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Relative Efficiencies of Optimal Designs in Four Dimensions Constructed Using Balanced Incomplete Block Designs

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Abstract

Experimentally, the best design gives estimates of the desired effects and contrasts with maximum precision. Efficiency as a discriminating factor enables comparison of designs. The goal of Response Surface Methodology (RSM) is the determination of the best settings of the in-put variables for a maximum (or a minimum) response within a region of interest, R. This calls for fitting a model that adequately represents the mean response since such a model, is then used to locate the optimum. D-, A-, E- and T-Optimal designs of a rotatable design of degree two in four dimensions constructed using balanced incomplete block designs (BIBD) when the number of replications is less than three times the number of pairs of treatments occur together in the design and their relative efficiencies to general designs are presented. D-optimal design had 88 runs after replicating the factorial part twice and the axial part thrice with an optimal variance of 0.6965612 giving an efficiency of 97.7% while for A- and T-optimal designs they are formed with 112 runs each obtained by replicating the factorial part two times and axial part six times. Their optimal variances are 0.05798174 and 1.29828 respectively, with efficiency of 71.8% for A-optimal and 87.5% for T-optimal design. E-optimal design was found to be the most efficient design with an only 32 runs comprising only of the factorial part and with an optimal variance of 0.4182000, attaining an efficiency of approximately 1%. This study proposes the adoption of the E-optimal design in estimating the parameters of a rotatable second-order degree model constructed using BIBD for less costs and time saving.

Keywords

Response Surface Methodology, Optimal Designs, Relative Efficiency

1. Introduction

Optimal designs are designs of fewer trials than non-optimal designs which are run in order to get an efficient design for fitting a reduced polynomials of degree two or more. Optimal designs are the obvious choice when the region of exploration is irregular probably due to factor levels constraints or existence of prior information to the experimenter on the process being a non-standard model with some terms of higher order or some interaction terms inclusion failure in the model and the objective is to obtain an efficient design [1]. Optimal designs have been suggested and are used frequently in practice [2]. Reference [3] proposed "OUCD 4" designs as good, space-filling and efficient. The best design among a set of designs, provides the estimate of effects and contrasts with maximum precision (efficiency), with a simple layout and analysis [4]. An appropriate experimental design entails finding the best optimality criterion with larger efficiency values, implying a better design [5]. D-optimal design was employed to study the significance and interactive effect of methanol-to-oil (M:O) molar ratio, catalyst concentration, reaction time, and mixing rate on bio-diesel yield [6]. The adoption of an appropriate experimental design for representing the response surface design influences the efficiency of a design [7]-[9]. Reference [10] studied the measure of efficiency of the design matrix under Latin squares and orthogonality properties of designs. Reference [11] reviewed some fundamentals of experimental design in particular orthogonality and balance, introducing the idea of design efficiency by comparing some widely available design softwares, such as Sawtooth Software's, CVA and SAS Institute's OPTEX programs. Reference [12] demonstrated the use of efficient design (of RSM) as a method to optimize experiments so as to capture better synergies between species and conditions, their work indicated the suitability of the approach to model carbon dioxide corrosion at pH 4 - 5.5.

2. Methodology

Rotatable designs are a class of three-level designs for estimating second-order response surfaces [13]. The designs are rotatable or nearly so with a reduced number of experimental runs by the 3^n designs. They combine 2^n designs with incomplete block designs. Reference [14] gave the conditions for blocking second order response surface designs so that the block effects do not affect the estimates of the parameters for the response surface equation. Spherical variance of the estimation of the response surface, demands that design points within the experimental region satisfy the following conditions

$$\sum_{u=1}^{N} x_{iu} = 0, \sum_{u=1}^{N} x_{iu} x_{ju} = 0,$$
$$\sum_{u=1}^{N} x_{iu}^2 = \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = constant \quad \text{and}$$
$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 \quad \forall i = j \tag{1}$$

A general second degree rotatable design in four factors constructed using balanced incomplete blocks design, when replications (r) are less than three λ (where λ is the number of pairs of treatments occurring together in the design) was put forward by [14], with the coded levels being ±1.137 and ±2.116 for factorial and the axial parts respectively. During parameters (β's) estimations, the $X_{N\times p}$ matrix of levels of independent variables known as the model matrix with $p=k+1$ variables and N being the number of runs is related to the response variable y by the equation

$$y = X\beta + \epsilon \tag{2}$$

where $X = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N1} & x_{N2} & \cdots & x_{Nk} \end{bmatrix}_{N\times p}$ is the model matrix.

Reference [15] worked out design matrix for the second-degree rotatable design constructed using BIBD shown in Table 1.

Table 1. Model matrix.

X	X1	X2	X3	X4	X1X2	X1X3	X1X4	X2X3	X2X4	X3X4	X1^2	X2^2	X3^2	X4^2
1	−1.137	0	−1.137	−1.137	0	1.2928	1.2928	0	0	1.2928	1.2928	0	1.2928	1.2928
1	1.137	0	−1.137	−1.137	0	−1.2928	−1.2928	0	0	1.2928	1.2928	0	1.2928	1.2928
1	−1.137	0	1.137	−1.137	0	−1.2928	1.2928	0	0	−1.2928	1.2928	0	1.2928	1.2928
1	1.137	0	1.137	−1.137	0	1.2928	−1.2928	0	0	−1.2928	1.2928	0	1.2928	1.2928
1	−1.137	0	−1.137	1.137	0	1.2928	−1.2928	0	0	−1.2928	1.2928	0	1.2928	1.2928
1	1.137	0	−1.137	1.137	0	−1.2928	1.2928	0	0	−1.2928	1.2928	0	1.2928	1.2928
1	−1.137	0	1.137	1.137	0	−1.2928	−1.2928	0	0	1.2928	1.2928	0	1.2928	1.2928
1	1.137	0	1.137	1.137	0	1.2928	1.2928	0	0	1.2928	1.2928	0	1.2928	1.2928
1	−1.137	−1.137	0	−1.137	1.2928	0	1.2928	0	1.2928	0	1.2928	1.2928	0	1.2928
1	1.137	−1.137	0	−1.137	−1.2928	0	−1.2928	0	1.2928	0	1.2928	1.2928	0	1.2928
1	−1.137	1.137	0	−1.137	−1.2928	0	1.2928	0	−1.2928	0	1.2928	1.2928	0	1.2928
1	1.137	1.137	0	−1.137	1.2928	0	−1.2928	0	−1.2928	0	1.2928	1.2928	0	1.2928
1	−1.137	−1.137	0	1.137	1.2928	0	−1.2928	0	−1.2928	0	1.2928	1.2928	0	1.2928
1	1.137	−1.137	0	1.137	−1.2928	0	1.2928	0	−1.2928	0	1.2928	1.2928	0	1.2928
1	−1.137	1.137	0	1.137	−1.2928	0	−1.2928	0	1.2928	0	1.2928	1.2928	0	1.2928
1	1.137	1.137	0	1.137	1.2928	0	1.2928	0	1.2928	0	1.2928	1.2928	0	1.2928
1	−1.137	−1.137	−1.137	0	1.2928	1.2928	0	1.2928	0	0	1.2928	1.2928	1.2928	0
1	1.137	−1.137	−1.137	0	−1.2928	−1.2928	0	1.2928	0	0	1.2928	1.2928	1.2928	0
1	−1.137	1.137	−1.137	0	−1.2928	1.2928	0	−1.2928	0	0	1.2928	1.2928	1.2928	0
1	1.137	1.137	−1.137	0	1.2928	−1.2928	0	−1.2928	0	0	1.2928	1.2928	1.2928	0

Continued

1	−1.137	−1.137	1.137	0	1.2928	−1.2928	0	−1.2928	0	0	1.2928	1.2928	1.2928	0
1	1.137	−1.137	1.137	0	−1.2928	1.2928	0	−1.2928	0	0	1.2928	1.2928	1.2928	0
1	−1.137	1.137	1.137	0	−1.2928	−1.2928	0	1.2928	0	0	1.2928	1.2928	1.2928	0
1	1.137	1.137	1.137	0	1.2928	1.2928	0	1.2928	0	0	1.2928	1.2928	1.2928	0
1	0	−1.137	−1.137	−1.137	0	0	0	1.2928	1.2928	1.2928	0	1.2928	1.2928	1.2928
1	0	1.137	−1.137	−1.137	0	0	0	−1.2928	−1.2928	1.2928	0	1.2928	1.2928	1.2928
1	0	−1.137	1.137	−1.137	0	0	0	−1.2829	1.2928	−1.2928	0	1.2928	1.2928	1.2928
1	0	1.137	1.137	−1.137	0	0	0	1.2928	−1.2928	−1.2928	0	1.2928	1.2928	1.2928
1	0	−1.137	−1.137	1.137	0	0	0	1.2928	−1.2928	−1.2928	0	1.2928	1.2928	1.2928
1	0	1.137	−1.137	1.137	0	0	0	−1.2928	1.2928	−1.2928	0	1.2928	1.2928	1.2928
1	0	−1.137	1.137	1.137	0	0	0	−1.2928	−1.2928	1.2928	0	1.2928	1.2928	1.2928
1	0	1.137	1.137	1.137	0	0	0	1.2928	1.2928	1.2928	0	1.2928	1.2928	1.2928
1	2.116	0	0	0	0	0	0	0	0	0	4.4775	0	0	0
1	−2.116	0	0	0	0	0	0	0	0	0	4.4775	0	0	0
1	0	2.116	0	0	0	0	0	0	0	0	0	4.4775	0	0
1	0	−2.116	0	0	0	0	0	0	0	0	0	4.4775	0	0
1	0	0	2.116	0	0	0	0	0	0	0	0	0	4.4775	0
1	0	0	−2.116	0	0	0	0	0	0	0	0	0	4.4775	0
1	0	0	0	2.116	0	0	0	0	0	0	0	0	0	4.4775
1	0	0	0	−2.116	0	0	0	0	0	0	0	0	0	4.4775

The moment matrix of the design is given as

$$M = \frac{X'X}{N} \tag{3}$$

Assuming N is fixed, solutions to parameter estimations consist of developing criterion based on the model to obtain optimal designs [1]. There are many optimality criteria, sometimes called alphabetical optimality criteria and they are simply single number criteria capturing an aspect of the "goodness" of a design and classified into either information-based criteria, distance-based criteria, compound design criteria, etc. Information-based criteria concern the information matrix $X'X$ of the design, which is proportional to the inverse of the variance-covariance matrix for the least-squares estimates of the linear parameters of the model. Further [15] gave details of determination of the moment matrix and the D-, A-, E- and T-optimal values of a general second degree rotatable design in four factors constructed using BIBD as 0.6796529, 0.04104631, 0.002856958 and 1.135448 respectively. According to [10], the efficiency and sensitivity of a design may be very much affected by the choice of the design matrix X. Given $S_{p\times p} = X'X$ a real non-negative symmetric matrix of rank $r \le p$. Let t be a column vector with ρ components not all zero such that the equation $S_\rho = t$ has solution for ρ .If ρ is any solution for $S_\rho = t$ then the inequality

$$\frac{1}{\lambda_{\max}} \le \frac{\rho' S \rho}{t't} \le \frac{1}{\lambda_{\min}}. \tag{4}$$

holds where $\lambda_{\max}$ and $\lambda_{\min}$ are the non-zero characteristics roots of *S*. If we denote with *u*, the maximum value of $\lambda_{\min}$ for $x_{\alpha i}$ in *T* such that *T* is an orthogonal matrix satisfying

$$T'AT = \begin{bmatrix} \lambda_1 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \lambda_n \end{bmatrix}. \tag{5}$$

where *A* is a $k \times k$ matrix such that

$$S^{-1} = \begin{bmatrix} A & B \\ B' & D \end{bmatrix}. \tag{6}$$

When *S* is of full rank, Equation (4) becomes

$$\frac{1}{\lambda_{\max}} \le \frac{t'S^{-1}t}{t't} \le \frac{1}{\lambda_{\min}}. \tag{7}$$

For testing linear hypothesis of the regression coefficient $H_0 : \beta_1 = \beta_2 = \cdots = \beta_r = 0$ when $r \le p$ for a model, the efficiency is a ratio $\frac{\lambda_{\min}}{u}$ and the design is most efficient if the efficiency of the design is equal to one [16]. A uniform design has all regression vectors run an equal number of times. By varying the proportion that a particular vector is run, a design can be made better. Reference [17], outlines the procedure for obtaining the optimal weights of a design using matrix means ϕ_p with $p \in (-\infty, 1]$ which satisfy

$$w_i = \frac{\sqrt{b_{ii}}}{\sum_{j \le N} \sqrt{b_{jj}}} \quad \text{for all } i = 1, \cdots, N. \tag{8}$$

where $b_{11}, \cdots, b_{NN}$ are the diagonal eentries of matrix *B* given as equation (9)

$$B = UC^{p+1}U'. \tag{9}$$

where $U = (XX')^{-1} XK$, *C* being the information matrix, *K* is coefficient matrix and *N* regression vectors $x_1, \cdots, x_N$ forming rows of the design matrix *X*. The w_i forms the proportion each regression vector is run to obtain D-, A-, E- and T-optimal designs. The optimal value (if $p \ne 0$), for $p \in (-\infty, 1]$ for the design is given by Equation (10)

$$V(\phi_P) = \left(\frac{1}{S} trace\, C^P\right)^{1/p} = \left(\frac{1}{S}\left(\sum_{j \le N} \sqrt{b_{jj}}\right)^2\right)^{1/p} \tag{10}$$

2.1. D-Optimal Efficiency

For the corresponding weights, Equation (8) is used by setting $p = 0$ in Equation (9) matrix B_d is obtained and for optimal variance, use Equation (10),

$$B_d = UCU'. \tag{11}$$

Factorial and axial weight corresponding to D-optimal are $D_{fw} = \frac{\sqrt{b_{11}}}{\sum_{j\le N}\sqrt{b_{jj}}}$ and $D_{aw} = \frac{\sqrt{b_{NN}}}{\sum_{j\le N}\sqrt{b_{jj}}}$ respectively with $b_{11},\cdots,b_{NN}$ being diagonal entries of matrix B_d. Replicating the factorial and axial parts as per the weights gives the design matrix X_d. The D-optimal moment matrix $C_d = \frac{X_d' X_d}{N_d}$ with N_d being the number of runs in the design. The corresponding optimal variance for the design is given as $V(\phi_0) = (trace\ C_d)^{\frac{1}{S}}$. S being the number of parameters in the model. Given two designs X_1 and X_2, their relative efficiency as per D-criterion is;

$$D_e = \left[\frac{\left|\left(X_2' X_2\right)^{-1}\right|}{\left|\left(X_1' X_1\right)^{-1}\right|}\right]^{\frac{1}{S}}. \tag{12}$$

S is the number of model parameters [1]. Relative efficiency of the general to D-optimal design is

$$\phi_{Deff(\xi)} = \frac{(\det C)^{\frac{1}{S}}}{(\det C_d)^{\frac{1}{S}}}. \tag{13}$$

2.2. E-Optimal Efficiency

E-optimality aims at minimizing the largest eigen value of $(X'X)^{-1}$ [18]. Let the minimum eigen value of the general design be $\lambda_{\min}(C)$ and the normalized eigen vector be *Z*. If $\lambda_{\min}(C)$ has a multiplicity of one, then matrix $E = \frac{zz'}{z}$ such that trace(*E*) = 1 and $x_i' E x_i \le \lambda_{\min}(C)$ for all $x_i \in \chi$ for $\phi_{-\infty}$-optimal is used to determine the E-optimal weights for factorial and axial parts respectively [17]. The optimal variance is the minimum eigen value of the resulting moment matrix of the design *i.e.* $\lambda_{\min}(C_e)$ where $C_e = E'E/N$ and *N* is the number of rows of matrix *E*. E-efficiency becomes

$$\phi_{Eeff} = \frac{\lambda_{\min}(C)}{\lambda_{\min}(C_e)}. \tag{14}$$

2.3. A-Optimal Efficiency

Substituting $p = -1$ in Equation (9), matrix B_a for computing the A-Optimal weights is

$$B_a = UU'. \tag{15}$$

The factorial (A_{fw}) and axial (A_{aw}) weights are $\frac{\sqrt{b_{11}}}{\sum_{j\le N}\sqrt{b_{jj}}}$ and $\frac{\sqrt{b_{NN}}}{\sum_{j\le N}\sqrt{b_{jj}}}$ respectively with $b_{11},\cdots,b_{NN}$ being diagonal entries of B_a. A-optimal design is by replicating the factorial and axial parts as per the weights to give the design matrix X_a. The moment matrix $C_a=\frac{X_a' X_a}{N_a}$ with N_a runs in the design. Optimal variance for the design is

$$V\left(\phi_{-1}\right)=\left(\frac{1}{15}trace\ C_a^{-1}\right)^{-1}=\left(\frac{1}{15}\left(\sum_{j\le N}\sqrt{b_{jj}}\right)^2\right)^{-1} \tag{16}$$

Relative efficiency of the general to the A-Optimal design is

$$\varphi_{Aeff}=\frac{\left(\frac{1}{15}trace\ C^{-1}\right)^{-1}}{\left(\frac{1}{15}trace\ C_a^{-1}\right)^{-1}}. \tag{17}$$

2.4. T-Optimal Efficiency

By putting $p=1$ in Equation (9), matrix B_t is given as

$$B_t=UC^2U'. \tag{18}$$

Factorial (T_{fw}) and axial (T_{aw}) weights are $\frac{\sqrt{b_{11}}}{\sum_{j\le N}\sqrt{b_{jj}}}$ and $\frac{\sqrt{b_{NN}}}{\sum_{j\le N}\sqrt{b_{jj}}}$ respectively with $b_{11},\cdots,b_{NN}$ being diagonal elements of matrix B_t. T-optimal design is by replicating the factorial and axial parts as per the weights to give the design matrix X_t which is employed to obtain the moment matrix $C_t=\frac{X_t' X_t}{N_t}$ with N_t being the number of runs in the design. Corresponding optimal variance is

$$\begin{aligned}V\left(\phi_1\right)&=\left(\frac{1}{S}trace\ C_t^1\right)^1\\&=\left(\frac{1}{S}\left(\sum_{j\le N}\sqrt{b_{jj}}\right)^2\right)^1 32\times2+8\times3\\&=88\end{aligned} \tag{19}$$

The T-efficiency for T-optimal design is

$$\frac{\phi(C)}{\phi(C_t)}=\frac{\left\{\frac{1}{S}trace\,C\right\}}{\left\{\frac{1}{S}trace\,C_t\right\}}. \tag{20}$$

3. Results

3.1. D-Optimal Design and Its Efficiency

Factorial and axial weights are $D_{fw}=0.02387036$ $D_{aw}=0.02951858$. Hence a

D-optimal design has factorial part replicated twice *i.e.* ($n_f = 2$) and axial part thrice (*i.e.* $n_a = 3$) with matrix *X* having runs. The corresponding D-optimal moment matrix $M_d = \frac{1}{88} X'X$, as shown in Table 2.

Table 2. Moment matrix for the D-Optimal design.

1	0	0	0	0	1.01	0	0	0	1.011	0	0	1.011	0	1.01
0	1.011	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1.011	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1.011	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1.011	0	0	0	0	0	0	0	0	0	0
1.011	0	0	0	0	2.281	0	0	0	0.608	0	0	0.608	0	0.608
0	0	0	0	0	0	0.608	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0.608	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0.608	0	0	0	0	0	0
1.011	0	0	0	0	0.608	0	0	0	2.281	0	0	0.608	0	0.608
0	0	0	0	0	0	0	0	0	0	0.608	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0.608	0	0	0
1.011	0	0	0	0	0.608	0	0	0	0.608	0	0	2.281	0	0.608
0	0	0	0	0	0	0	0	0	0	0	0	0	0.608	0
1.011	0	0	0	0	0.608	0	0	0	0.608	0	0	0.608	0	2.281

The D-optimal value is $V(\phi_0) = (\det M_d)^{1/s}$ *i.e.* $V(\phi_0) = 0.6965612$, the general design value is 0.6796529 according to [15]. Relative efficiency of the general design to the D-optimal design is

$$\phi_{eff(\xi)} = \frac{0.6796529}{0.6965612} = 97.71\% \cong 98\% . \quad (21)$$

3.2. E-Optimal Design and Its Efficiency

The smallest eigen value of general design is $\lambda_{\min}(M_G) = 0.002893284$ with a normalized eigenvector:

$$Z' = [0.895 \quad 0.00 \quad 0.00 \quad 0.00 \quad 0.00 \quad -0.233 \quad 0.00 \quad 0.00 \quad 0.00 \quad -0.233 \quad 0.00 \quad 0.00 \quad -0.233 \quad 0.00 \quad -0.233]$$

Matrix $E = \frac{zz'}{z}$ of trace one and $x'Ex \le \lambda_{\min}(M_G)$ for all $x \in \chi$ for $\phi_{-\infty}$-optimal for all x_i where $i = 1, \cdots, 32$, such that $x'Ex = 0.0009351244$ and for all x_i and $i = 33, \cdots, 40$. $x'Ex = 0.01089277 > \lambda_{\min}(M_G) = 0.002893284$. E-optimal design allocates a weight of 1 to factorial and 0 to the axial part of the design. The $\phi_{-\infty}$-optimal design moment matrix is $M_e = \frac{1}{32} X'X$, see Table 3.

M_e an optimal variance of 0.4182000. The E-efficiency is given by

$$\phi_{Eeff} = \frac{\lambda_{\min}(C)}{\lambda_{\min}(C_e)} = \frac{0.002856958}{0.4182000} = 0.0068315590626 \cong 1\%$$

Table 3. E-Optimal (M_e) moment matrix.

1	0	0	0	0	0.97	0	0	0	0.97	0	0	0.97	0	0.97
0	0.97	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0.97	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.97	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.97	0	0	0	0	0	0	0	0	0	0
0.97	0	0	0	0	1.255	0	0	0	0.836	0	0	0.836	0	0.836
0	0	0	0	0	0	0.836	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0.836	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0.836	0	0	0	0	0	0
0.97	0	0	0		0.836	0	0	0	1.255	0	0	0.836	0	0.836
0	0	0	0	0	0	0	0	0	0	0.836	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0.836	0	0	0
0.97	0	0	0	0	0.836	0	0		0.836	0	0	1.255	0	0.836
0	0	0	0	0	0	0	0	0	0	0	0	0	0.836	0
0.97	0	0	0	0	0.836	0	0	0	0.836	0	0	0.836	0	1.255

3.3. A-Optimal Design and Its Efficiency

The factorial and axial weights are 0.01704711 and 0.05681391 respectively. The optimal design is formed by setting $n_f = 2$ and $n_a = 6$ with $32 \times 2 + 8 \times 6 = 112$ being the total number of runs and A-optimal moment matrix is $M_A = \frac{1}{112} X'X$.

The A-Optimal value is $V_{-1} = \left(\frac{1}{15} trace\, M_a^{-1}\right)^{-1} = 0.05798174$ while the optimal value for the general design is 0.04154701. Giving a relative efficiency of $0.71655334938 \cong 71.7\%$.

3.4. T-Optimal and Its Efficiency

Factorial, $T_{fw} = 0.01691205$ and axial $T_{aw} = 0.05735179$ weights after setting $p = 1$ in Equations (9). Again the T-optimal design is formed by replicating factorial twice ($n_f = 2$) and axial six times ($n_a = 6$) for a total of 112 runs, with a moment matrix as shown in Table 4. T-optimal value $V_1 = \frac{1}{15} trace(M_G) = 1.29828$, giving a relative efficiency of $\frac{1.136227}{1.29828} \cong 87.5\%$.

Table 4. A-Optimal design moment matrix.

1	0	0	0	0	1.034	0	0	0	1.034	0	0	1.034	0	1.034
0	1.034	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1.034	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1.034	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1.034	0	0	0	0	0	0	0	0	0	0
1.034	0	0	0	0	2.867	0	0	0	0.478	0	0	0.478	0	0.478
0	0	0	0	0	0	0.478	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0.478	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0.478	0		0	0	0	0
1.034	0	0	0	0	0.478	0	0	0	2.867	0	0	0.478	0	0.478
0	0	0	0	0	0	0	0	0	0	0.478		0	0	0
0	0	0	0	0	0	0	0	0	0	0	0.478	0	0	0
1.034	0	0	0	0	0.478	0	0	0	0.478	0	0	2.867	0	0.478
0	0	0	0	0	0	0	0	0	0	0	0	0	0.478	0
1.034	0	0	0	0	0.478	0	0	0	0.478	0	0	0.478	0	2.867

4. Conclusion

The weights corresponding to D-, E-, A- and T-optimal designs and the corresponding optimal variances of the optimal designs were determined. The number of runs for the D-optimal design was 88 after replicating the factorial part twice and the axial part thrice with an efficiency of 98% while for A- and T-optimal designs had 112 runs each obtained by replicating the factorial part two times and axial part six times with efficiencies of 71.8% and 87.5% respectively. E-optimal had a relative efficiency of approximately 1% to the general design. Only the factorial part of the general design is carried without replication as per the weights giving only 32 runs, which is the least number of experiments that are required in order to estimate the parameters of the model, thereby cutting costs and the time required in conducting the experiments. Hence to model a process with four input variables using this design constructed using BIBD, E-optimal design is proposed.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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