

Shape and Topological Derivative Using Minmax Method: Application to Linear Thermoelasticity Problem

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Abstract

We consider shape and topological optimization problems for thermoelasticity. A model of thermoelasticity problem is first proposed before presenting the models for which a mathematical analysis will be provided. Under mild conditions, we establish shape and topological derivative results using the minmax method with constrained partial differential equations from the model in the stationary case.

Keywords

Shape Derivative, Topological Derivative, Minmax, Thermoelasticity Problem, Shape Gradient

1. Introduction

In this paper, we deal with thermoelasticity problem by using shape and topological optimization problem and their application in solar energy.

For what purpose, it is important to study one of many efficiency criteria of the material: we want to point out the geometrical and topological properties of the materials. The performance of the used material is linked to its physical properties. But there is a relationship between physical and mechanical aspects of the material. For the solids it is interesting to get information on the deformations field of the solid studied.

The topology of the material may play a principal role for a selected efficiency

criteria. For instance, the homogenization theory allows us to get good physical and topological properties for a material. For thermal solar or photo voltaic energy one of the efficiency criteria is to know which deformation for the used material in order to get the best performance in the output sense.

Topological optimization gives an opportunity to get important informations on the topology of the considered domain in order to optimize at least a criteria. For our study, it permits us to know the optimal deformation in the geometrical and topological point of view in order to have the best distribution of the temperature in the domain. And then, the hope is to improve the output of thermal and photo voltaic systems.

We study the geometrical and topological of solid materials by using tools of partial differential equation (PDE) and the topological optimization. And this may lead to a good selection for the choice of the material in physics, in industry.

Domain optimization is used today in many industrial environments, such as Airbus, for the reduction of structures, the improvement of resistance to vibrations, and many other areas of physics [1]-[3].

In [4], the authors address a shape optimization problem for a thermoelasticity model with uncertainties in the Robin boundary condition. The problem was formulated as the minimization of the volume of the body under an inequality constraint on the expectation. They derived analytical expressions of the shape functional to obtain the shape derivative via second order correlations. An efficient numerical method based on the low rank approximation was proposed. The solution of the optimization problem was implemented numerically via the level method.

The isogeometric approach has been adopted in research areas where sophisticated geometric representations are demanding, such as shell analysis [5], fluid-structure interaction [6], robust mesh [7] and shape design optimization [8] [9]. With respect with thermoelastic behavior, the thermomechanical contact of the mortar problem [10] and material distribution of functionally graded structures [11] [12] were studied using the isogeometric approach. For more information see [13].

In 1995, Rodrigues and Fernandes [14] attempted for the first time solve the problem of optimizing the topology of the thermoelastic structure. Focused on the problem of minimum conformity, they used the homogenization method and the augmented Lagrange method. Li *et al.* [15] applied a scalable thickness design to the displacement minimization problem. Cho and Choi [16] developed a design sensitivity analysis method for weakly coupled thermoelastic elasticity problems and applied this method to solve the minimum conformance topology problem. In the same year, a similar method was used by Zuo *et al.* [17] and was applied to the topological design of thermally actuated folding micro-mechanisms. Xia and Wang [18] applied the method of defining levels with the augmented Lagrange multiplier method to the problem of minimal and comparative conformity results with other methods. Sun and Zhang [19] used independent interpolation models in mechanics and thermal fields for better results in 2009. For more informations

see [20].

Chung *et al.* [21] studied the topological optimization of structures subjected to large deformations due to thermal and mechanical loads, which demonstrated how temperature changes affected the optimized design of large deformation structures. Considering the effect of temperature changes, Deng *et al.* [22] and Yan *et al.* [23] multi-scale simultaneous use formulations to optimize macro-scale topology and micro-scale material configurations. Li *et al.* [24] studied multi-scale optimization based on the level set approach in thermomechanics. Environment and indicated that the porous material proves systematically favored for a coupled multi-physics problem. Zhu *et al.* [25] proposed a temperature-constrained topology optimization for coupled thermomechanical problems and revealed that temperature constraints play an important role in relevant issues. For more reviews on thermoelastic design optimization, readers can refer to Wu *et al.* [26] and [27].

The main objective in this article is to determine the shape and topological derivative of the functional $J(\Omega_t) = J(\Omega_t, u_t)$, where the perturbed domain Ω_t of Ω is defined by $\Omega_t = T_t(\Omega)$ or $\Omega_t = \Omega \setminus \bar{\omega}_t$, depending on the derivative to be calculated.

The paper is organized as follows: In the first section we give the introduction. In the second section we give a modeling of linear thermoelasticity problem and the presentation of the models problem. The section 3 is devoted to shape and topological optimization. This section gives theoretical results in asymptotic analysis. These results allow us to get ideas and information about the topological variation of the domain.

And in the section 4, we give the conclusion and some extensions.

2. Modeling

The model is essentially based on the principle of conservation of the mass and the momentum and the general law of thermoelasticity.

Let Ω be a domain included in big ball \mathcal{D}_0 at time t_0 which is in movement and becomes at time t , $\Omega_t \subset \mathcal{D}_t$. Let $a = (a_1, a_2, a_3)$ be a point of Ω which is $x = (x_1, x_2, x_3)$ in Ω_t . Let $\mathbf{u} = (u_1, u_2, u_3)$ the displacement vector (the deformation) where u_i is given by

$$u_i(a, t) = x_i - a_\alpha \delta_{i\alpha}.$$

Using conservation of mass and the momentum we have

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \text{div} \bar{\bar{\sigma}} + \mathbf{f} \quad (2.1)$$

and

$$\text{div} \mathbf{u} = \frac{\rho_0 - \rho}{\rho} \text{ in } \mathcal{D}_0 \quad (2.2)$$

where ρ_0 (resp. ρ) the volume mass of \mathcal{D}_0 (resp. \mathcal{D}_t).

Suppose that \mathcal{D} is an homogenous medium. The general law of thermoelasticity

is given by

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - 3k\alpha \delta_{ij}.$$

The general law of thermoelasticity, combining with law conservation gives directly in the permanent case

$$-\mu \Delta \mathbf{u} - (\lambda + \mu) \operatorname{grad}(\operatorname{div}(\mathbf{u})) + 3k\alpha \operatorname{grad} \theta = \mathbf{f} \text{ in } \Omega. \tag{2.3}$$

The equation (2.3) relates a thermoelasticity problem. We will add some boundary condition. The temperature θ is solution of a boundary value problem (in our case it is solution of the Neumann Laplacian problem) see the following section.

In the following section we will study these partial differential equations with boundaries conditions. In what follows, we consider the following optimization problem

$$\min \{ J(\Omega), \Omega \subset \mathbb{R}^N \} \tag{2.4}$$

where J is the functional defined by

$$J(\Omega) = \alpha \int_{\Omega} |u_{\Omega} - u_0|^2 \, dx + \beta \int_{\Omega} |\nabla u_{\Omega}|^2 \, dx \tag{2.5}$$

with $u_{\Omega} = (u_1, u_2, u_3)$ is solution to:

$$\begin{cases} -\mu \Delta u_{\Omega} - (\lambda + \mu) \nabla(\operatorname{div} u_{\Omega}) + 3k\alpha \nabla \theta_{\Omega} = \mathbf{f} & \text{in } \Omega \\ -\Delta \theta_{\Omega} = g & \text{in } \Omega \\ \frac{\partial \theta_{\Omega}}{\partial n} = h & \text{on } \partial \Omega \\ \frac{\partial u_{\Omega}}{\partial n} = v_0 & \text{on } \partial \Omega \end{cases} \tag{2.6}$$

$$f \in H^1(\mathbb{R}^3), g \in H^{-1}(\Omega).$$

3. Main Results

In this part, we give the main results of the paper. We first start with the theorem giving the shape derivative and the theorem giving the topological derivative. For this purpose, we rely on the work of the pioneers [28]-[33] on the minmax method.

3.1. Shape Derivative

3.1.1. Preliminaries for the Shape Derivative

We will need some notations. The reader can confer to [34]-[37] for further information.

Notations: For $V \in C^0([0, \tau]; \mathcal{C}_1^0(\mathbb{R}^N, \mathbb{R}^N))$ and the diffeomorphism $T_{\epsilon}(V) = T_{\epsilon}$

$$\frac{d}{d\epsilon} T_{\epsilon}(X) = V(\epsilon, T_{\epsilon}(X)), T_0(X) = X, \frac{dT_{\epsilon}}{d\epsilon} = V(\epsilon) \circ T_{\epsilon}, T_0 = I,$$

where $V(\epsilon)(X) = V(\epsilon, X)$ and I is the identity matrix on \mathbb{R}^N . Moreover

$$\frac{d}{d\epsilon} DT_{\epsilon} = DV(\epsilon) \circ T_{\epsilon} DT_{\epsilon}, DT_0 = I, \frac{d}{d\epsilon} J_{\epsilon} = \operatorname{div} V(\epsilon) \circ T_{\epsilon} J_{\epsilon}, J_0 = 1,$$

where $DV(\epsilon)$ and DT_ϵ are the jacobian matrix of $V(\epsilon)$ and T_ϵ . For $k \geq 1$, $(C_0^k(\mathbb{R}^N, \mathbb{R}^N))$ is the space of k times continuously differentiable functions from \mathbb{R}^N to \mathbb{R}^N going to zero at infinity; for $k = 0$, $(C_0^k(\mathbb{R}^N, \mathbb{R}^N))$ is the space of continuous functions from \mathbb{R}^N to \mathbb{R}^N going to zero at infinity. We shall also use the notation

$$V(\epsilon) := V(\epsilon) \circ T_\epsilon, V\epsilon(X) = V(\epsilon, T_\epsilon(X)), g(\epsilon) := T_\epsilon - I, g(\epsilon)(X) = T_\epsilon(X) - X.$$

Lemma 3.1 Assume that $V \in C^0$, then $f_1 \in C^1$.

For $\tau > 0$ sufficiently small, $J_\epsilon = \det DT_\epsilon = |\det DT_\epsilon| = |J_\epsilon|, 0 \leq \epsilon \leq \tau$, and there exist constants $0 < \alpha < \beta$ such that for all $\xi \in \mathbb{R}^N$, $\alpha|\xi|^2 \leq B(\epsilon)\xi \cdot \xi \leq \beta|\xi|^2$ et $\alpha \leq J_\epsilon \leq \beta$. Moreover, one has:

(i) When ϵ goes to zero, $V_\epsilon \rightarrow V(0)$ in $C_0^1(\mathbb{R}^N, \mathbb{R}^N)$, $DT_\epsilon \rightarrow I$ in $C_0^0(\mathbb{R}^N, \mathbb{R}^N)$, $J_\epsilon \rightarrow 1$ in $C_0^0(\mathbb{R}^N)$, $\frac{DT_\epsilon - I}{\epsilon}$ is bounded in $C_0^0(\mathbb{R}^N, \mathbb{R}^N)$, $\frac{J_\epsilon - 1}{\epsilon}$ is bounded in $C_0^0(\mathbb{R}^N)$.

(ii) When ϵ goes to zero,

$$B(\epsilon) \rightarrow I \text{ in } C_0^0(\mathbb{R}^N, \mathbb{R}^N), \frac{B(\epsilon) - I}{\epsilon} \text{ is bounded in } C_0^0(\mathbb{R}^N, \mathbb{R}^N),$$

$$\begin{aligned} B'(v) &= \text{div}V_\epsilon I - DV_\epsilon - (DV_\epsilon)^* \\ &\rightarrow B'(0) = \text{div}V(0) - DV(0) - (DV(0))^* \text{ in } C_0^0(\mathbb{R}^N, \mathbb{R}^N). \end{aligned}$$

where DV_ϵ is the jacobian matrix of V_ϵ and DV_ϵ^* is the transpose of DV_ϵ .

(iii) Given $h \in H^1(\mathbb{R}^N)$, as ϵ goes to zero,

$$h \circ T_\epsilon \rightarrow h \text{ in } L^2(\Omega), \frac{h \circ T_\epsilon - h}{\epsilon} \text{ is bounded in } L^2(\Omega), \nabla h \cdot V_\epsilon \rightarrow \nabla h \cdot V(0) \text{ in } L^2(\Omega).$$

Proof. see [30] [31].

In this section, we recall the framework used in [37] and extended in [32] for the multivalued case.

Definition 3.1 A Lagrangian is a function of the form:

$$(\epsilon, x, y) \mapsto L(\epsilon, x, y) : [0, \tau] \times X \times Y \rightarrow \mathbb{R}, \tau > 0, \tag{3.1}$$

where Y is a vector space, X is a non-empty subset of a vector space, and $y \mapsto L(\epsilon, x, y)$ is affine. Associate with the parameter ϵ the parametrized min-imax:

$$\begin{aligned} g(\epsilon) &= \inf_{x \in X} \sup_{y \in Y} L(\epsilon, x, y) : [0, \tau] \rightarrow \mathbb{R}, \\ dg(0) &= \lim_{\epsilon \rightarrow 0^+} \frac{g(\epsilon) - g(0)}{\epsilon}. \end{aligned} \tag{3.2}$$

When the limits exist, we shall use the following compact notation:

$$d_\epsilon L(0, x, y) = \lim_{\epsilon \rightarrow 0^+} \frac{L(\epsilon, x, y) - L(0, x, y)}{\epsilon}, \tag{3.3}$$

$$d_x L(\epsilon, x, y; \varphi) = \lim_{\theta \rightarrow 0^+} \frac{L(\epsilon, x + \theta\varphi, y) - L(\epsilon, x, y)}{\theta}, \varphi \in X, \tag{3.4}$$

$$d_y L(\epsilon, x, y; \psi) = \lim_{\theta \rightarrow 0^+} \frac{L(\epsilon, x, y + \theta\psi) - L(\epsilon, x, y)}{\theta}, \quad \psi \in Y. \quad (3.5)$$

The notation $\epsilon \rightarrow 0^+$ and $\theta \rightarrow 0^+$ means that ϵ and θ go to 0 through strictly positive values. Since $L(\epsilon, x, y)$ is affine in y , for all $(\epsilon, x) \in [0, \tau] \times X$,

$$\forall y, \psi \in Y, \quad d_y L(\epsilon, x, y; \psi) = L(\epsilon, x, \psi) - L(\epsilon, x, 0) = d_y L(\epsilon, x, 0; \psi). \quad (3.6)$$

We define the state equation as follows : for all $\epsilon \geq 0$

$$\text{Find } x^\epsilon \in X \text{ such that for all } \psi \in Y, \quad d_y L(\epsilon, x^\epsilon, 0; \psi) = 0. \quad (3.7)$$

The set of solutions (states) x^ϵ at $\epsilon \geq 0$ is denoted:

$$E(\epsilon) = \left\{ x^\epsilon \in X : \forall \psi \in Y, \quad d_y L(\epsilon, x^\epsilon, 0; \psi) = 0 \right\}. \quad (3.8)$$

The adjoint state equation is defined as follows: for all $\epsilon \geq 0$ is:

$$\text{Find } p^\epsilon \in Y \text{ such that for all } \varphi \in X, \quad d_x L(\epsilon, x^\epsilon, p^\epsilon; \varphi) = 0, \quad (3.9)$$

and the set of solutions is denoted $Y(\epsilon, x^\epsilon)$. Finally, the set of minimizers for the minimax is given by:

$$X(\epsilon) = \left\{ x^\epsilon \in X : g(\epsilon) = \inf_{x \in X} \sup_{y \in Y} L(\epsilon, x, y) = \sup_{y \in Y} L(\epsilon, x^\epsilon, y) \right\}. \quad (3.10)$$

Lemma 3.2 *The following properties are satisfied*

1. $\inf_{x \in X} \sup_{y \in Y} L(\epsilon, x, y) = \inf_{x \in E(\epsilon)} L(\epsilon, x, 0)$.
2. The minimax $g(\epsilon) = +\infty$ if and only if $E(\epsilon) = \emptyset$. Hence, $X(\epsilon) = X$.
3. If $E(\epsilon) \neq \emptyset$, then $g(\epsilon) < +\infty$ and

$$X(\epsilon) = \left\{ x^\epsilon \in E(\epsilon) : L(\epsilon, x^\epsilon, 0) = \inf_{x \in E(\epsilon)} L(\epsilon, x, 0) \right\} \subset E(\epsilon). \quad (3.11)$$

Proof. See [32] [37].

Theorem 3.3 *Consider the Lagrangian functional*

$$(\epsilon, x, y) \mapsto L(\epsilon, x, y) : [0, \tau] \times X \times Y \rightarrow \mathbb{R}, \quad \tau > 0, \quad (3.12)$$

where X and Y are vector spaces, and the function $y \mapsto L(\epsilon, x, y)$ is affine.

Let the following hypotheses be satisfied:

- (H0) X is a vector space.
- (H1) For all $\epsilon \in [0, \tau]$, $g(\epsilon)$ is finite, $X(\epsilon) = \{x^\epsilon\}$ and $Y(\epsilon, x^0, x^\epsilon) = \{y^\epsilon\}$ are singletons.
- (H2) $d_\epsilon L(0, x^0, y^0)$ exists for all $\epsilon \in [0, \tau]$ and $y \in Y$.
- (H3) The following limit exists:

$$\lim_{s \rightarrow 0^+, \epsilon \rightarrow 0^+} d_\epsilon L(s, x^0, y^\epsilon) = d_\epsilon L(0, x^0, y^0). \quad (3.13)$$

$$\text{Then } dg(0) \text{ exists and } dg(0) = d_\epsilon L(0, x^0, y^0). \quad (3.14)$$

Proof. See [32] [37].

Theorem 3.4 *Consider the Lagrangian functional*

$$(\epsilon, x, y) \mapsto L(\epsilon, x, y) : [0, \tau] \times X \times Y \rightarrow \mathbb{R}, \quad \tau > 0, \quad (3.15)$$

where X and Y are vector spaces, and the function $y \mapsto L(\epsilon, x, y)$ is affine.

Let the following hypotheses be satisfied:

- (H0) X is a vector space.
- (H1) For all $\epsilon \in [0, \tau]$, $g(\epsilon)$ is finite, $X(\epsilon) = \{x^\epsilon\}$ and $Y(\epsilon, x^0, x^\epsilon) = \{y^\epsilon\}$ are singletons.
- (H2') $d_\epsilon L(0, x^0, y^0)$ exists;
- (H3') the following limit exists

$$R(x^0, y^0) = \lim_{\epsilon \rightarrow 0} d_y L\left(\epsilon, x^0, 0, \frac{y^\epsilon - y^0}{\epsilon}\right).$$

Then, the differential $dg(0)$ exists and $dg(0) = d_\epsilon L(0, x^0, y^0) + R(x^0, y^0)$. Notice that, under condition (H2'), condition (H3') is optimal since

$$dg(0) \text{ exists} \Leftrightarrow \lim_{\epsilon \rightarrow 0} d_y L\left(\epsilon, x^0, 0, \frac{y^\epsilon - y^0}{\epsilon}\right) \text{ exists.}$$

Hypotheses (H2') and (H3') are weaker and more general than (H2) and (H3). Indeed, it is readily seen that if (H2) and (H3) are verified, then (H2') and (H3') are verified with $R(x^0, y^0) = 0$.

Proof. See [32] [37].

In the following we will use a new condition with the standard adjoint at $\epsilon = 0$. The use of the averaged adjoint revealed the possible occurrence of an extra term and provided a simpler expression for the former hypothesis (H3). It turns out that the extra term can also be obtained by using the standard adjoint at $\epsilon = 0$, significantly simplifying the checking of that condition.

Theorem 3.5 Consider the Lagrangian functional

$$L(\epsilon, x, y) : [0, \tau] \times X \times Y \rightarrow \mathbb{R}, \tau > 0,$$

where X and Y are vector spaces and the function $y \rightarrow L(\epsilon, x, y)$ is affine. Let (H0) and the following hypotheses be satisfied:

- (H1): For all $\epsilon \in [0, \tau]$, $g(\epsilon)$ is finite, $X(\epsilon) = \{x^\epsilon\}$, and $Y(0, x^0) = \{p^0\}$ are singletons.
- (H2''): $d_\epsilon L(0, x^0, p^0)$ exists.
- (H3''): The following limit exists:

$$R(x^0, p^0) = \lim_{\epsilon \rightarrow 0^+} \int_0^1 d_x L\left(\epsilon, x^0, \theta(x^\epsilon - x^0), p^0, \frac{x^\epsilon - x^0}{\epsilon}\right) d\theta.$$

Then, $dg(0)$ exists and $dg(0) = d_\epsilon L(0, x^0, p^0) + R(x^0, p^0)$.

Proof.

Recalling that $g(\epsilon) = L(\epsilon, x^\epsilon, y)$ and $g(0) = L(0, x^0, y)$ for any $y \in Y$, then for the standard adjoint state p^0 at $\epsilon = 0$:

$$g(\epsilon) - g(0) = L(\epsilon, x^\epsilon, p^0) - L(\epsilon, x^0, p^0) + (L(\epsilon, x^0, p^0) - L(0, x^0, p^0)).$$

Dividing by $\epsilon > 0$:

$$\frac{g(\epsilon) - g(0)}{\epsilon} = \frac{L(\epsilon, x^\epsilon, p^0) - L(\epsilon, x^0, p^0)}{\epsilon} + \frac{L(\epsilon, x^0, p^0) - L(0, x^0, p^0)}{\epsilon}.$$

Therefore, in view of Hypothesis (H2ⁿ), the limit $dg(0)$ exists if and only if the limit of the first term exists. Thus,

$$dg(0) = \lim_{\epsilon \rightarrow 0^+} \int_0^1 d_x L \left(\epsilon, x^0, \theta(x^\epsilon - x^0), p^0, \frac{x^\epsilon - x^0}{\epsilon} \right) d\theta + d_\epsilon L(0, x^0, p^0)$$

and the existence of the limit of the first term replaces hypothesis (H3ⁿ).

3.1.2. Shape Derivative for Funtional

In what follows, we will apply the previous results to our model problem and calculate the derivative associated with the functional (2.5). This, therefore, requires first verifying the existence of solutions to the model given by the partial differential equation (2.6) and verifying the Lagragian differentiability associated with the form funtional (2.5).

The following theorem gives the main result of the form derivative of the functional.

Theorem 3.6 *The shape derivative exists if and only if $\lim_{\epsilon \rightarrow 0} R(\epsilon)$ exists with*

$$\begin{aligned} R(\epsilon) = & \int_{\Omega} 2\beta \left[\frac{B(\epsilon) - I}{\epsilon} \right] \nabla \left(\frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} \right) \nabla (u_{\Omega}^{\epsilon} - u_{\Omega}^0) + \int_{\Omega} \mu \left[\frac{B(\epsilon) - I}{\epsilon} \right] \nabla (u_{\Omega}^{\epsilon} - u_{\Omega}^0) \nabla p^0 dx \\ & - \int_{\Omega} \beta \left| \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\sqrt{\epsilon}} \right) \right|^2 dx + \int_{\Omega} 2\alpha \left[\frac{J(\epsilon) - 1}{\epsilon} \right] \left[\frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} - u_0 \circ T_{\epsilon} \right] (u_{\Omega}^{\epsilon} - u_{\Omega}^0) dx \\ & + \int_{\Omega} 2\alpha [u_0 \circ T_{\epsilon} - u_0] \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) dx - \alpha \int_{\Omega} \left| \frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\sqrt{\epsilon}} \right|^2 dx. \end{aligned}$$

If $R(\epsilon) \rightarrow 0$ the shape derivative is given by the expression:

$$\begin{aligned} DJ(\Omega, V(0)) = & \int_{\Omega} \left[\beta |\nabla u_{\Omega}^0|^2 \operatorname{div} V(0) + \alpha |u_{\Omega}^0 - u_0|^2 \operatorname{div} V(0) - 2(u_{\Omega}^0 - u_0) \nabla u_0 V(0) \right] dx \\ & + \int_{\Omega} \mu B'(0) \nabla u_{\Omega}^0 \nabla p^0 dx - \int_{\Omega} [fp^0 + 3k\alpha \nabla \theta_{\Omega} p^0] \operatorname{div} V(0) dx. \end{aligned}$$

Proof. The shape functional associated the perforated domain is given by

$$j(\chi_{\epsilon}(x_0)) = J(\Omega) = \alpha \int_{\Omega_{\epsilon}} |u_{\Omega_{\epsilon}} - u_0|^2 dx + \beta \int_{\Omega_{\epsilon}} |\nabla u_{\Omega_{\epsilon}}|^2 dx \tag{3.16}$$

where $u_{\Omega_{\epsilon}}$ is solution the variational problem

$$\mu \int_{\Omega_{\epsilon}} \nabla u_{\Omega_{\epsilon}} \cdot \nabla v dx - (\lambda + \mu) \int_{\Omega_{\epsilon}} \nabla (v \cdot u_{\Omega_{\epsilon}}) v dx - 3k\alpha \int_{\Omega_{\epsilon}} \nabla \theta_{\Omega_{\epsilon}} v = \int_{\Omega_{\epsilon}} f v dx. \tag{3.17}$$

and

$$\int_{\Omega_{\epsilon}} \nabla \theta_{\Omega_{\epsilon}} \nabla \varphi dx = \int_{\Omega_{\epsilon}} g \varphi dx \tag{3.18}$$

for all $v \in H_{\operatorname{div}}(\Omega, \mathbb{R}^3)$ and $\varphi \in H_{\operatorname{div}}$ for some given functions $f \in H^1(\mathbb{R}^3)$, $g \in H^{-1}(\Omega)$.

In the case where $\operatorname{div} u_{\Omega} = 0$ in (3.17) is reduced to the following variational problem

$$\mu \int_{\Omega_{\epsilon}} \nabla u_{\Omega_{\epsilon}} \cdot \nabla \theta v dx = \int_{\Omega_{\epsilon}} f v dx + 3k\alpha \int_{\Omega_{\epsilon}} \nabla \theta_{\Omega_{\epsilon}} v dx \tag{3.19}$$

and θ_Ω solution to (3.18). Before starting the formal derivative, will first need to define the following set.

$$H_{div(\Omega_\epsilon, \mathbb{R}^3)} = \left\{ \phi \circ T_\epsilon^{-1}, \phi \in H_{div(\Omega, \mathbb{R}^3)} \right\}.$$

We also introduce

$$u_\Omega^\epsilon = u_{\Omega_\epsilon} \circ T_\epsilon \quad v_0 = v \circ T_\epsilon.$$

So, we aim to find $u_\Omega^\epsilon \in H_{div(\Omega, \mathbb{R}^3)}$ thus that

$$\int_{\Omega_\epsilon} \mu \nabla (u_\Omega^\epsilon \circ T_\epsilon^{-1}) \cdot \nabla (v_0 \circ T_\epsilon^{-1}) - f(v_0 \circ T_\epsilon^{-1}) - 3k\alpha \nabla \theta_\Omega (v_0 \circ T_\epsilon^{-1}) \, dx = 0 \quad \forall v_0 \in H_{div(\Omega, \mathbb{R}^3)}. \tag{3.20}$$

By applying the change of variable formula, we have

$$\int_\Omega \left[\mu \nabla (u_\Omega^\epsilon \circ T_\epsilon^{-1}) \cdot \nabla (v_0 \circ T_\epsilon^{-1}) - f(v_0 \circ T_\epsilon^{-1}) - 3k\alpha \nabla \theta_\Omega (v_0 \circ T_\epsilon^{-1}) \right] \circ T_\epsilon J_\epsilon \, dx = 0.$$

And on the other hand, we also have

$$\begin{aligned} & \int_\Omega \left[f(v_0 \circ T_\epsilon^{-1}) + 3k\alpha \nabla \theta_\Omega (v_0 \circ T_\epsilon^{-1}) \right] \circ T_\epsilon J_\epsilon \, dx \\ &= \int_\Omega \left[f(v_0 \circ T_\epsilon^{-1} \circ T_\epsilon) + 3k\alpha \nabla \theta_\Omega (v_0 \circ T_\epsilon^{-1} \circ T_\epsilon) \right] J_\epsilon \, dx. \end{aligned}$$

And thus, we obtain

$$\int_\Omega \left[f(v_0 \circ T_\epsilon^{-1}) + 3k\alpha \nabla \theta_\Omega (v_0 \circ T_\epsilon^{-1}) \right] \circ T_\epsilon J_\epsilon \, dx = \int_\Omega [f v_0 + 3k\alpha \nabla \theta_\Omega v_0] J_\epsilon \, dx.$$

So, we have

$$\begin{aligned} & \int_{\Omega_\epsilon} \left[\mu \nabla (u_\Omega^\epsilon \circ T_\epsilon^{-1}) \cdot \nabla (v_0 \circ T_\epsilon^{-1}) \right] \circ T_\epsilon J_\epsilon \, dx = \int_\Omega \mu D T_\epsilon^{-1} (D T_\epsilon^{-1})^* \nabla u_\Omega^\epsilon \nabla v_0 J_\epsilon \, dx, \\ & \int_{\Omega_\epsilon} \left[\mu \nabla (u_\Omega^\epsilon \circ T_\epsilon^{-1}) \cdot \nabla (v_0 \circ T_\epsilon^{-1}) \right] \circ T_\epsilon J_\epsilon \, dx = \int_\Omega \mu B(\epsilon) \nabla u_\Omega^\epsilon \nabla v_0 \, dx \end{aligned}$$

with $B(\epsilon) = D T_\epsilon^{-1} (D T_\epsilon^{-1})^* J_\epsilon, J_\epsilon = \det D T_\epsilon$

$D T_\epsilon$ denotes the Jacobian matrix of T_ϵ . And we have

$$\int_\Omega \mu B(\epsilon) \nabla u_\Omega^\epsilon \nabla v_0 \, dx - \int_\Omega [f v_0 + 3k\alpha \nabla \theta_\Omega v_0] J_\epsilon \, dx. \tag{3.21}$$

Let us recall that the shape functional is given by

$$\begin{aligned} J(\Omega_\epsilon) &= \beta \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^2 \, dx + \alpha \int_{\Omega_\epsilon} |u_{\Omega_\epsilon} - u_0|^2 \, dx \\ J(\Omega_\epsilon) &= \beta \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^2 J_\epsilon \, dx + \alpha \int_{\Omega_\epsilon} |u_{\Omega_\epsilon} \circ T_\epsilon - u_0 \circ T_\epsilon|^2 J_\epsilon \, dx. \end{aligned}$$

And on the other hand, we also have

$$J(\Omega_\epsilon) = \int_\Omega \beta |\nabla u_\Omega^\epsilon|^2 J_\epsilon \, dx + \int_\Omega |\alpha u_\Omega^\epsilon - u_0 \circ T_\epsilon|^2 J_\epsilon \, dx.$$

We now define the Lagrangian in terms of ϵ .

$$\begin{aligned} L(\epsilon, \phi, \Phi) &= \int_\Omega \left[\beta |\nabla \phi|^2 J_\epsilon + \alpha |\phi - u_0 \circ T_\epsilon|^2 J_\epsilon \right] \, dx + \int_\Omega \mu B(\epsilon) \nabla \phi \nabla \Phi \, dx \\ &\quad - \int_\Omega [f \Phi + 3k\alpha \nabla \theta_\Omega \Phi] J_\epsilon \, dx. \end{aligned}$$

$$g(\epsilon) = \inf_{\phi \in H_{div}} \sup_{\Phi \in H_{div}} L(\epsilon, \phi, \Phi), \quad dg(0) = \lim_{\epsilon \rightarrow 0} \frac{g(\epsilon) - g(0)}{\epsilon} = dJ(\Omega, V(0)).$$

Let evaluate the derivative of the Lagrangian by first setting

$$V_\epsilon = V(\epsilon) \circ T_\epsilon, \quad V_\epsilon(X) = V(\epsilon, T_\epsilon(X)), \quad g(\epsilon) = T_\epsilon - I.$$

$$\begin{aligned} d_\epsilon L(\epsilon, \phi, \Phi) &= \int_\Omega \left[\beta |\nabla \phi|^2 \operatorname{div} V(\epsilon) \circ T_\epsilon J_\epsilon + \alpha |\phi - u_0 \circ T_\epsilon|^2 \operatorname{div} V(\epsilon) \circ T_\epsilon J_\epsilon \right] dx \\ &\quad - \int_\Omega \left[2(\phi - u_0 \circ T_\epsilon) \nabla u_0 V(\epsilon) \circ T_\epsilon J_\epsilon \right] dx + \int_\Omega \mu B'(\epsilon) \nabla \phi \nabla \Phi dx \\ &\quad - \int_\Omega \left[f \Phi + 3k\alpha \nabla \theta_\Omega \Phi \right] \operatorname{div} V(\epsilon) \circ T_\epsilon J_\epsilon dx. \end{aligned}$$

By taking $\phi = u_\Omega^0$ and $\Phi = p^0$ we have

$$\begin{aligned} d_\epsilon L(\epsilon, u_\Omega^0, p^0) &= \int_\Omega \left[\beta |\nabla u_\Omega^0|^2 \operatorname{div} V(\epsilon) \circ T_\epsilon J_\epsilon + \alpha |u_\Omega^0 - u_0 \circ T_\epsilon|^2 \operatorname{div} V(\epsilon) \circ T_\epsilon J_\epsilon \right] dx \\ &\quad - \int_\Omega \left[2(u_\Omega^0 - u_0 \circ T_\epsilon) \nabla u_0 V(\epsilon) \circ T_\epsilon J_\epsilon \right] dx + \int_\Omega \mu B'(\epsilon) \nabla u_\Omega^0 \nabla p^0 dx \\ &\quad - \int_\Omega \left[f p^0 + 3k\alpha \nabla \theta_\Omega p^0 \right] \operatorname{div} V(\epsilon) \circ T_\epsilon J_\epsilon dx. \end{aligned}$$

So, we obtain

$$\begin{aligned} &d_\epsilon L(0, u_\Omega^0, p^0) \\ &= \int_\Omega \left[\beta |\nabla u_\Omega^0|^2 \operatorname{div} V_0 + \alpha |u_\Omega^0 - u_0|^2 \operatorname{div} V(0) - 2(u_\Omega^0 - u_0) \nabla u_0 V(0) \right] dx \quad (3.22) \\ &\quad + \int_\Omega \mu B'(0) \nabla u_\Omega^0 \nabla p^0 dx - \int_\Omega \left[f p^0 + 3k\alpha \nabla \theta_\Omega p^0 \right] \operatorname{div} V(0) dx. \end{aligned}$$

For the derivative of the Lagrangian with respect to ϕ we have

$$\begin{aligned} &d_\phi L(\epsilon, \phi, \Phi, \phi') \\ &= \int_\Omega \left[2\beta B(\epsilon) \nabla \phi \nabla \phi' + 2\alpha (\phi - u_0 \circ T_\epsilon) \phi' J_\epsilon \right] dx + \int_\Omega \mu B(\epsilon) \nabla \phi' \nabla \Phi dx. \quad (3.23) \end{aligned}$$

And on the other hand, the derivative of the Lagrangian with respect to Φ is given by

$$d_\Phi L(\epsilon, \phi, \Phi, \Phi') = \int_\Omega \mu B(\epsilon) \nabla \phi \nabla \Phi' dx - \int_\Omega \left[f \Phi' + 3k\alpha \nabla \theta_\Omega \Phi' \right] J_\epsilon dx. \quad (3.24)$$

So, we have the state equation for $\epsilon \geq 0$ is given by

$$u_\Omega^\epsilon \in H_{div}, \quad \nabla \Phi' \in H_{div}, \quad \int_\Omega \left[\mu B(\epsilon) \nabla u_\Omega^\epsilon \nabla \Phi' - (f \Phi' + 3k\alpha \nabla \theta_\Omega \Phi') J_\epsilon \right] dx = 0. \quad (3.25)$$

And the adjoint state equation for $\epsilon = 0$ is also given by

$$p^0 \in H_{div}, \quad \forall \phi' \in H_{div}, \quad \int_\Omega \left[2\beta \nabla u_\Omega^0 \nabla \phi' + 2\alpha (u_\Omega^0 - u_0) \phi' + \mu \nabla \phi' \nabla p^0 \right] dx = 0. \quad (3.26)$$

$$R(\epsilon) = \int_0^1 d_\phi L \left(\epsilon, u_{\Omega_0} + \Psi(u_\Omega^\epsilon - u_\Omega^0), p_0, \left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon} \right) \right) d\Psi.$$

$$\begin{aligned} R(\epsilon) &= \int_\Omega 2\beta B(\epsilon) \nabla \left(\frac{u_\Omega^\epsilon + u_\Omega^0}{2} \right) \nabla \left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon} \right) \\ &\quad + 2\alpha \left[\left(\frac{u_\Omega^\epsilon + u_\Omega^0}{2} \right) - u_0 \circ T_\epsilon \right] \left(\frac{u_\Omega^\epsilon - u_\Omega^0}{\epsilon} \right) J_\epsilon dx \end{aligned}$$

$$+ \int_{\Omega} \mu B(\epsilon) \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) \nabla p^0 dx.$$

By substituting $\phi' = \frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon}$ into the adjoint equation for p^0 , we obtain:

$$\int_{\Omega} 2\beta \nabla u_{\Omega}^0 \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) + 2\alpha (u_{\Omega}^0 - u_0) \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) + \mu \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) \nabla p^0 dx = 0.$$

So, we have

$$\begin{aligned} & \int_{\Omega} 2\beta \left[\nabla \left(\frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} \right) - \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{2} \right) \right] \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) + \mu \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) \nabla p^0 dx \\ & + \int_{\Omega} 2\alpha \left[\frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} - \frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{2} - u_0 \circ T_{\epsilon} + u_0 \circ T_{\epsilon} - u_0 \right] \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) = 0. \end{aligned}$$

As a result, we have

$$\begin{aligned} & \int_{\Omega} 2\beta \nabla \left(\frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} \right) \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) + \int_{\Omega} \mu \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) \nabla p^0 dx - \alpha \int_{\Omega} \left| \frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\sqrt{\epsilon}} \right|^2 \\ & - \int_{\Omega} \beta \left| \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\sqrt{\epsilon}} \right) \right|^2 + \int_{\Omega} 2\alpha \left[\frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} - u_0 \circ T_{\epsilon} \right] \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) \\ & + \int_{\Omega} 2\alpha [u_0 \circ T_{\epsilon} - u_0] \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) = 0. \end{aligned}$$

We can now rewrite the expression of $R(\epsilon)$ as follows.

$$\begin{aligned} R(\epsilon) &= \int_{\Omega} 2\beta \left[\frac{B(\epsilon) - I}{\epsilon} \right] \nabla \left(\frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} \right) \nabla (u_{\Omega}^{\epsilon} - u_{\Omega}^0) \\ & + \int_{\Omega} \mu \left[\frac{B(\epsilon) - I}{\epsilon} \right] \nabla (u_{\Omega}^{\epsilon} - u_{\Omega}^0) \nabla p^0 dx - \alpha \int_{\Omega} \left| \frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\sqrt{\epsilon}} \right|^2 \\ & - \int_{\Omega} \beta \left| \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\sqrt{\epsilon}} \right) \right|^2 + \int_{\Omega} 2\alpha \left[\frac{J(\epsilon) - 1}{\epsilon} \right] \left[\frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} - u_0 \circ T_{\epsilon} \right] (u_{\Omega}^{\epsilon} - u_{\Omega}^0) dx \\ & + \int_{\Omega} 2\alpha [u_0 \circ T_{\epsilon} - u_0] \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) dx. \end{aligned}$$

From this, we have the following estimate

$$\begin{aligned} |R(\epsilon)| &\leq \int_{\Omega} 2|\beta| \left\| \frac{B(\epsilon) - I}{\epsilon} \right\| \left\| \nabla \left(\frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} \right) \right\| \left\| \nabla (u_{\Omega}^{\epsilon} - u_{\Omega}^0) \right\| \\ & + \int_{\Omega} |\mu| \left\| \frac{B(\epsilon) - I}{\epsilon} \right\| \left\| \nabla (u_{\Omega}^{\epsilon} - u_{\Omega}^0) \right\| \left\| \nabla p^0 \right\| dx \\ & + |\alpha| \int_{\Omega} \left\| \frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\sqrt{\epsilon}} \right\|^2 + \int_{\Omega} |\beta| \left\| \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\sqrt{\epsilon}} \right) \right\|^2 \\ & + \int_{\Omega} 2|\alpha| \left\| \frac{J(\epsilon) - 1}{\epsilon} \right\| \left\| \frac{u_{\Omega}^{\epsilon} + u_{\Omega}^0}{2} - u_0 \circ T_{\epsilon} \right\| \left\| u_{\Omega}^{\epsilon} - u_{\Omega}^0 \right\| \end{aligned}$$

$$+ \int_{\Omega} 2|\alpha| \|u_0 \circ T_{\epsilon} - u_0\| \left\| \frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right\| dx.$$

By lemma 3.1 the terms $\frac{B(\epsilon)-I}{\epsilon}$ and $\frac{J(\epsilon)-1}{\epsilon}$ are uniformly bounded. To conclude that the limit of $R(\epsilon)$ exists and is zero, it remains to show that $u_{\Omega}^{\epsilon} \rightarrow u_{\Omega}^0$ in H_{div} -strong and that the L^2 norm of $\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon}$ is bounded. From the state equation (3.25) of u_{Ω}^{ϵ} and u_{Ω}^0 , for $\Phi' \in H_{div}$:

$$\begin{aligned} \int_{\Omega} \nabla(u_{\Omega}^{\epsilon} - u_{\Omega}^0) \nabla \Phi' &= \int_{\Omega} 3k\alpha(J(\epsilon)-1) \nabla \theta_{\Omega} \Phi' dx + \int_{\Omega} f \Phi' dx \\ &\quad - \int_{\Omega} \mu(B(\epsilon)-I) \nabla u_{\Omega}^{\epsilon} \nabla \Phi' dx. \end{aligned}$$

Substitute $\Phi' = u_{\Omega}^{\epsilon} - u_{\Omega}^0$ to obtain the following estimate

$$\begin{aligned} \left\| \nabla(u_{\Omega}^{\epsilon} - u_{\Omega}^0) \right\|^2 &\leq 3|k\alpha| \|J(\epsilon)-1\| \|\nabla \theta_{\Omega}\| \|u_{\Omega}^{\epsilon} - u_{\Omega}^0\| + \|f\| \|u_{\Omega}^{\epsilon} - u_{\Omega}^0\| \\ &\quad + |\mu| \|B(\epsilon)-I\| \|u_{\Omega}^{\epsilon} - u_{\Omega}^0\| \left\| \nabla(u_{\Omega}^{\epsilon}) \right\|. \end{aligned}$$

So Ω is a bounded open lipschitzian domain, there exists a constant such that

$$\|u_{\Omega}^{\epsilon} - u_{\Omega}^0\| \leq C(\Omega) \left\| \nabla(u_{\Omega}^{\epsilon} - u_{\Omega}^0) \right\|$$

and we have

$$\begin{aligned} \left\| \nabla(u_{\Omega}^{\epsilon} - u_{\Omega}^0) \right\| &\leq 3|k\alpha| \|J(\epsilon)-1\| \|\nabla \theta_{\Omega}\| C(\Omega) + \|f\| C(\Omega) \\ &\quad + |\mu| \|B(\epsilon)-I\| C(\Omega) \left\| \nabla(u_{\Omega}^{\epsilon}) \right\| \end{aligned}$$

but the right-hand side of this inequality goes to zero as ϵ goes to zero. Therefore $u_{\Omega}^{\epsilon} \rightarrow u_{\Omega}^0$ in H_{div} . Finally, going back to the inequality and dividing by $\epsilon > 0$ and we have

$$\begin{aligned} \left\| \nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right) \right\| &\leq 3|k\alpha| \left\| \frac{J(\epsilon)-1}{\epsilon} \right\| \|\nabla \theta_{\Omega}\| C(\Omega) + \left\| \frac{f}{\epsilon} \right\| C(\Omega) \\ &\quad + |\mu| \left\| \frac{B(\epsilon)-I}{\epsilon} \right\| C(\Omega) \left\| \nabla(u_{\Omega}^{\epsilon}) \right\|. \end{aligned}$$

Since the right hand of the above inequality is bounded, $\nabla \left(\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon} \right)$ is bounded. This means that $\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon}$ is bounded in H_{div} and hence $\frac{u_{\Omega}^{\epsilon} - u_{\Omega}^0}{\epsilon}$ is bounded in L^2 . And the term $R(\epsilon)$ is zero.

The shape derivative is given by

$$\begin{aligned} &DJ(\Omega, V(0)) \\ &= \int_{\Omega} \left[\beta |\nabla u_{\Omega}^0|^2 \operatorname{div} V_0 + \alpha |u_{\Omega}^0 - u_0|^2 \operatorname{div} V(0) - 2(u_{\Omega}^0 - u_0) \nabla u_0 V(0) \right] dx \\ &\quad + \int_{\Omega} \mu B'(0) \nabla u_{\Omega}^0 \nabla p^0 dx - \int_{\Omega} \left[f p^0 + 3k\alpha \nabla \theta_{\Omega} p^0 \right] \operatorname{div} V(0) dx. \end{aligned}$$

3.2. Topological Derivative

In this part we give the fundamental results of the topological derivative of the functional. Before tackling this main result, we will need the following preliminary results.. For more informations, the reader can refer to Delfour papers [29] [32] [37]-[41].

3.2.1. Preliminaries for Topological Derivative

Definition 3.2 Given d , $0 \leq d \leq N$, the d -dimensional upper and lower Minkowski contents of a set E are defined through an r -dilatation of the set E as follows:

$$M^{*d}(E) = \limsup_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d} r^{N-d}}, \quad M_*^d(E) = \liminf_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d} r^{N-d}}.$$

where m_N is the Lebesgue measure in \mathbb{R}^N and α_{N-d} is the volume of the ball of radius one in \mathbb{R}^{N-d} . When the two limits exist and are equal, we say that E admits a d -dimensional Minkowski content, and their common value will be denoted $M^d(E)$.

Definition 3.3 Let E be a subset of a metric space X . We say that E is d -rectifiable if it is the image of a compact subset K of \mathbb{R}^d by a Lipschitz continuous function $f: \mathbb{R}^d \rightarrow X$.

Definition 3.4 Let $E \subset \mathbb{R}^N$ be H^d -measurable. We say that E is

1) countably d -rectifiable if there exist countably many Lipschitzian functions $f_i: \mathbb{R}^d \rightarrow \mathbb{R}^N$ such that $E \subset \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^d)$.

2) countably H^d -rectifiable if there exist countably many Lipschitz functions $f_i: \mathbb{R}^d \rightarrow \mathbb{R}^N$ such that $E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^d)$ is H^d -negligible:

$$H^d\left(E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^d)\right) = 0.$$

3) H^d -rectifiable if it is countably H^d -rectifiable and $H^d(E) < \infty$.

Definition 3.5 Let E be a closed subset of \mathbb{R}^N .

1. The set of points of \mathbb{R}^N which have a unique projection on E .

$$\text{Unp}(E) = \{y \in \mathbb{R}^N : \exists \text{ an unique } x \in E \text{ such that } d_E(y) = \|y - x\|\},$$

where $d_E(y)$ is the distance function from a point y to E .

2. The reach of a point $x \in E$ and the reach of E are respectively defined as follows:’

$$\text{reach}(E, x) = \sup\{r > 0 : B(x, r) \subset \text{Unp}(E)\}, \quad \text{reach}(E) = \inf_{x \in E} \text{reach}(E, x).$$

Remark 3.1 We say that E is a positive reach if $\text{reach}(E) > 0$.

The definition of $\text{Unp}(E)$ implies the existence of a projection on E , a function $p_E: \text{Unp}(E) \rightarrow E$ which associates $x \in \text{Unp}(E)$ such that $d_E(x) = \|x - p_E(x)\|$.

Definition 3.6 Let E be a closed subset of \mathbb{R}^N such that $E_r \subset \Omega$ for some $r > 0$, and f be a continuous function on a bounded open subset Ω of \mathbb{R}^N . The d -dimensional topological derivative of the volume functional

$$V(\chi_\Omega) = \int_{\mathbb{R}^N} f \chi_\Omega dx = \int_\Omega f dx,$$

with respect to E is defined as follows:

$$dV(\chi_\Omega; \delta E) = \lim_{r \searrow 0} \frac{V(\chi_\Omega \setminus E_r) - V(\chi_\Omega)}{\alpha_{N-d} r^{N-d}},$$

whenever the limit exists.

Theorem 3.7 Let E be a compact subset of \mathbb{R}^N and $0 \leq d < N$ an integer.

Let α_k denote the volume of the unit ball in \mathbb{R}^k . We assume that the following properties hold:

- 1) E is a d -rectifiable subset of \mathbb{R}^N such that $\partial E = E$ and $0 < H^d(E) < \infty$;
- 2) E has positive reach; that is, there exists $R > 0$ such that $d_E^2 \in C^{1,1}(E_R)$;
- 3) f is continuous in E_R .

$$\int_E f dH^d = \lim_{r \searrow 0} \frac{1}{(N-d)\alpha_{N-d} r^{N-d}} \int_{E_r} f d\psi = \lim_{r \searrow 0} \frac{1}{(N-d)\alpha_{N-d} r^{N-d}} \int_{E_r} f \circ p_E d\psi.$$

$$\int_E f dH^d = \lim_{r \searrow 0} \frac{1}{(N-d)\alpha_{N-d} r^{N-d-1}} \int_{\partial E_r} f dH^{N-1}.$$

In the section of topological derivative we recall the notions of Minkowski content, d -rectifiability, and positive reach that are used to define the topological derivative of an objective function with respect to perturbations by a d -dimensional closed subset \bar{w} of \mathbb{R}^N . Given an open domain Ω with boundary $\partial\Omega$, the perturbed domain $\Omega_\epsilon = \Omega \setminus \bar{w}_\epsilon$ is obtained by removing the ϵ -dilatation \bar{w}_ϵ of the set \bar{w} . The boundary of Ω_ϵ is made up of two disjoint parts $\partial\Omega_\epsilon = \partial\Omega \cup \partial\bar{w}_\epsilon$. The notion of positive reach for a nowhere dense closed set \bar{w} ensures that the boundary $\partial\bar{w}_\epsilon$ of the hole is of class $C^{1,1}$ and that $\Omega \setminus \bar{w} = \Omega$.

Before tackling the theorem of the topological derivative, we also need the following hypotheses. For the conception of these hypotheses, we relied on Novotny's papers [38] [42] [43], and on Delfour's papers [29]-[33].

Let us associate with ϵ , $0 < \epsilon \leq r$, the perturbed domain $\Omega_\epsilon = \Omega \setminus \bar{w}_\epsilon$, where by assumption, $\partial\Omega_\epsilon = \partial\Omega \cup \partial\bar{w}_\epsilon$, and $\partial\Omega \cap \partial\bar{w}_\epsilon = \emptyset$ and $\partial\bar{w}_\epsilon \in C^{1,1}$.

Let Ω_ϵ^m denote the component of Ω_ϵ for which $\partial\Omega$ is part of its boundary. Let Ω_ϵ^b denote the blind component of Ω_ϵ whose boundary has no intersection with $\partial\Omega$. The function u_{Ω_ϵ} is divided between the two

components Ω_ϵ^b and Ω_ϵ^m as follows: $u_{\Omega_\epsilon} = u_{\Omega_0}$ within Ω_ϵ^b , and $\frac{\partial u_r}{\partial n_A} = 0$ on

$\partial\Omega_\epsilon^m \cap \partial\bar{w}_\epsilon$ as $\partial\bar{w}_\epsilon$ consists of the two disjoint boundaries Ω_ϵ^b and $\partial\Omega_\epsilon^b$ at $\partial\Omega_\epsilon^m \cap \partial\bar{w}_\epsilon$. We can construct an extension of Ω by defining the solution as follows: $u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon}$ on $\partial\Omega_\epsilon^m \cap \partial\bar{w}_\epsilon$, $u_{\Omega_\epsilon}^0 = u_{\Omega_0}$ on $\partial\Omega_\epsilon^b$.

For more information, the reader can refer to [29].

In the following theorem N denotes the dimension of the workspace, d the dimension of a subset E of \mathbb{R}^N , r the radius of the ball and α_{N-d} is the volume of the unit ball in \mathbb{R}^{N-d} .

3.2.2. Topological Derivative for the Functional

In what follows, we establish the main result of the topological derivative.

Theorem 3.8 *Let $0 \leq d < N$, $E = \bar{w}$ and $s = \alpha_{N-d} r^{N-d}$. The topological derivative exists if and only if the following limit:*

$$L = \lim_{\epsilon \rightarrow 0} (L_0(\epsilon) + L_1(\epsilon)),$$

exists with

$$L_0(\epsilon) = \int_{\Omega_\epsilon} \beta \left| \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right) \right|^2 + \alpha \left| \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right|^2$$

and

$$L_1(\epsilon) = \frac{1}{s} \left[\int_{\partial\Omega_\epsilon^m \cap \partial\bar{w}_\epsilon} \nabla u_{\Omega_0} \cdot \nabla d\bar{w} p_0 dH^{N-1} \right] dx - \frac{1}{s} \int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \cdot \nabla p_\epsilon dx.$$

Moreover, the topological derivative of the function is given by the expression:

$$DJ(\Omega) = \lim_{\epsilon \rightarrow 0} \frac{J(\Omega_\epsilon) - J(\Omega)}{\alpha_{N-d} r^{N-d}} = L - \left[\int_{\bar{w}} \beta |\nabla u_{\Omega_0}|^2 + \alpha |u_{\Omega_0} - u_0|^2 + \mu \nabla u_{\Omega_0} \cdot \nabla p_0 - f p_0 - 3k\alpha \nabla \theta_{\Omega} p_0 \right] dH^d.$$

where p_0, u_{Ω_0} are solutions of systems

$$\int_{\Omega} 2\beta \nabla u_{\Omega_0} \cdot \nabla \phi' dx + \int_{\Omega} 2\alpha (u_{\Omega_0} - u_0) \phi' dx + \int_{\Omega} \mu \nabla \phi' \cdot \nabla p_0 dx = 0.$$

Proof. In the following, consider also then functional defined in Ω_ϵ , by

$$J(\Omega_\epsilon) = \beta \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^2 dx + \alpha \int_{\Omega_\epsilon} |u_{\Omega_\epsilon} - u_0|^2 dx \tag{3.27}$$

where u_{Ω_ϵ} be the solution to the following problem

$$\begin{cases} -\mu \Delta u_{\Omega_\epsilon} - (\lambda + \mu) \nabla(\operatorname{div} u_{\Omega_\epsilon}) + 3k\alpha \nabla \theta_{\Omega_\epsilon} = \mathbf{f} & \text{in } \Omega_\epsilon \\ -\Delta \theta_{\Omega_\epsilon} = g & \text{in } \Omega_\epsilon \\ \frac{\partial \theta}{\partial n} = h & \text{on } \partial\Omega \\ \frac{\partial \theta}{\partial n} = 0 & \text{on } \partial\bar{w}_\epsilon \\ \frac{\partial u_{\Omega_\epsilon}}{\partial n} = v_0 & \text{on } \partial\Omega \\ \frac{\partial u_{\Omega_\epsilon}}{\partial n} = 0 & \text{on } \partial\bar{w}_\epsilon \end{cases} \tag{3.28}$$

Considering a shape function J defined by

$$J(\Omega) = \alpha \int_{\Omega} |u_{\Omega} - u_0|^2 dx + \beta \int_{\Omega} |\nabla u_{\Omega}|^2 dx \tag{3.29}$$

where $u_{\Omega} \in H_{\operatorname{div}}(\Omega, \mathbb{R}^3)$ is solution to the variational problem

$$\mu \int_{\Omega} \nabla u_{\Omega} \cdot \nabla v dx - (\lambda + \mu) \int_{\Omega} \nabla(\nabla \cdot u_{\Omega}) v dx - 3k\alpha \int_{\Omega} \nabla \theta_{\Omega} v dx = \int_{\Omega} f v dx \tag{3.30}$$

and

$$\int_{\Omega} \nabla \theta_{\Omega} \nabla \varphi dx = \int_{\Omega} g \varphi dx \tag{3.31}$$

for all $v \in H^{div}(\Omega, \mathbb{R}^3)$ and $\varphi \in \mathcal{D}(\Omega)$ for some given functions $f \in H^1(\mathbb{R}^3)$, $g \in H^{-1}(\Omega)$.

In the case where $div u_\Omega = 0$ in Ω , (3.30) is reduced to the following variational problem

$$\mu \int_{\Omega} \nabla u_\Omega \cdot \nabla v \, dx = \int_{\Omega} f v \, dx + 3k\alpha \int_{\Omega} \nabla \theta_\Omega v \, dx \tag{3.32}$$

and θ_Ω solution to (3.31). The shape functional associated the perforated domain is given by

$$j(\chi_\epsilon(x_0)) = J(\Omega_\epsilon) = \alpha \int_{\Omega_\epsilon} |u_{\Omega_\epsilon} - u_0|^2 \, dx + \beta \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^2 \, dx \tag{3.33}$$

where u_{Ω_ϵ} is solution the variational problem

$$\mu \int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon} \cdot \nabla v \, dx - (\lambda + \mu) \int_{\Omega_\epsilon} \nabla(\nabla \cdot u_{\Omega_\epsilon}) v \, dx - 3k\alpha \int_{\Omega_\epsilon} \nabla \theta_{\Omega_\epsilon} v \, dx = \int_{\Omega_\epsilon} f v \, dx. \tag{3.34}$$

We aim to compute the topological derivative of the functional $J(\Omega_\epsilon)$

$$DJ = \lim_{\epsilon \rightarrow 0} \frac{J(\Omega_\epsilon) - J(\Omega)}{\alpha_{N-d} \epsilon^{N-d}}.$$

And for this purpose, we define the following set:

$$V_H = \left\{ u_{\Omega_\epsilon} \in (L^2(\Omega))^d, div(u_{\Omega_\epsilon}) \in L^2(\Omega) \right\}. \tag{3.35}$$

Our variational formulation (3.28) consists of finding $u_{\Omega_\epsilon} \in V_H$ such that

$$\mu \int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon} \cdot \nabla v \, dx - (\lambda + \mu) \int_{\Omega_\epsilon} \nabla(\nabla \cdot u_{\Omega_\epsilon}) v \, dx - 3k\alpha \int_{\Omega_\epsilon} \nabla \theta_{\Omega_\epsilon} v \, dx = \int_{\Omega_\epsilon} f v \, dx. \tag{3.36}$$

In the case where $div(u_{\Omega_\epsilon}) = 0$ in Ω_ϵ we have

$$\mu \int_{\Omega_\epsilon} \nabla u_{\Omega_\epsilon} \cdot \nabla v \, dx - 3k\alpha \int_{\Omega_\epsilon} \nabla \theta_{\Omega_\epsilon} v \, dx = \int_{\Omega_\epsilon} f v \, dx, \quad \forall v \in V_H. \tag{3.37}$$

Thus, the Lagrangian dependent on ϵ will be written in the form:

$$L(\epsilon, \phi, \Phi) = \beta \int_{\Omega_\epsilon} |\nabla \phi|^2 \, dx + \alpha \int_{\Omega_\epsilon} |\phi - u_0|^2 \, dx + \mu \int_{\Omega_\epsilon} \nabla \phi \cdot \nabla \Phi \, dx - 3k\alpha \int_{\Omega_\epsilon} \nabla \theta_{\Omega_\epsilon} \Phi \, dx - \int_{\Omega_\epsilon} f \Phi \, dx \tag{3.38}$$

$$J(\Omega_\epsilon) = \inf_{\phi \in V_H} \sup_{\Phi \in V_H} L(\epsilon, \phi, \Phi).$$

From this, we can now evaluate the derivative of the Lagrangian, dependent on ϵ , with respect to ϕ .

$$d_\phi L(\epsilon, \phi, \Phi, \phi') = \int_{\Omega_\epsilon} 2\beta \nabla \phi \cdot \nabla \phi' \, dx + \int_{\Omega_\epsilon} 2\alpha (\phi - u_0) \phi' \, dx + \int_{\Omega_\epsilon} \mu \nabla \phi' \cdot \nabla \Phi \, dx.$$

Subsequently, we obtain the variational formulation of the adjoint state equation given by

$d_\phi L(0, u_{\Omega_0}, p_0, \phi') = 0$, where $u_{\Omega_0} = u_{\Omega_\epsilon}$ for $\epsilon = 0$. Find $p_0 \in H_0^1(\Omega)$ such that

$$\int_{\Omega} 2\beta \nabla u_{\Omega_0} \cdot \nabla \phi' \, dx + \int_{\Omega} 2\alpha (u_{\Omega_0} - u_0) \phi' \, dx + \int_{\Omega} \mu \nabla \phi' \cdot \nabla p_0 \, dx = 0. \tag{3.39}$$

Next, we derive the Lagrangian with respect to Φ .

$$d_\Phi L(\epsilon, \phi, \Phi, \Phi') = \int_{\Omega_\epsilon} \mu \nabla \phi \cdot \nabla \Phi' \, dx - \int_{\Omega_\epsilon} f \Phi' \, dx - 3k\alpha \int_{\Omega_\epsilon} \nabla \theta_{\Omega_\epsilon} \Phi' \, dx.$$

The initial state $u_{\Omega_0} = u_{\Omega}$ is a solution of $d_{\Phi}L(0, u_{\Omega_0}, 0, \Phi') = 0 \forall \Phi' \in H_0^1$ and in this case, we have:

$$\int_{\Omega} \mu \nabla u_{\Omega_0} \cdot \nabla \Phi' dx - \int_{\Omega} f \Phi' dx - 3k\alpha \int_{\Omega} \nabla \theta_{\Omega} \Phi' dx = 0.$$

Then, we have:

$$\int_{\Omega} [\mu \nabla u_{\Omega_0} \cdot \nabla \Phi' - f \Phi' - 3k\alpha \nabla \theta_{\Omega} \Phi'] dx = 0.$$

The state $u_{\Omega_{\epsilon}}$ for all $\epsilon \geq 0$ satisfies

$$\int_{\Omega_{\epsilon}} [\mu \nabla u_{\Omega_{\epsilon}} \cdot \nabla \Phi' - f \Phi' - 3k\alpha \nabla \theta_{\Omega} \Phi'] dx = 0, \forall \Phi' \in V_H.$$

In the following, we aim to find the derivative of the Lagrangian, with respect to ϵ . To achieve this, let us first compute the quotient

$$\frac{L(\epsilon, \phi, \Phi) - L(0, \phi, \Phi)}{s}$$

$$\begin{aligned} & L(\epsilon, \phi, \Phi) - L(0, \phi, \Phi) \\ &= \beta \int_{\Omega_{\epsilon}} |\nabla \phi|^2 dx + \alpha \int_{\Omega_{\epsilon}} |\phi - u_0|^2 dx + \mu \int_{\Omega_{\epsilon}} \nabla \phi \cdot \nabla \Phi dx - 3k\alpha \int_{\Omega_{\epsilon}} \nabla \theta_{\Omega} \Phi dx \\ &\quad - \int_{\Omega_{\epsilon}} f \Phi dx - \left[\beta \int_{\Omega} |\nabla \phi|^2 dx + \alpha \int_{\Omega} |\phi - u_0|^2 dx \right] \\ &\quad - \left[\mu \int_{\Omega} \nabla \phi \cdot \nabla \Phi dx - 3k\alpha \int_{\Omega} \nabla \theta_{\Omega} \Phi dx - \int_{\Omega} f \Phi dx \right] \\ &= \frac{L(\epsilon, \phi, \Phi) - L(0, \phi, \Phi)}{s} \\ &= - \left[\int_{\bar{\omega}_{\epsilon}} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu \nabla \phi \cdot \nabla \Phi - f \Phi - 3k\alpha \nabla \theta_{\Omega} \Phi \right] dx. \end{aligned}$$

For $d = 0$, $\bar{\omega} = \{x_0\}$, $\bar{\omega}_{\epsilon} = \{x \in \mathbb{R}^N : |x - x_0| \leq \epsilon\} = \bar{B}(x_0, \epsilon)$.

$$\begin{aligned} & d_s L(0, \phi, \Phi) \\ &= \lim_{s \rightarrow 0} - \frac{1}{|B(x_0, \epsilon)|} \left[\int_{B(x_0, \epsilon)} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu \nabla \phi \cdot \nabla \Phi - f \Phi - 3k\alpha \nabla \theta_{\Omega} \Phi \right] dx \\ &= -\beta |\nabla \phi(x_0)|^2 - \alpha |\phi(x_0) - u_0(x_0)|^2 - \mu \nabla \phi(x_0) \cdot \nabla \Phi(x_0) \\ &\quad + f \Phi(x_0) + 3k\alpha \nabla \theta_{\Omega} \Phi(x_0). \end{aligned}$$

By evaluating the last equation at the point u_{Ω_0}, p_0 , we obtain:

$$\begin{aligned} d_s L(0, u_{\Omega_0}, p_0) &= -\beta |\nabla u_{\Omega_0}(x_0)|^2 - \alpha |u_{\Omega_0}(x_0) - u_0(x_0)|^2 - \mu \nabla u_{\Omega_0}(x_0) \cdot \nabla p_0(x_0) \\ &\quad + f p_0(x_0) + 3k\alpha \nabla \theta_{\Omega} p_0(x_0). \end{aligned}$$

Hence, if $0 < d \leq N - 1$, we have:

$$\begin{aligned} & \frac{L(\epsilon, \phi, \Phi) - L(0, \phi, \Phi)}{s} \\ &= - \frac{1}{|\bar{\omega}_{\epsilon}|} \left[\int_{\bar{\omega}_{\epsilon}} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu \nabla \phi \cdot \nabla \Phi - f \Phi - 3k\alpha \nabla \theta_{\Omega} \Phi \right] dx \\ &= - \frac{1}{\alpha_{N-d} r^{N-d}} \left[\int_{\bar{\omega}_{\epsilon}} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu \nabla \phi \cdot \nabla \Phi - f \Phi - 3k\alpha \nabla \theta_{\Omega} \Phi \right] dx \\ &\rightarrow - \left[\int_{\bar{\omega}} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu \nabla \phi \cdot \nabla \Phi - f \Phi - 3k\alpha \nabla \theta_{\Omega} \Phi \right] dH^d. \end{aligned}$$

Therefore, taking the ast result at the point u_{Ω_0}, p_0 becomes:

$$d_s L(0, u_{\Omega_0}, p_0) = - \left[\int_{\bar{\omega}} \beta |\nabla u_{\Omega_0}|^2 + \alpha |u_{\Omega_0} - u_0|^2 + \mu \nabla u_{\Omega_0} \cdot \nabla p_0 - f p_0 - 3k\alpha \nabla \theta_{\Omega} p_0 \right] dH^d.$$

We will now define $R(\epsilon)$ by

$$R(\epsilon) = \int_0^1 d_x L \left(\epsilon, u_{\Omega_0} + \Psi(u_{\Omega_\epsilon} - u_{\Omega_0}), p_0, \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) \right) d\Psi.$$

By substituting $\phi' = \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\epsilon}$ and $\Psi = \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{2}$ into the adjoint equation

for p_0 , we obtain:

$$\begin{aligned} R(\epsilon) &= \int_{\Omega_\epsilon} 2\beta \nabla \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) \cdot \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) dx \\ &\quad + \int_{\Omega_\epsilon} 2\alpha \left[\left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) - u_0 \right] \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) dx + \int_{\Omega_\epsilon} \mu \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) \cdot \nabla p_0 dx \\ &= \frac{1}{s} \left[\int_{\Omega_\epsilon} 2\beta \nabla \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) \cdot \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) dx \right. \\ &\quad \left. + 2\alpha \left[\left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) - u_0 \right] (u_{\Omega_\epsilon} - u_{\Omega_0}) dx + \frac{1}{s} \left[\int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \cdot \nabla p_0 \right] dx \right] \\ &= \frac{1}{s} \left[\int_{\Omega_\epsilon} 2\beta \left(\nabla \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) - \nabla u_{\Omega_0} + \nabla u_{\Omega_0} \right) \cdot \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) dx \right. \\ &\quad \left. + \frac{2\alpha}{s} \int_{\Omega_\epsilon} \left[\left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) - u_{\Omega_0} + u_{\Omega_0} - u_0 \right] (u_{\Omega_\epsilon} - u_{\Omega_0}) dx \right. \\ &\quad \left. + \frac{1}{s} \left[\int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \cdot \nabla p_0 \right] dx \right] \\ &= \frac{1}{s} \left[\int_{\Omega_\epsilon} 2\beta \left(\nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{2} \right) + \nabla u_{\Omega_0} \right) \cdot \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) dx \right. \\ &\quad \left. + \frac{1}{s} \left[\int_{\Omega_\epsilon} 2\alpha \left[\left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{2} \right) + u_{\Omega_0} - u_0 \right] (u_{\Omega_\epsilon} - u_{\Omega_0}) dx \right. \right. \\ &\quad \left. \left. + \frac{1}{s} \left[\int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \cdot \nabla p_0 \right] dx \right] \right] \\ R(\epsilon) &= \int_{\Omega_\epsilon} \left(\beta \left| \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right) \right|^2 + \alpha \left| \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right|^2 \right) dx \\ &\quad + \frac{1}{s} \left[\int_{\Omega_\epsilon} 2\beta \nabla u_{\Omega_0} \cdot \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) + 2\alpha (u_{\Omega_\epsilon} - u_{\Omega_0})(u_{\Omega_0} - u_0) \right] dx \\ &\quad + \frac{1}{s} \left[\int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \cdot \nabla p_0 \right] dx. \end{aligned}$$

Thus, for all $u_{\Omega_0} \in H_{div(\Omega)}$, equation (3.37) becomes:

$$\int_{\Omega_\epsilon} (\mu \nabla u_{\Omega_\epsilon} \cdot \nabla v - f v - 3k\alpha \nabla \theta v) dx$$

$$\begin{aligned}
 &= - \left[\int_{\bar{\omega}_\epsilon} \left(\mu \nabla u_{\Omega_\epsilon} \cdot \nabla v - fv - 3k\alpha \nabla \theta v \right) dx \right] \\
 &= \int_{\partial \bar{\omega}_\epsilon} \frac{\partial u_{\Omega_0}}{\partial n} v dH^{N-1}, \quad \forall v \in H_{div}(\Omega).
 \end{aligned}$$

Now, considering the assumption about $\bar{\omega}$, $\partial \bar{\omega}_\epsilon \in \mathcal{C}^{1,1}$ and

$$\frac{\partial u_{\Omega_0}}{\partial n} = \nabla u_{\Omega_0} \cdot n \Omega_\epsilon = -\nabla u_{\Omega_0} \cdot \nabla d \bar{\omega} \text{ on } \bar{\omega}_\epsilon.$$

And in this case, we have:

$$\int_{\Omega_\epsilon} \left(\mu \nabla u_{\Omega_\epsilon} \cdot \nabla v - fv dx - 3k\alpha \nabla \theta v \right) dx = - \int_{\partial \bar{\omega}_\epsilon} \nabla u_{\Omega_0} \cdot \nabla d \bar{\omega} v dH^{N-1}. \quad (3.40)$$

By taking the difference between equation (3.37) and equation (3.40), we obtain:

$$\int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \cdot \nabla v = \int_{\partial \bar{\omega}_\epsilon} \nabla u_{\Omega_0} \cdot \nabla d \bar{\omega} v dH^{N-1}.$$

The adjoint equation for $\epsilon \geq 0$ yields:

$$\int_{\Omega_\epsilon} 2\beta \nabla u_{\Omega_0} \cdot \nabla \phi' dx + \int_{\Omega_\epsilon} 2\alpha (u_{\Omega_0} - u_0) \phi' dx + \int_{\Omega_\epsilon} \mu \nabla \phi' \cdot \nabla p_\epsilon dx = 0. \quad (3.41)$$

By taking $\phi' = u_{\Omega_\epsilon} - u_{\Omega_0}$ in equation (3.41), we have:

$$\begin{aligned}
 &\int_{\Omega_\epsilon} 2\beta \nabla u_{\Omega_0} \cdot \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) dx + \int_{\Omega_\epsilon} 2\alpha (u_{\Omega_0} - u_0) (u_{\Omega_\epsilon} - u_{\Omega_0}) dx \\
 &+ \int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \cdot \nabla p_\epsilon dx = 0.
 \end{aligned}$$

$$\begin{aligned}
 &\int_{\Omega_\epsilon} 2\beta \nabla u_{\Omega_0} \cdot \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) + 2\alpha (u_{\Omega_0} - u_0) (u_{\Omega_\epsilon} - u_{\Omega_0}) dx \\
 &= - \int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \cdot \nabla p_\epsilon dx.
 \end{aligned}$$

The final equation for $R(\epsilon)$ becomes:

$$\begin{aligned}
 R(\epsilon) &= \int_{\Omega_\epsilon} \left[\beta \left| \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right) \right|^2 + \alpha \left| \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right|^2 \right] dx \\
 &+ \frac{1}{s} \left[\int_{\partial \Omega_\epsilon^m \cap \partial \bar{\omega}_\epsilon} \nabla u_{\Omega_0} \cdot \nabla d \bar{\omega} p_0 dH^{N-1} \right] dx \\
 &- \frac{1}{s} \int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \cdot \nabla p_\epsilon dx.
 \end{aligned}$$

4. Conclusion and Extensions

In this article, we studied a linear thermoelasticity problem using optimization methods. After a thermoelasticity model using mass conservation was given, we established the derivative form. And to be sure, we have to recall a few notions about the minmax method. Finally, the topological derivative was established using the same method but with different basic tools. We then intend to study regularity problems as well as numerical methods of the problem proposed.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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