

of Yang-Hilbert-type inequalities, enhancing and promoting the classical Hardy-Hilbert inequality theory. In applications, he has formulated numerous Hilbert-type inequalities, connecting their optimal constant factors to special functions like the Riemann-zeta function. He has utilized the Euler-Maclaurin summation formula to establish various Hardy-Hilbert inequalities involving partial sums, extending them to half-discrete and integral inequalities that encompass partial sums, higher-order derivative functions, and multiple upper limit functions.

His prolific output includes over 600 papers published both domestically and internationally, with about 230 indexed in the SCI and 18 appearing in China's national authoritative journals. Additionally, he has authored 18 monographs with Springer and Science Press et al., and 18 monographs (including 22 chapters) with Springer Publishing.

In 2015, he was honored with the title of Science Chinese Person of the Year 2014. In 2021, a video highlighting his scientific research titled "Sticking to the Exploration of Microscopic, Not to Live Up to the Life of Mathematics" was uploaded on the Learning Strong Nation platform. On October 10, 2022, he was listed in Stanford University's Lifetime Impact Ranking (1960-2021) of the world's top $\frac{2}{100}$ of scientists. In December 2023, he was elected as a Life Fellow of the Royal Society for Arts (RSA Fellow no. 8288660). On October 10, 2024, he was listed in Stanford University's Lifetime Impact Ranking (1960-2023) of the world's top $\frac{2}{100}$ of scientists again.

Preface

Hilbert-type inequalities including Hilbert's inequalities (built-in 1908), Hardy-Hilbert-type inequalities (built-in 1934) and Yang-Hilbert-type inequalities (built-in 1998) played an important role in analysis and its applications, which are mainly divided into three classes of integral, discrete and half-discrete. In recent thirty years, there are many advances in research on Hilbert-type inequalities and some kinds of applications, especially in Yang-Hilbert-type inequalities.

In this book, by applying the weight functions, the idea of introduced parameters, the transfer formula, the Euler-Maclaurin summation formula and the techniques of real analysis and functional analysis, we use the well known Hardy's integral inequalities to provide two kinds of extended applications on multidimensional half-discrete Hilbert-type inequalities, involving partial sum, derivative function of higher-order, multiple upper limit function and multiple lower limit function et al., as well as the equivalent statements and the cases of reverses, which mostly published by the journals in recent about ten years.

There are three parts in this book. First part only includes Chapter 1, which introduces the history of Hilbert-type inequalities and some recent developments of them. Second

part includes Chapter 2-Chapter 4, which introduces first kind of extended applications on multidimensional half-discrete Hilbert-type inequalities involving one partial sum. Third part of this book includes Chapter 5-Chapter 10, which introduces second kind of extended applications on multidimensional half-discrete Hilbert-type inequalities involving multiple upper limit function, multiple lower limit function or derivative function of m-order.

We hope that this monograph will prove to be useful especially to graduate students of mathematics, physics, and engineering sciences.

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As long as a branch of knowledge offers an abundance of problems, it is full of vitality.

David Hilbert

... we have always found with most inequalities, that we have a little new to add.

... in a subject (inequalities) like this, which has applications in every part of mathematics but never been developed systematically.

G.H. Hardy

1 Introduction: Research on the Theory of Hilbert-Type Inequalities and Applications

In this chapter, we introduce the development history of Hilbert-type inequalities expressly in Hilbert's inequality, which involves a lot of published results. On the research of introduced parameters and the idea of weight functions in building Hilbert-type inequalities in recent about thirty years, it pushed forward enormously on a step further to explore in this field and applications. As an actor's opening words in this chapter, we list some details for explanation. Some cases of extended applications of Hardy-Hilbert's inequalities are listed.