequacy with the poverty line if not higher *i.e.*, Y_i must be higher than the poverty line. Therefore the constraint on the final objectives must be such that,

$$\left\| \sum_{i=1}^{Q} \left(a_i^{T-1} \right)^2 - NZ^2 \right\| \geq 1 \tag{7}$$

Regarding the system to control, $Y_{t+1} = g_t(Y_t, u_t)$, $t = t_0, t_1, \cdots, T-1$ for simplicity, we consider a linear system. Therefore,

$$Y_{t+1} = A_t Y_t + B_t u_t, t = t_0, t_1, \cdots, T-1 \tag{8}$$

where A_t is a square matrix of order Q , *i.e.*, $A_t = \left(a_{ij}^t \right)_{\substack{1 \leq i \leq Q \\ 1 \leq j \leq Q}}$, B_t is a matrix with Q rows and p columns, *i.e.*, $B_t = \left(b_{ij}^t \right)_{\substack{1 \leq i \leq Q \\ 1 \leq j \leq p}}$. These matrices can be identified using economic theory. Summarizing the discrete time problem we have (Troutman, 1980 [5]),

$$\begin{cases} \min J, J := \sum_{t=t_0}^{T-1} (Y_t - a_t)(Y_t - a_t)^t + u_t u_t^t \\ \text{s.t.} \\ Y_{t+1} = A_t Y_t + B_t u_t, \ t = t_0, t_1, t_2, \cdots, T-1 \\ a_i^t \geq \left(\frac{t}{T-1} \right) \sqrt{\left(\frac{1+NZ^2}{Q} \right)} \\ \left\| \sum_{i=1}^{Q} \left(a_i^{T-1} \right)^2 - NZ^2 \right\| \geq 1 \\ i \in \{1, \cdots, Q\}, \ t \in \{t_0, \cdots, T-1\} \end{cases} \tag{9}$$

2.2. The Problem in Continuous Time

Based on the above developments, by analogy and with some minor modifications, in continuous time, the dynamic problem can be formalized as follows,

$$\begin{cases} \min J : J = \int_{t_0}^{T} \left(\|Y(t) - a(t)\|^2 + \|u(t)\|^2 \right) dt \\ \text{s.t.} \\ \dfrac{dY(t)}{dt} = A(t)Y(t) + B(t)u(t) \\ a(t) \geq \dfrac{t}{T}\left(\dfrac{1+NZ^2}{Q} \right); t \in [t_0, T] \\ u : [t_0, T] \to \mathbb{R}^p \text{ a vector of continuous} \\ \text{step functions.} \\ \min J = -\max(-J) \end{cases} \tag{10}$$

Set $G = -J$ and

$$f_0(t_0, Y(t), u(t), a(t))$$
$$= -\|Y(t) - a(t)\|^2 - \|u(t)\|^2$$

2.3. Solution

To conserve space, the solution given here is related only to the continuous case. Deduction of the discrete time solution is then straightforward.

Let V_0 be the maximal value of the objective function. It is easy to verify that,

$$V_0(t, Y) = \max \int_{t_0}^{T} f_0(s_0, Y(s), u(s), a(s)) ds$$
$$+ V_0(t + \Delta, Y + \Delta)$$
$$\text{given } u : [t, t+\Delta] \to \mathbb{R}^p, \Delta \geq 0$$

$$V(t, Y) = -V_0(t, Y)$$
$$= \min \int_{t_0}^{T} -f_0(s_0, Y(s), u(s), a(s)) ds$$
$$- V(t + \Delta, Y + \Delta)$$
$$\text{given } u : [t, t+\Delta] \to \mathbb{R}^p, \Delta \geq 0$$

$$\frac{dY(s)}{dt} = A(s)Y(s) + B(s)u(s), \ Y(s) = Y \tag{11}$$

Let us assume that V is differentiable in t and Y. Then,

$$V(t, Y) = \min \left\{ \left(\|Y(t) - a(t)\|^2 + \|u(t)\|^2 \right) \Delta \right.$$
$$+ \Delta \frac{\partial V}{\partial Y} (A(t)Y(t) + B(t)u(t))$$
$$\left. + \frac{\partial V}{\partial t}(t, Y)\Delta + o(\Delta), u \in \mathbb{R}^p \right\}$$

with $\Delta > 0$ and where $\dfrac{\partial V}{\partial Y} = \nabla_Y$ and where ∇ is the gradient operator. Dividing by Δ and letting $\Delta \to 0$, we get the Hamilton-Jacobi-Bellman partial derivative equation of the form,

$$\frac{\partial V}{\partial t} + \min \left\{ \begin{array}{l} \left(\left\| Y(t) - a(t) \right\|^2 + \left\| u(t) \right\|^2 \right) \\ + \dfrac{\partial V}{\partial Y} \left(A(t) Y(t) + B(t) u(t) \right) \end{array} \right\} = 0 \quad (12)$$

given $u \in \mathbb{R}^p$

We then have the following theorem:

Theorem 1

Assuming there exists a differentiable function V: $[t_0, T] \times \mathbb{R}^Q \to \mathbb{R}$ which satisfies (12).

Assuming that: $\exists \Psi : [t_0, T] \times \mathbb{R}^Q \to \mathbb{R}^p$ with Ψ a continuous function in t and Lipschitz in Y and satisfying,

$$\left\| Y(t) - a(t) \right\|^2 + \left\| \Psi(t, Y) \right\|^2$$
$$+ \frac{\partial V}{\partial Y} \left(A(t) Y(t) + B(t) \Psi(t, Y) \right)$$
$$= \min \left\{ \left(\left\| Y(t) - a(t) \right\|^2 + \left\| u(t) \right\|^2 \right) \right. \quad (13)$$
$$\left. + \frac{\partial V}{\partial Y} \left(A(t) Y(t) + B(t) u(t) \right) \right\}$$

given $u \in \mathbb{R}^p$

Then, Ψ is a control optimal feedback for problem (13), i.e., V is the minimum of J.

Proof: (See the Appendix).

In the continuous case, by analogy to the discrete case, the general model proposed here is,

$$\left\{ \begin{array}{l} \text{Opt}: \quad J = \displaystyle\int_{t_0}^{T} k \left(Y(t), a(t), u(t), t \right) \mathrm{d}t \\ \text{s.t.} \\ \dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} = g \left(Y(t), u(t), t \right), \ t \in [t_0, T] \end{array} \right. \quad (14)$$

The specific model proposed is,

$$\left\{ \begin{array}{l} \min \text{J}: J = \displaystyle\int_{t_0}^{T} \left(\left\| Y(t) - a(t) \right\|^2 + \left\| u(t) \right\|^2 \right) \mathrm{d}t \\ \text{s.t.} \\ \dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} = A(t) Y(t) + B(t) u(t), t \in [t_0, T] \quad (15) \\ Y(T) = Y_i(T)_{i=1,\cdots,Q} \\ Y(t_0) = Y_0 \end{array} \right.$$

$a_i(T)$ is defined and is a square integrable function on $[t_0, T]$ and such that

$$\left\{ \begin{array}{l} a_i(t) \geq \dfrac{t}{T} \sqrt{\dfrac{1 + NZ^2}{Q}}, i \in \{1, \cdots, Q\}, t \in [t_0, T] \\ \left\| \displaystyle\sum_{i=1}^{Q} \left(a_i(T) \right)^2 - NZ^2 \right\| \geq 1 \end{array} \right. \quad (15)$$

where $Y_i(T) \geq Z$, $\forall i \in \{1, \cdots, Q\}$, $u(t) \in \mathbb{R}^p$ with u_j a continuous step function $\forall j = 1, \cdots, p$.

Theorem 2

Under the assumption of controllability of the above system, the problem of minimization admits an optimal solution. Paths and optimal controls are obtained by resolving the following system,

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = A(t) Y(t) - \dfrac{1}{2} B(t) B(t)^t \lambda(t) \\ \dfrac{\mathrm{d}\lambda}{\mathrm{d}t} = -\left(Y(t) - a(t) \right) - A(t)^t \lambda(t) \\ Y(t_0) = Y_0 \\ Y(T) = K, \text{ where } Y_i = Z \\ \lambda(T) \perp T^f \left(Y(T) \right) \end{array} \right. \quad (16)$$

where $\lambda(t)$ is the vector of Pontryagin multipliers and $T^f \left(Y(T) \right)$ the space containing conditions on the Pontryagin multipliers. $T^f \left(Y(T) \right)$ is defined in the Appendix.

Proof: (See the Appendix).

3. A Parametric Illustration

As an illustration we consider a general problem faces by a Government in determining dynamic poverty measures over time. To get a more interesting case, we consider a parametric problem.

3.1. The Problem

Consider a parametric dynamic optimization problem where government authorities have some flexibility on intermediary objectives as well as the control variables. The parametric dynamic optimization problem can be formalized as follows,

$$\left\{ \begin{array}{l} \min J = \displaystyle\int_{0}^{T} \left(\left\| Y(t) - m_1 a(t) \right\|^2 + m_2 \left\| u(t) \right\|^2 \right) \mathrm{d}t \\ \text{s.t.} \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = A(t) Y(t) + B(t) u(t) \\ a_i(t) \geq \dfrac{t}{T} \sqrt{\dfrac{1 + NZ^2}{Q}}, i = 1, \cdots, Q \\ Y(t_0) = Y_0 \\ Y(T) = \left(Y_i(T) \right), i = 1, \cdots, Q \end{array} \right.$$

where $a_i(t)$ a functional such that $\int_0^T |a_i(t)|^2 \, dt < +\infty$, $A \in M_{Q,Q}(\mathbb{R})$ and $B \in M_{Q,p}(\mathbb{R})$ and where for simplicity we assume that the coefficients of A and B are not time dependent. For the sake of flexibility, the objectives as well as the control variables are parameterized so as to account for possible changes during the implementation of Government economic policies; $m_1, m_2 \in \mathbb{R}^+$.

3.2. Solution

Using the Pontryagin principle, we get the following optimality system,

$$
\begin{cases}
\text{Min } J: J = \int_0^T \left(\|Y(t) - m_1 a(t)\|^2 + m_2 \|u(t)\|^2 \right) dt \\
s.t. \\
\dfrac{dY}{dt} = AY(t) - \dfrac{1}{2m_2} BB^t u(t) \\
\dfrac{d\lambda}{dt} = -2Y(t) - A^t \lambda(t) + 2m_1 a(t) \\
Y(t_0) = Y_0 \\
Y(T) = \left(Y_i(T) \right), i = 1, \cdots, Q \\
Y_i(T) = Z \\
\lambda(T) \perp T^f \left(Y(T) \right)
\end{cases}
$$

Set $X = \begin{pmatrix} Y \\ \lambda \end{pmatrix}$. We get,

$$
\begin{cases}
\dfrac{dX}{dt} = \begin{pmatrix} A & -\dfrac{1}{2m_2} BB^t \\ -2I_Q & -A^t \end{pmatrix} X + \begin{pmatrix} 0 \\ 2m_1 a(t) \end{pmatrix} \\
Y(t_0) = Y_0 \\
Y_i(T) = Z, \ i = 1, \cdots, Q
\end{cases}
$$

The above differential system can be written as,

$$
\frac{dX}{dt} = \underbrace{\begin{pmatrix} A & -\dfrac{1}{2m_2} BB^t \\ -2I_Q & -A^t \end{pmatrix}}_{Cm_2} X + \underbrace{\begin{pmatrix} 0 \\ 2m_1 a(t) \end{pmatrix}}_{F_{m_1}(t)}
$$

Or simply as, $\dfrac{dX}{dt} = Cm_2 X + F_{m_1}(t)$

where $C_{m_2} \in M_{2Q,2Q}(\mathbb{R})$ and $F_{m_1(t)} \in M_{2Q,1}(\mathbb{R})$.

Two cases must be considered depending on the fact that C_{m_2} is diagonalizable or not. To conserve space, we consider only the first case and when eigenvalues are real and distinct. Note that, the optimality system even though linear, is too general in its expression and in the sign of the second member. The resulting general results

will then be difficult to interpret. Let us consider a simple case where $Q = 1$ and $p = 1$. Again, for simplicity we assume that $A = (1)$ and $B = (1)$. The optimality system is therefore,

$$
\begin{cases}
\dfrac{dY}{dt} = Y(t) - \dfrac{1}{2m_2} \lambda(t), m_2 > 0 \\
\dfrac{d\lambda}{dt} = -2Y(t) - \lambda(t) + 2m_1 a_1(t) \\
Y(t_0) = 0 \\
\lambda(T) \perp T^f \left(Y(T) \right)
\end{cases}
$$

To fix ideas consider that $a_1(t) = \dfrac{t}{T} \sqrt{\dfrac{1+NZ^2}{Q}}$. The underlying matrices are

$$
C_{m_2} = \begin{pmatrix} 2 & -\dfrac{1}{2m_2} \\ -2 & -1 \end{pmatrix} \text{ and } F_{m_1}(t) = \begin{pmatrix} 0 \\ \dfrac{2m_1 t}{T} \sqrt{\dfrac{1+NZ^2}{Q}} \end{pmatrix}.
$$

The eigenvalues of C_{m_2} are

$$
\varphi_1 = \sqrt{1 + \frac{1}{m_2}} \text{ and } \varphi_2 = -\sqrt{1 + \frac{1}{m_2}} \text{ with } m_2 > 0. \ C_{m_2}
$$

is diagonalizable. The matrix of associated eigenvectors P is defined as,

$$
P = \begin{pmatrix} 1 & 1 \\ 2m_2 \left(1 - \sqrt{1 + \dfrac{1}{m_2}} \right) & 2m_2 \left(1 + \sqrt{1 + \dfrac{1}{m_2}} \right) \end{pmatrix} \quad \text{and}
$$

$$
P^{-1} = \begin{pmatrix} \dfrac{1 + \sqrt{1 + \dfrac{1}{m_2}}}{2\sqrt{1 + \dfrac{1}{m_2}}} & -\dfrac{1}{4m_2 \sqrt{1 + \dfrac{1}{m_2}}} \\ 1 - \dfrac{\left(1 + \sqrt{1 + \dfrac{1}{m_2}} \right)}{2\sqrt{1 + \dfrac{1}{m_2}}} & \dfrac{1}{4m_2 \sqrt{1 + \dfrac{1}{m_2}}} \end{pmatrix}
$$

The differential system is now equivalent to,

$$
\begin{pmatrix} \dfrac{du_1}{dt} \\ \dfrac{du_2}{dt} \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + P^{-1} F_{m_1}(t)
$$

Solving, we get

$$
u_1(t) = c_1 e^{\varphi_1 t} + \frac{m_1}{2\varphi_1 T m_2} \sqrt{\frac{1+NZ^2}{1 + \dfrac{1}{m_2}}} \left(t + \frac{1}{\varphi_1} \right),
$$

where c_1 is a constant of integration. and

$$u_2(t) = c_2 \mathrm{e}^{\varphi_2 t} - \frac{m_1}{2\varphi_2 T m_2} \sqrt{\frac{1 + NZ^2}{1 + \frac{1}{m_2}}} \left(t + \frac{1}{\varphi_2} \right)$$

c_2 being a constant of integration as well. Since,

$$U = P^{-1} X \Rightarrow X = PU \quad \text{where} \quad X = \begin{pmatrix} Y(t) \\ \lambda(t) \end{pmatrix}, \text{ we obtain,}$$

$$\begin{pmatrix} Y(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2m_2\left(1 - \sqrt{1 + \frac{1}{m_2}}\right) & 2m_2\left(1 + \sqrt{1 + \frac{1}{m_2}}\right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

After a bit of algebra we get,

$$\begin{cases} Y(t) = \left(c_1 \mathrm{e}^{\varphi_1 t} + c_2 \mathrm{e}^{-\varphi_1 t}\right) + \dfrac{m_1 t}{\varphi_1^2 T m_2} \sqrt{1 + NZ^2} \\ \lambda(t) = 2\left[\dfrac{-c_1 \mathrm{e}^{\varphi_1 t}}{\varphi_1 + 1} + \dfrac{c_2 \mathrm{e}^{-\varphi_1 t}}{\varphi_1 - 1} \right] + \dfrac{2m_1(t-1)\sqrt{1+NZ^2}}{T\varphi_1^2} \end{cases}$$

The optimal control $u(t)$ is given by $u(t) = -\dfrac{1}{2m_2}\lambda(t)$.

We now discuss several cases.

3.3. Discussion

Case 1: $m_1 \to 0^+$ and $m_2 \to 0^+$.

$$Y(t) \to \begin{cases} +\infty \text{ if } c_1 > 0 \\ -\infty \text{ if } c_1 < 0 \end{cases}, \quad \lambda(t) \to \begin{cases} +\infty \text{ if } c_1 < 0 \\ -\infty \text{ if } c_1 > 0 \end{cases}$$

and $$u(t) \to \begin{cases} +\infty \text{ if } c_1 < 0 \\ -\infty \text{ if } c_1 > 0 \end{cases}$$

Poverty can be greatly improved on the condition that intermediary objectives and control variables be realistic and comprehensive.

Case 2: $m_1 \to 0^+$ and m_2 fixed.

$$Y(t) \to c_1 \mathrm{e}^{\varphi_1 t} + c_2 \mathrm{e}^{-\varphi_1 t}, \quad \lambda(t) \to 2\left[\frac{-c_1 \mathrm{e}^{\varphi_1 t}}{\varphi_1 + 1} + \frac{c_2 \mathrm{e}^{-\varphi_1 t}}{\varphi_1 - 1} \right]$$

and $$u(t) \to \frac{1}{m_2}\left[\frac{c_1 \mathrm{e}^{\varphi_1 t}}{\varphi_1 + 1} - \frac{c_2 \mathrm{e}^{-\varphi_1 t}}{\varphi_1 - 1} \right]$$

Poverty can be gradually improved if intermediary objectives are reachable and control variables reasonably selected.

Case 3: $m_1 \to 0^+$ and $m_2 \to +\infty$.

$$Y(t) \to c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t}, \quad \lambda(t) \to \begin{cases} +\infty \text{ if } c_2 > 0 \\ -\infty \text{ if } c_2 < 0 \end{cases}$$

and $u(t) \to -2c_2 \mathrm{e}^{-t}$

Poverty can be gradually improved but many control variables can create entropy in the system.

Case 4: m_1 fixed and $m_2 \to 0^+$.

$$Y(t) \to \begin{cases} +\infty \text{ if } c_1 > 0 \\ -\infty \text{ if } c_1 < 0 \end{cases}, \quad \lambda(t) \to \begin{cases} +\infty \text{ if } c_1 < 0 \\ -\infty \text{ if } c_1 > 0 \end{cases}$$

and $$u(t) \to \begin{cases} +\infty \text{ if } c_1 < 0 \\ -\infty \text{ if } c_1 > 0 \end{cases}$$

Realistic intermediary objectives and well chosen control variables may result in positive impacts in terms of poverty alleviation.

Case 5: m_1 fixed and m_2 fixed.

The behaviors of $Y(t)$, $\lambda(t)$ and $u(t)$ depend on the specific values assigned to m_1 and m_2.

Case 6: m_1 fixed and $m_2 \to +\infty$.

$$Y(t) \to c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t}, \quad \lambda(t) \to \begin{cases} +\infty \text{ if } c_2 > 0 \\ -\infty \text{ if } c_2 < 0 \end{cases}$$

and $$u(t) \to \begin{cases} +\infty \text{ if } c_1 > 0 \\ -\infty \text{ if } c_1 < 0 \end{cases}$$

Fixed objectives can be beneficiary for poverty improvement but many control variables can negatively affect the system.

Case 7: $m_1 \to +\infty$ and $m_2 \to 0^+$.

$Y(t)$, $\lambda(t)$ and $u(t)$ oscillate to $\pm\infty$, a case of uncertainty.

Case 8: $m_1 \to +\infty$ and m_2 fixed.

$Y(t) \to +\infty$, $\lambda(t) \to +\infty$ and $u(t) \to -\infty$

Many objectives with reasonable control variables may result in poverty improvement.

Case 9: $m_1 \to +\infty$ and $m_2 \to +\infty$.

$Y(t)$, $\lambda(t)$ and $u(t)$ oscillate to $\pm\infty$, another case of uncertainty.

4. Final Remarks

How important are dynamic optimization techniques for poverty analysis? In dynamic settings, the paths of incomes are essential and the paper provides methods accordingly. It remains to establish optimality and stability criteria for the characterization of the various paths in a future research.

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Appendix: Theorems and Proofs

Theorem 1

Let $t \in [t_0, T]$, $Y \in \mathbb{R}^Q$ and $u_0 : [t_0, T] \to \mathbb{R}^p$ be a vector of functions which represents control variables and $Y_0(\tau)$ the solution to,

$$\begin{cases} \dot{Y}(\tau) = A(\tau)Y_0(\tau) + B(\tau)u_0(\tau). \\ \dot{Y}(t_0) = Y_0, \quad \tau \in [t_0, T] \end{cases}$$

Let $Y^*(\tau)$ be the solution to,

$$\begin{cases} \dot{Y}^*(\tau) = A(\tau)Y^*(\tau) + B(\tau)u^*(\tau) \\ \dot{Y}^*(t_0) = Y_0, \quad \tau \in [t_0, T] \end{cases}$$

where $u^*(\tau) = \Psi(\tau, Y^*(\tau))$. The assumption on Ψ insures the existence of the solution to the above system. To show that Ψ is an optimal feedback control, we need to prove that,

$$\int_{t_0}^T \left(\left\| Y^*(t) - a(t) \right\|^2 + \left\| u^*(t) \right\|^2 \right) dt$$
$$\leq \int_{t_0}^T \left(\left\| Y^*(t) - a(t) \right\|^2 + \left\| u_0(t) \right\|^2 \right) dt$$

Set, $f_0\left(t, Y^*(t), u^*(t)\right) = \left\| Y^*(t) - a(t) \right\|^2 + \left\| u^*(t) \right\|^2$ and

$$f_0\left(t, Y^*(t), u_0(t)\right) = \left(\left\| Y^*(t) - a(t) \right\|^2 + \left\| u_0(t) \right\|^2 \right)$$

On the one hand,

$$V\left(T, Y^*(T)\right) - V\left(t_0, Y^*(t_0)\right) = \int_{t_0}^T \frac{dV}{dt}\left(t, Y^*(t)\right) dt$$

$$\frac{dV}{dt}\left(t, Y^*(t)\right) = \frac{\partial V}{\partial t}\left(t, Y^*(t)\right) + \nabla_Y V\left(t, Y^*(t)\right)\frac{dY^*(t)}{dt}$$

and

$$\frac{dY^*(t)}{dt} = A(t)Y^*(t) + B(t)u^*(t)$$

where $u^*(t) = \Psi\left(t, Y^*(t)\right)$. Since by assumption,

$$f_0\left(t, Y^*(t), D\right)$$
$$= \min\left\{ f_0\left(t, Y(t), u(t)\right) \right.$$
$$\left. + \nabla_Y V\left(t, Y(t)\right)\left[\left(A(t)Y(t) + B(t)u(t)\right)\right] \right\}$$

given $u \in \mathbb{R}^p$

where
$$D = \Psi\left(t, Y^*(t)\right) + \nabla_Y V\left(t, Y^*(t)\right)$$
$$\times \left[A(t)Y^*(t) + B(t)\Psi\left(t, Y^*(t)\right) \right]$$

We have,
$$0 = \frac{\partial V}{\partial t} + f_0\left(t, Y^*(t), \Psi\left(t, Y^*(t)\right)\right)$$
$$+ \nabla_Y V\left(t, Y^*(t)\right)\left[A(t)Y^*(t) + B(t)\Psi\left(t, Y^*(t)\right) \right]$$
$$= \min\left\{ f_0\left(t, Y^*(t), \Psi\left(t, Y^*(t)\right)\right) \right.$$
$$\left. + \nabla_Y V\left(t, Y^*(t)\right)\left[A(t)Y^*(t) + B(t)\Psi\left(t, Y^*(t)\right) \right] \right\}$$

given $u \in \mathbb{R}^p$

Thus,
$$-f_0\left(t, Y^*(t), \Psi\left(t, Y^*(t)\right)\right)$$
$$= \frac{\partial V}{\partial t} + f_0\left(t, Y^*(t), \Psi\left(t, Y^*(t)\right)\right)$$
$$+ \nabla_Y V\left(t, Y^*(t)\right)\left[A(t)Y^*(t) + B(t)\Psi\left(t, Y^*(t)\right) \right]$$

Hence,
$$V\left(T, Y^*(T)\right) - V\left(t_0, Y^*(t_0)\right)$$
$$= -\int_{t_0}^T f_0\left(t, Y^*(t), \Psi\left(t, Y^*(t)\right)\right) dt$$

On the other hand,
$$V\left(T, Y_0(T)\right) - V\left(t_0, Y_0(t_0)\right)$$
$$= \int_{t_0}^T \frac{dV}{dt}\left(t, Y_0(t)\right) dt$$
$$= \int_{t_0}^T \left[\frac{\partial V\left(t, Y_0(t)\right)}{\partial t} + \nabla_Y V\left(t, Y_0(t)\right)\frac{dY_0(t)}{dt} \right] dt$$

We can immediately notice that $\forall u \in \mathbb{R}^p$,
$$f_0\left(t, Y_0(t), u(t)\right) + \nabla_Y V\left(t, Y_0(t)\right)\frac{dY_0(t)}{dt}$$
$$\geq \min\left\{ f_0\left(t, Y(t), u(t)\right) + \nabla_Y V\left(t, Y(t)\right)\frac{dY(t)}{dt} \right\} \text{ Hence,}$$
$$= f_0\left(t, Y^*(t), \Psi\left(t, Y^*(t)\right)\right)$$
$$+ \nabla_Y V\left(t, Y^*(t)\right)\frac{dY^*(t)}{dt}, \text{ given } u \in \mathbb{R}^p$$
$$V\left(T, Y_0(T)\right) - V\left(t_0, Y_0(t_0)\right)$$
$$\geq -\int_{t_0}^T f_0\left(t, Y_0(t), u^*(t)\right) dt$$

We then conclude using the Hamilton-Jacobi-Bellman equation and the fact that $Y^*(t_0) = Y_0(t_0) = Y_0$, that

$$\int_{t_0}^{T} f_0\big(t, Y^*(t), u^*(t)\big)\mathrm{d}t \le \int_{t_0}^{T} f_0\big(t, Y_0(t), u_0(t)\big)\mathrm{d}t$$

Theorem 2

We have,

$$\begin{cases} \min J : J = \int_{t_0}^{T}\Big(\|Y(t) - a(t)\|^2 + \|u(t)\|^2\Big)\mathrm{d}t \\ \text{s.t.} \\ \dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} = A(t)Y(t) + B(t)u(t), 0 \le t \le T \\ Y(0) = Y_0 \\ Y(t) = \big(Y_i(T)\big)_{1 \le i \le Q}, Y_i(T) = Z, u(t) \in \mathbb{R}^p \end{cases}$$

with $u(t)$ a continuous step function, $\forall j = 1, \cdots, p$. The Hamiltonian is,

$$H\big(t, Y(t), u(t), \lambda(t)\big)$$
$$= p_0 \|Y(t) - a(t)\|^2 + p_0 \|u(t)\|^2$$
$$+ \lambda(t)^t \big[A(t)Y(t) + B(t)u(t) \big]$$

where $\lambda(t)$ is the vector of Pontryagin multipliers. The first order conditions are,

$$\frac{\partial H\big(t, Y(t), u(t), \lambda(t)\big)}{\partial Y_i} = -\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}, \forall i = 1, \cdots, Q$$

$$\frac{\partial H\big(t, Y(t), u(t), \lambda(t)\big)}{\partial \lambda_i} = \frac{\mathrm{d}Y_i}{\mathrm{d}t}, \forall i = 1, \cdots, Q$$

Thus,

$$\frac{\partial H\big(t, Y(t), u(t), \lambda(t)\big)}{\partial Y_i}$$
$$= 2p_0\big(Y_i(t) - a_i(t)\big) + \frac{\partial}{\partial Y_i}\Big[{}^t \lambda(t) A(t)Y(t)\Big]$$
$$= 2p_0\big(Y_i(t) - a_i(t)\big) + \sum_{j=1}^{Q} \lambda_j \alpha_{ji}$$

where $\alpha_{ij_{1\le i, j \le Q}} = A(t)$. Finally, we get

$$\frac{\partial H\big(t, Y(t), u(t), \lambda(t)\big)}{\partial Y_i} = -\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}$$. We now have the following differential system,

$$\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = -2p_0\big(Y_i(t) - a_i(t)\big) - \sum_{j=1}^{Q} \lambda_j \alpha_{ij_{1\le i, j \le Q}}$$

and $\lambda \perp T^f(Y) = \big\{ \xi / G^f(Y(t))\xi = 0 \big\}$. In our case,

$$G \equiv I_d \big(G^f = Y(t) \big);$$

where

$$T^f = \left\{ \frac{x}{G^f} - b^f = 0 \right\},$$

$$T^f\big(Y(t)\big) = \left\{ \frac{\xi}{G^f}\big(Y(t)\big)\xi = 0 \right\},$$

$$G^f : \mathbb{R}^Q \to \mathbb{R}^Q,$$

$$T^0 = \left\{ \frac{x}{g^0(x)} - b^0 = 0 \right\}, \; T^0(Y_0) = \left\{ \frac{\xi}{G^0(Y_0)\xi} = 0 \right\}.$$

At the optimum, we have

$$\frac{\partial H(.,.)}{\partial u_j} = 2p_0 u_j + \lambda(t)^t \frac{\partial}{\partial u_j}\big(B(t)u(t)\big)$$
$$= 0, \forall j \in \{1, \cdots, p\}$$
$$\Leftrightarrow 2p_0 u_j(t) + B(t)^t \lambda(t) = 0.$$

1) If $p_0 \ne 0$, then $u(t) = -\dfrac{1}{2p_0}\big(B(t)^t \lambda(t)\big)$.

2) If $p_0 = 0$, then $B(t)^t \lambda(t) \equiv 0$. By assumption we have, $\dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} = A(t)Y(t) + B(t)u(t)$ which is controllable on $[0, T]$. Consider $\Phi(t, \tau)$ the function of transition matrices of $\dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} = A(t)Y(t)$. The assumption of controllability is equivalent to

$$\begin{array}{l} \forall \xi \in \mathbb{R}^Q, \xi^t \Phi(t, \tau) B(\tau) = 0, \\ 0 \le \tau \le T \Rightarrow \xi = 0 \end{array}.$$

We claim that if the system is controllable, then $p_0 \ne 0$, otherwise from $\dfrac{\mathrm{d}\lambda}{\mathrm{d}t} = -2p_0\big(Y(t) - a(t)\big) - A(t)^t \lambda(t)$ we would have deduced, $\lambda(t) = \Phi^t(T, t)\lambda(T)$, and $B(t)^t \lambda(t) = 0$. Thus, $\lambda(t)^t \Phi(T, t)B(t) = 0$, $0 \le t \le T$. Hence, $\lambda(T) = 0 \Rightarrow \lambda(t) = 0$, which is one condition since p_0 and $\lambda(t)$ cannot both be equal to zero at a time. Therefore, $p_0 \ne 0$. We can now normalize the system by setting $p_0 = 1$. The path and optimal control are obtained by solving the following system,

$$\begin{cases} \dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} = A(t)Y(t) - \dfrac{1}{2}B(t)^t B(t)\lambda(t) \\ \dfrac{\mathrm{d}\lambda}{\mathrm{d}t} = -2\big(Y(t) - a(t)\big) - A(t)^t \lambda(t) \\ Y(0) = Y_0, Y(t) = K = (Z, \cdots, Z)^t \\ Y_i(T) = Z, i = 1, \cdots, Q \end{cases}$$