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On the Regularization Method to Stable Approximate Solution of Equations of the First Kind with Unbounded Operators

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Abstract

Let $A:D(A)\subset X\to Y$ be a linear, closed, densely defined unbounded operator, where X and Y are Hilbert spaces. Assume that A is not boundedly invertible. If equation (1) $Au=f$ is solvable, and $\|f_\delta-f\|\le\delta$ then the following results are provided: Problem $F_{\alpha,\delta}(u):=\|Au-f_\delta\|^2+\alpha\|u\|^2$ has a unique global minimizer $u_{\alpha,\delta}$ for any $f_\delta\in Y$, and $u_{\alpha,\delta}=A^*\left(AA^*+\alpha I_Y\right)^{-1}f_\delta$. Then there is a function $\alpha(\delta)$, $\lim_{\delta\to0}\alpha(\delta)=0$ such that $\lim_{\delta\to0}\|u_{\alpha(\delta),\delta}-x_0\|=0$, where x_0 is the unique minimal-norm solution to (1). In this paper we introduce the regularization method solving Equation (1) with A being a linear, closed, densely defined unbounded operator. At the same time, an application is given to the weak derivative operator equation.

Keywords

Ill-Posed Problem, Regularization Method, Unbounded Linear Operator

1. Introduction

Let $A:D(A)\subset X\to Y$ be a linear, closed, densely defined unbounded operator, where X and Y are Hilbert spaces. Consider the equation

$$Au=f \tag{1}$$

Problem finding approximate solution of (1) is called ill-posed (instability) in the sense of Hadamard [1] if A is not boundedly invertible. This may happen if the null space $N(A)=\{u:Au=0\}$ is not trivial, *i.e.*, A is not injective, or if

A is injective but A^{-1} is unbounded, *i.e.*, the range of A, $R(A)$ is not closed [2].

If $\|A\|<\infty$, and f_δ, the noisy data, are given

$$\|f_\delta - f\| \le \delta \tag{2}$$

problem finding approximate solution of (1) has been extensively studied in the literature in detail [2]-[6] and references therein.

If A is a linear, closed, densely defined operator or is a linear unbounded operator, problem finding approximate solution of (1) has been some recent research [7]-[13], however, there are still many open problems such as parameter choice rules of regularization...

Our aim is to study problem finding approximate of (1) when unbounded operator A is a linear, closed, densely defined and the noisy data f_δ satisfies (2). We shall introduce the regularization method for solving the problem, and introduce a priori and a posteriori parameter choice rules of regularization; at the same time give an application to the weak derivative operator equation in Hilbert space of measurable functions, Lebesgue squares integrable $L^2[0,1]$.

2. The Some Main Results

Lemma 1. [2] Let $A: D(A) \subset X \to Y$ be a linear, closed, densely defined operator, where X and Y are Hilbert spaces, then

1) the operators $T = A^*A$ and $Q = AA^*$ are densely defined, self-adjoint;

2) A^* is closed, densely defined and $A^{**} = A$;

3) the operators $\tilde{A} := \left(I_X + A^*A\right)^{-1} : X \to X$, $A\tilde{A} : X \to Y$ are both defined on all of X and are bounded, $\sigma\left(\tilde{A}\right) \subseteq [0,1]$, with I_X be the identity mapping on X. Also, $\tilde{A}$ is self-adjoint;

4) the operator $\hat{A} := \left(I_Y + AA^*\right)^{-1} : Y \to Y$ is bounded and self-adjoint and $A^*\hat{A} : Y \to X$ is bounded, with I_Y be the identity mapping on Y.

Lemma 2. Let $A: D(A) \subset X \to Y$ be a linear, closed, densely defined operator, where X and Y are Hilbert spaces, if Equation (1) is solvable then minimal-norm solution of (1) is unique, we denote by x_0, furthermore $x_0 \perp N(A)$.

Proof. Because $A: D(A) \subset X \to Y$ is a linear operator, the set of solutions of (1) is a convex set in the Hilbert space X, which we denote by X_0. Furthermore, A is a closed operator then X_0 is a closed set. Indeed, suppose $x_n \in X_0$, $x_n \to x$. Because, $x_n \in X_0$, then $Ax_n = f, \forall n$. Therefore, $Ax_n \to f$, then $x \in D(A)$ and $Ax = f$ or $x \in X_0$. So, X_0 is a closed set in Hilbert X.

Because X_0 is a convex, closed set in Hilbert X, so it contains only one element with minimal norm x_0 [4] and $x_0 \perp N(A)$ [4].

Theorem 1. For any $f \in Y$, the problem

$$F_\alpha(u) = \|Au - f\|^2 + \alpha\|u\|^2 \to \min, \alpha = \text{const} > 0, \tag{3}$$

has a unique solution $u_\alpha = A^*\left(AA^* + \alpha I_Y\right)^{-1} f$, where I_Y is the identity operator on Y.

Proof. Consider the equation

$$\left(AA^* + \alpha I_Y\right) w_\alpha = f, \alpha = \text{const} > 0 \tag{4}$$

which is uniquely solvable $w_\alpha = \left(AA^* + \alpha I_Y\right)^{-1} f$ (Lemma 1). Let $u_\alpha = A^* w_\alpha$ then

$$\begin{aligned} Au_\alpha &= AA^*\left(AA^* + \alpha I_Y\right)^{-1} f \\ &= \left(AA^* + \alpha I_Y\right)\left(AA^* + \alpha I_Y\right)^{-1} f - \alpha I_Y \left(AA^* + \alpha I_Y\right)^{-1} f \\ &= f - \alpha w_\alpha, \end{aligned}$$

or

$$Au_\alpha - f = -\alpha w_\alpha.$$

We have

$$\begin{aligned} F_\alpha(u+v) &= \left\|A(u+v) - f\right\|^2 + \alpha\left\|u+v\right\|^2 = \left\|(Au-f) + Av\right\|^2 + \alpha\left\|u+v\right\|^2 \\ &= \left\|Au-f\right\|^2 + \left\|Av\right\|^2 + 2\operatorname{Re}\langle Au-f, Av\rangle + \alpha\left(\left\|u\right\|^2 + \left\|v\right\|^2 + 2\operatorname{Re}\langle u,v\rangle\right) \\ &= \left\|Au-f\right\|^2 + \left\|Av\right\|^2 + \alpha\left(\left\|u\right\|^2 + \left\|v\right\|^2\right) + 2\operatorname{Re}\left(\langle Au-f, Av\rangle + \alpha\langle u,v\rangle\right) \end{aligned} \tag{5}$$

for any $v \in D(A)$. If $u = u_\alpha$, then

$$\begin{aligned} \langle Au_\alpha - f, Av\rangle + \alpha\langle u_\alpha, v\rangle &= -\alpha\langle w_\alpha, Av\rangle + \alpha\langle u_\alpha, v\rangle \\ &= -\alpha A^*\langle w_\alpha, v\rangle + \alpha\langle u_\alpha, v\rangle \\ &= -\alpha\langle u_\alpha, v\rangle + \alpha\langle u_\alpha, v\rangle = 0. \end{aligned} \tag{6}$$

Thus (5) and (6) imply

$$F_\alpha(u_\alpha + v) = F_\alpha(u_\alpha) + \left\|Av\right\|^2 + \alpha\left\|v\right\|^2 \ge F_\alpha(u_\alpha)$$

and $F(u_\alpha + v) = F(u_\alpha)$ if and only if $v = 0$, so u_α is the unique minimizer of $F_\alpha(u)$.

Theorem 1 is proved.

Theorem 2. If x_0 is the unique minimal-norm solution of (1) and u_α is the unique minimizer of $F_\alpha(u)$ then

$$\lim_{\alpha\to 0}\left\|u_\alpha - x_0\right\| = 0, \text{ with } u_\alpha = A^*\left(AA^* + \alpha I_Y\right)^{-1} f. \tag{7}$$

Proof. Write (4) as $A\left(A^* w_\alpha - x_0\right) = -\alpha w_\alpha$. Apply A^*, which is possible because $w_\alpha \in D\left(A^*\right)$, we obtain

$$A^* A\left(u_\alpha - x_0\right) = -\alpha u_\alpha. \tag{8}$$

Multiply (8) by $u_\alpha - x_0$, we obtain

$$\left\langle A^* A\left(u_\alpha - x_0\right), u_\alpha - x_0\right\rangle = -\alpha\langle u_\alpha, u_\alpha - x_0\rangle$$

or

$$\left\|A\left(u_\alpha - x_0\right)\right\|^2 = -\alpha\left(\left\|u_\alpha\right\|^2 - u_\alpha, x_0\right). \tag{9}$$

Since $\alpha > 0$ this implies

$$\left\|u_\alpha\right\|^2 \le \langle u_\alpha, x_0\rangle,$$

So

$$\|u_\alpha\| \le \|x_0\|, \forall \alpha > 0.$$

Therefore, one may assume (taking a subsequence) that the sequence u_α weakly converges to an element z, denoted by $u_n := u_{\alpha_n} \rightharpoonup z$, as $\alpha_n \to 0$.

It follows from (9) that

$$\lim_{n\to\infty} \|A(u_n - x_0)\| = 0,\ i.e.\ \lim_{n\to\infty} \|Au_n - f\| = 0.$$

We shall prove that $z = x_0$.

Let γ run through the set such that $\{A^* A\gamma\}$ is dense in $N^\perp$, where $N := N(A)$. Note that $N(T) = N(A)$, where $T = A^* A$ [2].

Because of the formulas $X = \overline{R(T)} \oplus N(T)$ [2], then $\{\gamma\} = D(T)$ is dense in X, and the set $\{T\gamma\}$ is dense in $N^\perp$.

Multiply the equation $T(u_\alpha - x_0) = -\alpha u_\alpha$ by γ and pass to the limit $\alpha \to 0$. We obtain

$$(z - x_0, T\gamma) = 0.$$

We have $x_0 \perp N$. If $z \perp N$, then $z - x_0 \perp N$ and $z - x_0 \perp N^\perp$, so $z - x_0 = 0$.

One may always assume that $z \perp N$ because $Tu_\alpha = T\widetilde{u_\alpha}$, where $\widetilde{u_\alpha}$ is the orthogonal projection of u_α onto $N^\perp$.

Since the sequence $\{u_n\}$ converges weakly to z, and $\|u_n\| \le \|x_0\|$, so $\lim_{n\to\infty} \|u_n - x_0\| = 0$ [14].

For convenience for the reader, we prove this claim as follows:

Since $u_n := u_{\alpha_n} \rightharpoonup z$, one gets $\|x_0\| \le \underline{\lim}_{n\to\infty} \|u_n\|$. The inequality $\|u_n\| \le \|x_0\|$ implies $\overline{\lim}_{n\to\infty} \|u_n\| \le \|x_0\|$. Therefore $\lim_{n\to\infty} \|u_n\| = \|x_0\|$. This and the weakly converge $u_n := u_{\alpha_n} \rightharpoonup z$ imply strong convergence

$$\|u_n - x_0\|^2 = \|u_n\|^2 + \|x_0\|^2 - 2\operatorname{Re}\langle u_n, x_0\rangle \to 0,\ \text{as}\ n \to \infty.$$

Theorem 2 is proved.

Theorem 3. If $\|f_\delta - f\| \le \delta$, x_0 is the unique minimal-norm solution of (1), and

$$F_{\alpha,\delta}(u) = \|Au - f_\delta\|^2 + \alpha\|u\|^2 = \min, \tag{10}$$

then there exists a unique global minimizer $u_{\alpha,\delta}$ to (10) and $\lim_{\delta\to 0} \|u_\delta - x_0\| = 0$, where $u_\delta := u_{\alpha(\delta),\delta}$ and $\alpha(\delta)$ is properly chosen, in particular $\lim_{\delta\to 0} \alpha(\delta) = 0$.

Proof. The existence and uniqueness of the minimizer $u_{\alpha,\delta}$ of $F_\alpha(u)$ follows from Theorem 1 and $u_{\alpha,\delta} = A^*(Q + \alpha I_Y)^{-1} f_\delta$. We have

$$\|u_{\alpha,\delta} - x_0\| \le \|u_{\alpha,\delta} - u_\alpha\| + \|u_\alpha - x_0\|.$$

By Theorem 2, $\|u_\alpha - x_0\| := \eta(\alpha) \to 0$, as $\alpha \to 0$, with $u_\alpha = A^*(AA^* + \alpha I_Y)^{-1} f = A^*(Q + \alpha I_Y)^{-1} f$.

Let us estimate

$$\left\|u_{\alpha,\delta}-u_{\alpha}\right\|=\left\|A^{*}\left(Q+\alpha I_{Y}\right)^{-1}\left(f_{\delta}-f\right)\right\|\leq\delta\left\|A^{*}\left(Q+\alpha I_{Y}\right)^{-1}\right\|.$$

By the polar decomposition theorem [15] [16], one has $A^{*}=UQ^{1/2}$, where U is a partial isometry, so $\|U\|\leq 1$. One has,

$$\begin{aligned}\left\|A^{*}\left(Q+\alpha I_{Y}\right)^{-1}\right\|&=\left\|UQ^{1/2}\left(Q+\alpha I_{Y}\right)^{-1}\right\|\\&\leq\left\|Q^{1/2}\left(Q+\alpha I_{Y}\right)^{-1}\right\|\\&=\max_{\lambda\geq 0}\frac{\lambda^{1/2}}{\lambda+\alpha}=\frac{1}{2\sqrt{\alpha}},\end{aligned}$$

where the spectral representation for Q was used.

Thus

$$\left\|u_{\alpha,\delta}-x_{0}\right\|\leq\left\|u_{\alpha,\delta}-u_{\alpha}\right\|+\left\|u_{\alpha}-x_{0}\right\|\leq\frac{\delta}{2\sqrt{\alpha}}+\eta\left(\alpha\right). \tag{11}$$

For a fixed small $\delta>0$, choose $\alpha=\alpha\left(\delta\right)$ which minimizes the right side of (12). Then $\lim_{\delta\to 0}\alpha\left(\delta\right)=0$ and $\lim_{\delta\to 0}\left(\frac{\delta}{2\sqrt{\alpha\left(\delta\right)}}+\eta\left(\alpha\left(\delta\right)\right)\right)=0$.

Theorem 3 is proved.

Remark 1. From (11), we can also choose $\alpha\left(\delta\right)=c\delta^{k}$, with any $0<k<2$ and $c=\text{const}>0$. The constant c can be arbitrary.

We can also choose $\alpha\left(\delta\right)$ by a discrepancy principle. For example, consider the equation for finding $\alpha\left(\delta\right)$:

$$\left\|Au_{\alpha,\delta}-f_{\delta}\right\|=c\delta,c=\text{const}>1. \tag{12}$$

We assume that $\left\|f_{\delta}\right\|>c\delta$.

That is the content of the following theorem.

Theorem 4. The equation

$$\left\|Au_{\alpha,\delta}-f_{\delta}\right\|=c\delta,c=\text{const}>1,\left\|f_{\delta}\right\|>c\delta, \tag{13}$$

has a unique solution $\alpha=\alpha\left(\delta\right)>0$, $\lim_{\delta\to 0}\alpha\left(\delta\right)=0$, and if $u_{\delta}:=u_{\alpha(\delta),\delta}$, then $\lim_{\delta\to 0}\left\|u_{\delta}-x_{0}\right\|=0$.

Proof. Let us prove that Equation (13) has a unique root $\alpha\left(\delta\right)>0$, $\lim_{\delta\to 0}\alpha\left(\delta\right)=0$. Indeed, using the spectral theorem [15] [16], one gets

$$\begin{aligned}\left\|Au_{\alpha,\delta}-f_{\delta}\right\|^{2}&=\left\|\left[AA^{*}\left(Q+\alpha I\right)\right]^{-1}f_{\delta}\right\|^{2}\\&=\int_{0}^{\infty}\left|\frac{s}{s+\alpha}-1\right|^{2}d\left(E_{s},f_{\delta},f_{\delta}\right)\\&=\alpha^{2}\int_{0}^{\infty}\frac{d\left(E_{s},f_{\delta},f_{\delta}\right)}{\left(s+\alpha\right)^{2}}:=g\left(\alpha,\delta\right),\end{aligned}$$

where E_{s} is the resolution of the identity of Q.

One has $g(\infty,\delta)=\|f_\delta\|^2>c^2\delta^2$, and $g(0^+,\delta)=\|P_{N^*}f_\delta\|^2$, where P_{N^*} is the orthogonal projector onto the subspace $N^*=N(Q)=N(A^*)=R(A)^\perp$.

Since $f\in R(A)$ and $\|f_\delta-f\|\le\delta$, it follows that $\|P_{N^*}f_\delta\|\le\delta$, so $g(0^+,\delta)\le\delta^2$. The function $g(\alpha,\delta)$ for a fixed $\delta>0$ is a continuous strictly increasing function of α on $[0,\infty)$. Therefore, there exists a unique $\alpha=\alpha(\delta)>0$ which solves Eq. (13) if $\|f_\delta\|>c\delta$ and $c>1$. Clearly $\lim_{\delta\to0}\alpha(\delta)=0$, because $\lim_{\delta\to0}c\alpha(\delta)=0$ and the relation $\lim_{\delta\to0}\alpha^2(\delta)\int_0^\infty\frac{d(E_s,f_\delta,f_\delta)}{(s+\alpha(\delta))^2}=0$ implies $\lim_{\delta\to0}\alpha(\delta)=0$. The function $\alpha=\alpha(\delta)$ is a monotonically growing function of δ with $\alpha(0^+)=0$.

Let us prove that $\lim_{\delta\to0}\|u_\delta-x_0\|=0$, where $u_\delta:=u_{\alpha(\delta),\delta}$, and $\alpha(\delta)$ solves Equation (13). By the definition of u_δ, we get

$$\|Au_\delta-f_\delta\|^2+\alpha(\delta)\|u_\delta\|^2\le\|Ax_0-f_\delta\|^2+\alpha(\delta)\|x_0\|^2=\delta^2+\alpha(\delta)\|x_0\|^2.$$

Since $\|Au_\delta-f_\delta\|^2=c^2\delta^2>\delta^2$, it follows that $\|u_\delta\|\le\|x_0\|$. Thus the sequence u_δ weakly converges to z, denoted by $u_\delta\rightharpoonup z$, and similar argument as in proof of Theorem 2, we obtain $z=x_0$ and $\lim_{\delta\to0}\|u_\delta-x_0\|=0$.

Theorem 4 is proved.

Remark 2. Theorems 1-4 are well known in the case of a bounded linear operator A.

If A is bounded, then a necessary condition for the minimum of the functional $F_\alpha(u)=\|Au-f\|^2+\alpha\|u\|^2$ is the equation

$$A^*Au+\alpha u=A^*f. \tag{14}$$

Hence in this case conditions are required $f\in D(A^*)$.

If A is unbounded, by the above method, then f does not necessarily belong to $D(A^*)$. Therefore, some changes in the usual theory are necessary. The changes are given in this paper. We prove, among other things, that for any $f\in Y$, in particular for $f\notin D(A^*)$, the element $u_\alpha=A^*(AA^*+\alpha I_Y)^{-!}f$ is well defined for any $\alpha=\text{const}>0$, provided that A is a closed, linear, densely defined operator in Hilbert space (Theorem 1).

According to [8] [12], we can replace x_0 is unique minimal-norm solution of (1) by y is solution of (1) satisfying condition $y\perp N(A)$, then Theorems 1-4 are still true.

3. Applications

We will give a concrete example applying the regularization method presented in Section 2.

Let $X=L^2[0,1]$ to be a Hilbert space of measurable functions, Lebesgue squares integrable, with scalar product

$$\langle x,y\rangle=\int_0^1 x(t)\overline{y(t)}\,\mathrm{d}t$$

We define the operator $A:D(A)\subset X\to X$ be the weak derivative operator in $X=L^2[0,1]$, denoted by

$$Ax=\frac{\mathrm{d}x}{\mathrm{d}t},x\in D(A),$$

with domain

$$D(A)=\left\{x\in X:x(t)\text{ is absolutely continuous on }[0,1]\text{ and }\frac{\mathrm{d}x}{\mathrm{d}t}\in X\right\}.$$

Then, A is a linear, closed operator with dense domain.

Indeed $D(A)$ is dense in X since it contains the complete orthonormal set $\{\sin n\pi t\}_{n=1}^{\infty}$.

Clearly, A is a linear operator.

We now prove that A is a closed operator in Hilbert X. Indeed, for suppose $\{x_n\}\subset D(A)$ and $x_n\to x$ and $x_n'\to g$, in each case the convergence being in the $L^2[0,1]$ norm. Since

$$x_n(t)=x_n(0)+\int_0^t x_n'(\xi)\,\mathrm{d}\xi,$$

we see that the sequence of constant functions $\{x_n(0)\}$ converges in $L^2[0,1]$ and hence the numerical sequence $\{x_n(0)\}$ converges to some real number C.

Now define $h\in D(A)$ by $h(t)=C+\int_0^t g(\xi)\,\mathrm{d}\xi$. Then, for any $t\in[0,1]$, we have the Cauchy-Schwarz inequality

$$\begin{aligned}|x_n(t)-h(t)|&=\left|x_n(0)-C+\int_0^t\left(x_n'(\xi)-g(\xi)\right)\mathrm{d}\xi\right|\\&\le|x_n(0)-C|+\int_0^t|x_n'(\xi)-g(\xi)|\,\mathrm{d}\xi\\&\le|x_n(0)-C|+\|x_n'-g\|\end{aligned}$$

and hence $x_n\to h$ uniformly. Therefore, $x=h\in D(A)$ and $Ax=x'=h'=g$, verifying that the operator A is closed, linear, densely defined in $L^2[0,1]$.

Let

$$D^*=\{g\in D(A):g(0)=g(1)=0\}.$$

Then for $x\in D(A)$ and $g\in D^*$, we have

$$\langle Ax,g\rangle=\int_0^1 x'(t)g(t)\,\mathrm{d}t=x(t)g(t)\Big|_0^1-\int_0^1 x(t)g'(t)\,\mathrm{d}t=\langle x,-g'\rangle$$

Therefore $D^*\subset D(A^*)$ and $A^*g=-g'$, for $g\in D^*$.

On the other hand, if $g\in D(A^*)$, let $g^*=A^*g$. Then

$$\langle Ax,g\rangle=\langle x,g^*\rangle$$

for all $x\in D(A)$. In particular, for $x\equiv 1$, we find that $\int_0^1 g^*(t)\,\mathrm{d}t=0$.

Now let

$$h(t)=-\int_0^t g^*(s)\,\mathrm{d}s.$$

Then $h\in D^*$ and $A^*h=g^*=A^*g$ and hence $h-g\in N(A^*)$. Therefore,

$\langle Ax, h-g \rangle = 0$, for all $x \in D(A)$. But $R(A)$ contains all continuous function and hence $g = h \in D^*$.

We conclude that

$$D(A^*) = D^* \text{ and } A^* g = -g'.$$

If the equation

$$Ax = f \tag{15}$$

is solvable, then (15) has the unique minimal-normal solution x_0. (Lemma 2)

According to Theorem 1, for any $f \in X = L^2[0,1]$, the problem

$$F_\alpha(u) = \|Au - f\|^2 + \alpha\|u\|^2 \to \min, \alpha = \text{const} > 0,$$

has a unique solution $u_\alpha = A^*(AA^* + \alpha I_X)^{-1} f$, where I_X is the identity operator on $X = L^2[0,1]$. $f \in X = L^2[0,1]$ does not necessarily belong to $D(A^*)$.

It follows from Theorem 2, then

$$\lim_{\alpha \to 0} \|u_\alpha - x_0\| = 0, u_\alpha = A^*(AA^* + \alpha I)^{-1} f.$$

It follows from Theorem 3, that if $\|f_\delta - f\| \le \delta$ and

$$F_{\alpha,\delta}(u) = \|Au - f_\delta\|^2 + \alpha\|u\|^2 = \min, \tag{16}$$

then there exists a unique global minimizer $u_{\alpha,\delta}$ to (16) and $\lim_{\delta \to 0} \|u_\delta - x_0\| = 0$, where $u_\delta := u_{\alpha(\delta),\delta}$ and $\alpha(\delta)$ is properly chosen, in particular $\lim_{\delta \to 0} \alpha(\delta) = 0$.

It follows from Theorem 4, that the equation

$$\|Au_{\alpha,\delta} - f_\delta\| = c\delta, c = \text{const} > 1, \|f_\delta\| > c\delta,$$

has a unique solution $\alpha = \alpha(\delta) > 0$, $\lim_{\delta \to 0} \alpha(\delta) = 0$, and if $u_\delta := u_{\alpha(\delta),\delta}$, then $\lim_{\delta \to 0} \|u_\delta - x_0\| = 0$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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