Preface

Walking on a snowy, narrow road together with my wife in my home-valley, Beiarn, north of the arctic circle, this idea came to me: put two elementary exponential functions with equal exponent, but opposite sign, into the solution. One exponential function outside an integral and one inside the same integral. These two exponential functions will disappear under the derivation, but the exponent will be an important parameter in the system of ODEs.

Coming home I tested this idea, and it worked, first with the Shimizu-Morioka system and then with the Lorenz system. Solution to the chaotic Lorenz system was obtained. A big moment. A long journey had ended.

I invite you to come with me on a mathematical journey from the Jacobi elliptic functions to the expo-elliptic functions, the trig-elliptic functions, the amplitude functions, the one- and two-integral amplitude functions that provide solutions to some well-known second order nonlinear ODEs and systems of ODEs that are exhibiting chaotic behavior. The main goal for this journey is to define a function that provide solution to the chaotic Lorenz system. On our mathematical journey, we will make some necessary stops defining new non-elementary functions.

In this book, we define an exponential function whose exponent is the product of a real number and the upper limit of integration in a non-elementary integral that can be arbitrary. We are using Abel's methods, described by Armitage and Eberlein. The key is to start with a non-elementary integral function, differentiating and inverting, and then define a set of functions. Differentiating these functions twice give second order nonlinear ODEs, that have the defined set of functions as solutions. We will give some examples.

Later in this book, we define a group of solutions $x(t)$ that are sine and cosine to the upper limit of integration in a non-elementary integral. We will also define a group of solutions $x(t)$ that are equal to the amplitude. This is a generalized non-elementary amplitude function. We will also define an exponential function whose exponent is the product of a complex number and the upper limit of integration in a non-elementary integral that can be arbitrary. At least three groups of non-elementary functions are special cases of this complex function.

And finally, we will define an extension to the non-elementary amplitude function, where the first derivative contains one or two integrals, and we will show how these functions can serve as solutions to the forced and damped pendulum equation, the forced and damped Duffing equation, the Ueda chaotic oscillator, the forced Van der Pol equation, the Lorenz system, Rössler system and some more systems that are exhibiting chaotic behavior.

If it is not possible to find a solution to a distinct second order nonlinear ODE, working in the traditional ways, why not just define a solution? How must this solution look like, and where in the landscape of functions can we place this solution? If we are thinking that every reasonable problem has a solution, let us try to define solutions to some of the well-known nonlinear ODEs, like the forced and damped pendulum equation or the Lorenz system with chaotic behavior.

Studying the examples, we will show how these functions can serve as solutions to different kind of ODEs. Some of the functions defined in this book have some amazing and surprising qualities, as we can see in the phase portraits of the second order nonlinear ODEs or systems of ODEs.