

# Binding Energy, Root-Mean Square Radius and Magnetic Dipole Moment of the Nucleus ${}^6\text{Li}$

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## Abstract

In this work, we have applied the translation invariant shell model with number of quanta of excitations  $N = 2, 4, 6, 8$  and 10 to define the ground-state eigenenergies and their corresponding normalized eigenstates, the root mean-square radius, and the magnetic dipole moment of the nucleus  ${}^6\text{Li}$ . We have computed the necessary two-particle orbital fractional parentage coefficients for nuclei with mass number  $A = 6$  and number of quanta of excitations  $N = 10$ , which are not available in the literature. In addition, we have used our previous findings on the nucleon-nucleon interaction with Gaussian radial dependencies, which fits the deuteron characteristics as well as the triton binding energy, root-mean square radius and magnetic dipole moment. The numerical results obtained in this work are in excellent agreement with the corresponding experimental data and the previously published theoretical results in the literature.

## Keywords

Nuclear Structure, The Nucleus  ${}^6\text{Li}$ , The Translation Invariant Shell Model, Binding Energy, Root-Mean Square Radius, Magnetic Dipole Moment

## 1. Introduction

Research on the nucleus  ${}^6\text{Li}$  and its various properties has gained a lot of attention in the current literature. It has a unique structure among all the p-shell nuclei; the interaction of the four nucleons in the s-shell with the two valence nucleons in the p-shell is very weak. The relation among these valence nucleons plays a crucial role. If  ${}^6\text{Li}$  has the structure  ${}^4\text{He}1 p_{3/2}^2$ , based on the empirical law for producing moments in odd-odd nuclei, its spin should be 3. Since the two nucleons beyond the core of the nucleus add up to a total spin of 1, it may be inferred that this empirical rule is not followed. Theoretically, such a system possesses a magnetic

moment of 0.63 N.M., whereas the experimental value is 0.82 N.M. On the other hand, a large quadrupole moment is expected for the two nucleons in a  $1 p_{3/2}$  state, whereas the experimental value is  $-0.00083$  e barns. The irregularity in filling the energy levels of the nucleus  ${}^6\text{Li}$  lies behind the development of many models to analyze it. Among these models are the intermediate-coupling shell model [1], the cluster model [2], the Bethe-Goldstone integral equation method [3], the Hartree-Fock method [4] [5], the method of deformed orbit [6], the large-basis shell-model [7]-[9], and the translation invariant-shell model [10]-[12].

The translation invariant shell model [11] [12] (TISM), also referred to as the unitary scheme model, enables us to use algebraic techniques to determine the matrix elements of operators that correspond to physical quantities by considering the nucleus as a system of non-interacting quasi-particles. The construction of the basis functions of this model ensures that they possess a certain symmetry under the interchange of particles and a definite orbital angular momentum. This model has shown good results for the structure of light nuclei with mass number  $2 \leq A \leq 7$  [13]-[25].

In the present paper, the ground state wave function of the nucleus  ${}^6\text{Li}$  is expanded in series in terms of the TISM basis functions corresponding to number of quanta of excitations  $N = 2, 4, 6, 8$  and 10. Moreover, the necessary two-particle orbital fractional parentage coefficients (OFPCs) for nuclei with mass number and number of quanta of excitations  $N = 10$  are also calculated to the first time in the present paper. Furthermore, our previously obtained four-parameters nucleon-nucleon interaction [26], with Gaussian radial dependencies, which fits the ground-state characteristics of the two nuclei deuteron and triton is used to calculate the ground-state energy eigenvalue and the corresponding eigenfunction of the nucleus  ${}^6\text{Li}$  as functions of the oscillator parameter  $\hbar\omega$ , which is varied in a wide energy range in order to obtain the best fit to the ground-state energy eigenvalue of this nucleus. The root mean-square radius of  ${}^6\text{Li}$  has also been calculated in this case as function of  $\hbar\omega$ . Moreover, the nuclear supermultiplet model [11] is applied to calculate the magnetic dipole moment of the nucleus  ${}^6\text{Li}$  by using the basis functions of the TISM as functions of  $\hbar\omega$ .

The calculated values of the binding energy, the root-mean square radius and the magnetic dipole moment of the nucleus  ${}^6\text{Li}$  are in excellent agreement with the corresponding experimental and previously published values, which show that our previously obtained four parameters nucleon-nucleon interaction is also suitable for calculations of the ground-state characteristics of the nucleus  ${}^6\text{Li}$ .

## 2. The Hamiltonian and the Nuclear Wave Function

The TISM Hamiltonian, which corresponds to the internal motions of a nucleus consisting of  $A$  nucleons, interacting via nucleon-nucleon interactions, can be written in the form [12] [13] [15]

$$H = H^{(0)} + V' \quad (2.1)$$

where:

$$H^{(0)} = \frac{1}{A} \sum_{i=1}^A \left[ \frac{(\mathbf{p}_i - \mathbf{p}_j)^2}{2m} + \frac{1}{2} m\omega^2 (\mathbf{r}_i - \mathbf{r}_j)^2 \right] \quad (2.2)$$

is the TISM-Hamiltonian, and

$$V' = \sum_{1=i<j}^A \left[ V(|\mathbf{r}_i - \mathbf{r}_j|) - \frac{m\omega^2}{2} (\mathbf{r}_i - \mathbf{r}_j)^2 \right] = \sum_{1=i<j}^A V'_{ij}, \quad (2.3)$$

is the residual interaction.

The Hamiltonian operator (2.2) can be rewritten, with the usual notations, in the form [11] [12]:

$$H^{(0)} = \left[ \sum_{\alpha=1}^3 \sum_{i=1}^{A-1} a_{\alpha i}^+ a_{\alpha i} + \frac{3}{2}(A-1) \right] \hbar\omega. \quad (2.4)$$

It is easy to verify that the Hamiltonian operator (2.4) is invariant with respect to the transformations of the  $3(A-1)$ -dimensional unitary group  $U_{3(A-1)}$ .

The eigenfunctions of the Hamiltonian (2.4) are [11] [12]:

$$\varphi_{\alpha_1 i_1, \dots, \alpha_N i_N} = C a_{\alpha_1 i_1}^+ a_{\alpha_2 i_2}^+ \cdots a_{\alpha_N i_N}^+ \exp \left\{ -\frac{m\omega}{2\hbar} \sum_{\varepsilon=1}^3 \sum_{i=1}^{A-1} \varepsilon^2 \right\}, \quad (2.5)$$

and the corresponding eigenvalues are given by:

$$E_N^{(0)} = \left[ N + \frac{3}{2}(A-1) \right] \hbar\omega. \quad (2.6)$$

The method of calculating the ground state energy eigenvalues and eigenfunctions by using the basis functions of the TISM and the two-particle orbital and spin-isospin fractional parentage coefficients [12] [27] is given in [13]-[21]. Since the functions (2.5) are symmetric with respect to permutations of any pair of their indices, they may be used as basis for irreducible representation (IR) of a symmetric tensor of the rank  $N$ . The Young Scheme  $\{\mathcal{N}\}$  is useful for obtaining such IR. The dimension of the representation  $\{\mathcal{N}\}$  of the group  $U_{3(A-1)}$  is equal to the number of functions  $\varphi_{\alpha_1 i_1, \dots, \alpha_N i_N}$ . The basis functions (2.5) are usually represented by [12] [20] [21]:

$$|A\Gamma M_L; \Gamma_S M_S M_T\rangle \equiv |AN\{\rho\}(v)\alpha[f](\lambda\mu)LM_L; [\tilde{f}]STM_S M_T\rangle, \quad (2.7)$$

where  $\Gamma$  and  $\Gamma_S$  are the sets of all orbital and spin-isospin quantum numbers characterizing the states, respectively. The total number of quanta of excitations  $N$  is an IR of the group  $U_{3(A-1)}$ . The IR of the two groups  $U_3$  and  $U_{A-1}$  are set by the same symbols  $\{\rho\} = \{\rho_1, \rho_2, \rho_3\}$ , where  $\rho_1 \geq \rho_2 \geq \rho_3 \geq 0$  are any integers satisfying the requirements  $\rho_1 + \rho_2 + \rho_3 = N$ . The symbol  $(\lambda\mu)$  of the  $SU_3$  symmetry is determined by the relations  $\lambda = \rho_1 - \rho_2$  and  $\mu = \rho_2 - \rho_3$ , which enable us to find the values of the total orbital angular momentum  $L$ , by using Elliott's rule [10]-[12]. According to this rule:

$L = K, K+1, \dots, K+B$ ;  $K = C, C-2, \dots, 1$  or  $0$  for  $K \neq 0$ ,  $L = B, B-\dots, 1$  or  $0$  if  $K = 0$ , where  $C = \min(\lambda, \mu)$  and  $B = \max(\lambda, \mu)$ .

The allowed Young Schemes  $\{\rho\}$  for the representation  $\{\rho_1, \rho_2, \rho_3\}$  of the

group  $U_{A-1}$  may be found using the formalism of plethysm, which has been described in detail in [11]. In (2.7),  $M_L$  stands for the IR of the group  $SO_2$ . The representation ( $\nu$ ) is an IR of the group  $O_{A-1}$  and the representation  $[f]$  is an IR of the symmetric group  $S_A$ . The quantum numbers  $S, M_S$  are the spin, its projection and  $T, M_T$  are the isospin, its projection, which are IRs of the direct product of the groups  $SU_2 \times SU_2$ . Among all the possible Young schemes  $[f]$ , only those comprising not more than four columns should be selected. Finally, if the values of the total spin and isotopic spin  $S$  and  $T$  are to be taken for the conjugated Young diagrams  $[\tilde{f}]$ , we shall obtain the total list of the TISM states with given number of quanta of excitations  $N$ . The nuclear wave function of a state with total angular momentum  $J$ , isotopic spin  $T$ , and parity  $\pi$  is expanded in series in terms of the basis functions of the TISM, the functions (2.7), as given in [20]. Accordingly, wave function with given total quantum numbers  $J, M_J, T, M_T$ , and parity  $\pi$  can be constructed from the functions (2.7) as follows [20]:

$$|AJ^\pi M_J T M_T\rangle = \sum_{\Gamma M_L; \Gamma_S M_S} C_\Gamma^{J^\pi T} (LM_L, SM_S | JM_J) |A\Gamma M_L; \Gamma_S M_S M_T\rangle \quad (2.8)$$

where  $C_\Gamma^{J^\pi T}$  are the state expansion coefficients and  $(LM_L, SM_S | JM_J)$  are Clebsch-Gordan coefficients (CGC,s).

### 3. The Nucleon-Nucleon Interactions

For the two-nucleon states with orbital angular momentum  $\ell$ , spin momentum  $s$ , and isotopic spin  $t$ , our nucleon-nucleon interaction has the form [26] [28] [29]:

$$V(r) = {}^t X \{V_C(r) + V_T(r)S_{12} + V_{LS}(r)\ell \cdot s + V_{LL}(r)L_{12}\} \quad (3.1)$$

The central, tensor, spin-orbit and quadratic spin-orbit terms are standard and given in details in [28]. The operator  ${}^t X$  has the form:

$${}^t X = C_W + (-1)^{s+t+1} C_M + (-1)^{s+1} C_B + (-1)^{t+1} C_H, \quad (3.2)$$

where  $C_W, C_M, C_B$  and  $C_H$  are the Wigner, the Majorana, the Bartlett and the Heisenberg exchange constants, respectively. Each term of the interaction is expressed as a sum of Gaussian functions in the form

$$V_\alpha(r) = \sum_{k=1}^{2,3,4} V_{\alpha k} e^{-r^2/t_{\alpha k}^2} \quad (3.3)$$

where  $\alpha = C, T, LS$  and  $LL$ .

Two sets of values are usually considered for the exchange constants, namely:  $C_W = 0.1333$ ,  $C_M = -0.9333$ ,  $C_B = -0.4667$ ,  $C_H = -0.2667$ , which are known as the Rosenfeld constants [11], and belong to the symmetric case, and  $C_W = -0.41$ ,  $C_M = -0.41$ ,  $C_B = -0.09$ ,  $C_H = 0.09$  which belong to the Serber case. For the triplet-even state ( $t = 0$ ,  $s = 1$ ), which is the case for the ground-state of the deuteron nucleus, and from the normalization condition of the exchange constants, the operator  ${}^t X$  equals  $-1$  for both of the symmetric and the Serber cases so that the two types of the exchange constants will produce the same results for the ground-state characteristics of the deuteron nucleus.

#### 4. The Two-Particle Orbital Fractional Parentage Coefficients in the TISM

Doma, Kopaleyshvili and Machabeli [12] modified the recurrence method, introduced by Vanagas [11], for calculating the two-particle OFPCs. for nuclei with any mass number  $A$  and they applied the new method for selected basis functions of the TISM corresponding to nuclei with  $A = 6$  and number of quanta if excitations  $N = 2$ , and 4. Furthermore, Doma [13] [16] [19] constructed the basis functions of the TISM for nuclei with  $A = 4$ ,  $N = 0, 2, 4, 6, 8$  and nuclei with  $A = 6$  and  $N = 2, 4$ , and 6. Moreover, a general and direct method for calculating the two-particle spin-isospin FPCs. has been introduced by Doma, Kopaleyshvili and Machabeli [12] and then applied to the calculation of some coefficients of the two-particle spin-isospin FPCs. for nuclei with  $A = 6$ . The resulting two-particle FPCs. are very useful in calculating the matrix elements of any kind of the two-particle operators, such as the central, the tensor, the spin orbit and the quadratic spin orbit operators.

In the following we are interested in the case where the quantum number  $\Gamma$  of the supermultiplet function  $\varphi$  assumes the following classification [11] [12]:

$$\Gamma = [f] \Gamma_0 \Gamma_S, \quad (4.1)$$

where  $[f]$  is an IR of the symmetric group of  $A$  objects,  $S_A, \Gamma_0$  and  $\Gamma_S$  are the sets of all the other orbital and spin-isospin quantum numbers, respectively. Following the method introduced by Vanagas [11], the expansion coefficient of the decomposition of the nuclear wave function with  $A$  particles into its two sub-functions corresponding to  $A'$  and  $A''$  particles, the coefficient  $B_{\Gamma \Gamma_0 \Gamma^* \Gamma}$ , can be factorized in terms of product of orbital and spin-isospin parts as follows:

$$B_{[f] \Gamma_0 \Gamma_S [f'] \Gamma_0 \Gamma_S^* [f] \Gamma_0 \Gamma_S} = \sum_{\alpha_0 \tilde{\alpha}_0} A_{\Gamma_0 \Gamma_0 [f] \alpha_0 \Gamma_0}^{[f] [f']} C_{\Gamma_S^* \Gamma_S^*}^{[f'] [\tilde{f}^*] \tilde{\alpha}_0 [f]} \times C_{\alpha_0 [f'] [\tilde{f}^*] \tilde{\alpha}_0 [f] [f']}^{[f] [f]} \quad (4.2)$$

Here  $\alpha_0$  and  $\tilde{\alpha}_0$  are repetition indices in the orbital and the spin-isospin states, respectively and the last factor under the sum is the isoscaler factor of the C.G.C. of the symmetric group  $S_A$ . The second factor in the right-hand side of Equation (4.2) is a CGC of the unitary group in four dimensions  $U_4$ . The first factor in Equation (4.2) is the orbital FPC. This factor can be factorized as follows [11] [12]:

$$A_{\Gamma_0 \Gamma_0 [f] \alpha_0 \Gamma_0}^{[f] [f']} = \delta_{\rho' + \rho'' \rho} \sum_{\beta} A_{\{\rho'\} \{\rho''\} \{\rho\} (v') \beta, (v) [f] \alpha_0}^{\{\rho\} [f] [f']} \times C_{\gamma' L' M_L'}^{\{\rho'\} \{\rho''\} \beta \{\rho\}} \quad (4.3)$$

where  $\{\rho'\}, \{\rho''\}$  and  $\{\rho\}$  are IRs of the unitary groups in  $3(A'-1)$ ,  $3(A''-1)$  and  $3(A-1)$  dimensions, respectively. They are also IRs of the group  $SU_3$ , simultaneously. The quantum numbers  $L', L''$  and  $L$  are orbital-angular momentum quantum numbers of the sets of  $A', A''$  and  $A$  particles, respectively. Also,  $M_L', M_L''$  and  $M_L$  are the z-projections of  $L', L''$  and  $L$ . The symbol  $\beta$  shows how many times  $\{\rho\}$  appears in the multiplication  $\{\rho'\} \times \{\rho''\}$ . Similarly, the repetition index  $\gamma$  shows how many times  $L$  appears in  $\{\rho\}$ , and

the same situations are also for  $\gamma'$  and  $\gamma''$ . Concerning the repetition index  $\alpha_0$  of Equation (4.2) it shows how many times the IR  $[f]$  appears in  $(\nu)$ , where  $(\nu)$  is an IR of the orthogonal group  $O_A$ . The last factor in (4.3) represents the CGC of the group  $SU_3$ . Substituting from Equations (4.3) into (4.2), we get:

$$B_{[f]\Gamma_0\Gamma_S [f']\Gamma_0\Gamma_S [f]\Gamma_0\Gamma_S} = \sum_{\alpha_0 \tilde{\alpha}_0} \delta_{\rho'+\rho'',\rho} \sum_{\beta} A_{\{\rho'\}(\nu)\{\rho''\}(\nu')\beta,(\nu)[f]\alpha_0}^{(\{\rho\} [f'] [f''])} \quad (4.4)$$

$$\times C_{\gamma' L' M_L' \gamma'' L'' M_L''}^{\{\rho'\} \beta \{\rho''\}} C_{\Gamma_S}^{[\tilde{f}][\tilde{f}'']\tilde{\alpha}_0} C_{\Gamma_S}^{[\tilde{f}][f]\alpha_0} C_{\alpha_0 [f][f']\tilde{\alpha}_0}^{[\tilde{f}][\tilde{f}'']}$$

Equation (4.4) gives the total many-particle FPC of the supermultiplet of the nuclear wave function.

For  $A'' = 2$  and  $A' = A - 2$ , the two-particle total FPC has the form:

$$B_{[\tilde{f}]\bar{\Gamma}_0\bar{\Gamma}_S, [f_{12}]\varepsilon \ell m_\ell s m_s t m_t; [f]\Gamma_0\Gamma_S} \quad (4.5)$$

$$= A_{\{\bar{\rho}\}(\nu)\{\varepsilon\}(\nu_{12});(\nu)[f]}^{(\{\rho\} [\tilde{f}] [f_{12}])} C_{L\bar{M}_L \ell m_\ell L M_L}^{\{\bar{\rho}\} \{\varepsilon\} \{\rho\}} \times C_{\bar{S}\bar{M}_S \bar{T}\bar{M}_T s m_s t m_t S M_S T M_T}^{[\tilde{f}] [f_{12}] [\tilde{f}]} \sqrt{\frac{d_{[\tilde{f}]}}{d_{[f]}}}$$

where  $[f_{12}] = [2]$  or  $[11]$ ,  $\Gamma_0 \equiv N\{\rho\}(\nu)LML$  and  $\Gamma_S \equiv [\tilde{f}]SM_S TM_T$ . The quantities  $d_{[\tilde{f}]}$  and  $d_{[f]}$  are the dimensions of the IRs  $[\tilde{f}]$  and  $[f]$ , respectively. The first coefficient in the right-hand side of Equation (4.5), which is usually called the two-particle orbital FPC is calculated as follows:

$$A_{\{\bar{\rho}\}(\nu)\{\varepsilon\}(\nu_{12});(\nu)[f]}^{(\{\rho\} [\tilde{f}] [f_{12}])} \equiv \sum_{\bar{\rho}'} \sum_{i=1,2} \hat{\lambda}_{\{\bar{\rho}'\}(\nu')(\nu)[f_i]}^{(\{\rho\} [\tilde{f}])} \times U_{[\tilde{f}][f_{12}]}^{([\tilde{f}][\tilde{f}])} D_{\{\bar{\rho}'\},\{\bar{\rho}\}}^{\{\rho\}}(a'), \quad (4.6)$$

where:

$$D_{\bar{\rho},\bar{\rho}'}^{\rho}(a') = D_{m'm}^j(a') \langle (\bar{\rho}(\varepsilon_2 \varepsilon_1) \rho_{12}) \rho | ((\bar{\rho} \varepsilon_2) \bar{\rho}' \varepsilon_1) \rho \rangle \quad (4.7)$$

$$\times \langle ((\bar{\rho} \varepsilon_s) \bar{\rho} \varepsilon_a) \rho | (\bar{\rho}(\varepsilon_s \varepsilon_a) \rho_{12}) \rho \rangle.$$

Here,

$$\varepsilon_1 = \rho - \bar{\rho}', \varepsilon_2 = \bar{\rho}' - \bar{\rho}, \varepsilon_s = \bar{\rho} - \bar{\rho}', \varepsilon_a = \rho - \bar{\rho}, \rho_{12} = \{\rho_1, \rho_2\} \subset \varepsilon_2 \times \varepsilon_1,$$

$$j = \frac{1}{2}(\rho_1 - \rho_2), m' = \frac{1}{2}(\varepsilon_2 - \varepsilon_1), m = \frac{1}{2}(\varepsilon_2 - \varepsilon_1) \text{ and}$$

$$a' = \begin{vmatrix} a'_1 & a'_2 \\ -a'_2 & a'_1 \end{vmatrix}, a'_1 = \sqrt{\frac{A}{2(A-1)}}, a'_2 = -\sqrt{\frac{A-2}{2(A-1)}}$$

In Equation (4.7),  $D_{m'm}^j(a')$  are the matrix elements of the IR of the group  $SU_2$ . and are given by [11]:

$$D_{m'm}^j = \sum_i \frac{[(j+m')!(j-m')!(j+m)!(j-m)!]^{1/2}}{i!(j-m'-i)!(j+m-i)!(i-m+m')!} \times a_1^{j+m-i} a_2^i a_3^{i-m+m'} a_4^{j-m'-i} \quad (4.8)$$

The last two factors in the right-hand side of Equation (4.7) are the recoupling matrix elements of the IR of the  $SU_3$  group and are given by Vanagas [11]. The factors  $U_{[\tilde{f}][f_{12}]}^{([\tilde{f}][\tilde{f}])}$  in (4.6) are the recoupling matrix elements of the IR of the group  $S_A$  and are given by Vanagas [11].

The coefficients  $\hat{\lambda}_{\{\bar{\rho}'\}(\nu')(\nu)[f_i]}^{(\{\rho\} [\tilde{f}])}$ , of Equation (4.6) are orthonormal solutions

of the following system of linear homogeneous equations:

$$\begin{aligned} & \sum_{\bar{\rho}'} \left\langle \left\{ \rho \right\} \left\{ \bar{\rho} \right\} \left\{ \bar{\rho} \right\} \left| P_{A-1,A} \left| \left\{ \rho \right\} \left\{ \bar{\rho} \right\} \left\{ \bar{\rho} \right\} \right\rangle \right\rangle \dot{\Lambda}_{\left\{ \rho \right\} \left\{ \bar{\rho} \right\} (v), (v) [f] [\bar{f}]}^{\left\{ \rho \right\} [\bar{f}]} \\ & = \sum_f D_{\left[ \bar{f} \right] \left[ \bar{f} \right], \left[ \bar{f} \right] \left[ \bar{f} \right]}^{\left[ f \right]} \left( P_{A-1,A} \right) \dot{\Lambda}_{\left\{ \rho \right\} \left\{ \bar{\rho} \right\} (v), (v) [f] [\bar{f}]}^{\left\{ \rho \right\} [\bar{f}]} \end{aligned} \quad (4.9)$$

where the matrix elements  $\left\langle \left\{ \rho \right\} \left\{ \bar{\rho} \right\} \left\{ \bar{\rho} \right\} \left| P_{A-1,A} \left| \left\{ \rho \right\} \left\{ \bar{\rho} \right\} \left\{ \bar{\rho} \right\} \right\rangle \right\rangle$  can be obtained from  $D_{\bar{\rho}, \bar{\rho}'}^{\rho} (a')$  of Equation (4.7), by replacing the determinant  $a'$  by  $a$ , where:

$$a = \begin{vmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{vmatrix}, \quad a_1 = \frac{1}{A-1} \quad \text{and} \quad a_2 = \frac{\sqrt{A(A-2)}}{A-1}$$

The second factor in the right-hand side of Equation (4.5) is the CGC of the  $SU_3$  group and can be calculated from the chain of groups  $SU_3 \supset R_3$  where  $R_3$  is the rotational group in 3-dimensions, as follows:

$$C_{\bar{L} \bar{M}_L \ell m_\ell | L M_L}^{\left\{ \bar{\rho} \right\} \left\{ \varepsilon \right\} \left\{ \rho \right\}} = \left( \bar{L} \bar{M}_L, \ell m_\ell | L M_L \right) C_{\bar{L} \bar{\ell} \bar{L}}^{\left\{ \bar{\rho} \right\} \left\{ \varepsilon \right\} \left\{ \rho \right\}}, \quad (4.10)$$

where  $\left( \bar{L} \bar{M}_L, \ell m_\ell | L M_L \right)$  is a CGC of the group  $R_3$  and  $C_{\bar{L} \bar{\ell} \bar{L}}^{\left\{ \bar{\rho} \right\} \left\{ \varepsilon \right\} \left\{ \rho \right\}}$  is an isoscalar factor of the  $SU_3$  group. The isoscalar factor of Equation (4.10) can be rewritten in the form:

$$C_{\bar{L} \bar{\ell} \bar{L}}^{\left\{ \bar{\rho} \right\} \left\{ \varepsilon \right\} \left\{ \rho \right\}} = \left\langle \left( \bar{\lambda} \bar{\mu} \right) \bar{L}, (\lambda_a \mu_a) \ell \left| (\lambda \mu) L \right\rangle, \quad (4.11)$$

where  $\lambda_a = \varepsilon_1 - \varepsilon_2$ ,  $\mu_a = \varepsilon_2$ ,  $\bar{\lambda} = \bar{\rho}_1 - \bar{\rho}_2$ ,  $\bar{\mu} = \bar{\rho}_2 - \bar{\rho}_3$ ,  $\lambda = \rho_1 - \rho_2$  and  $\mu = \rho_2 - \rho_3$ . All the isoscalar factors needed in our calculations can be found in refs. [11] [12].

## 5. The Method of Calculations

The matrix elements of the residual two-body interaction  $V'$  with respect to the basis (2.8) are given by [13] [16] [19].

$$\begin{aligned} & \left\langle A J^\pi M_J T M_T \left| V' \right| A J^\pi M_J T M_T \right\rangle \\ & = \frac{A(A-1)}{2} \sum_{\bar{\Gamma} \varepsilon \ell s j t \ell'} C_{\Gamma}^{J^\pi T} C_{\Gamma'}^{J^\pi T} (L M_L, S M_S | J M_J) (L' M'_L, S' M'_S | J M_J) \\ & \quad \times \left\langle A \Gamma J T \left| A - 2 \bar{\Gamma}; 2 \varepsilon \ell s j t \right\rangle \times \left\langle A \Gamma' J T \left| A - 2 \bar{\Gamma}; 2 \varepsilon' \ell' s' j t \right\rangle \begin{pmatrix} \bar{L} & \bar{S} & \bar{J} \\ \ell & s & j \\ L & S & J \end{pmatrix} \right. \\ & \quad \times \begin{pmatrix} \bar{L} & \bar{S} & \bar{J} \\ \ell' & s' & j' \\ L' & S' & J' \end{pmatrix} \left. \left\langle (\varepsilon \ell s) j t \left| V'_{A-1,A} \right| (\varepsilon' \ell' s') j t \right\rangle \right. \end{aligned} \quad (5.1)$$

Here,  $\bar{\Gamma}$  stands for the set of all orbital and spin-isospin quantum numbers characterizing the set of  $A-2$  particles. The number of quanta of excitations for the two-particle wave function is  $\varepsilon = 2n + \ell$ , in which  $n$  is the radial quantum number of the inter-particle distance joining the last pair. The quantum numbers  $\ell, s, j$  and  $t$  are the orbital, the spin, the total spin and the isospin quantum numbers of the last pair, respectively. In Equation (5.1),  $\left\langle A \Gamma J T \left| A - 2 \bar{\Gamma}; 2 \varepsilon \ell s j t \right\rangle$  are

the two-particle total FPCs which are products of orbital and spin-isospin coefficients. The last elements in Equation (5.1) are the two-particle matrix elements of the residual interaction where:

$$V'_{A-1,A} = V'(|r_{A-1} - r_A|) \text{ and } \begin{pmatrix} J_1 & J_2 & J_{12} \\ J_3 & J_4 & J_{34} \\ J_{13} & J_{24} & J \end{pmatrix} \text{ are the normalized } 9j\text{-symbols.}$$

Thus, the energy matrices can be constructed according to Equations (2.6) and (5.1) for the different states of a nucleus with mass number  $A$  and for each residual interaction, as functions of the oscillator parameter  $\hbar\omega$ . These matrices are diagonalized with respect to  $\hbar\omega$  which is allowed to vary in a wide range of values,  $8 \leq \hbar\omega \leq 30$  MeV, to obtain the best fit to the ground-state energy eigenvalue of this nucleus. Hence, the energy eigenvalues and the corresponding eigenfunctions of the ground-state of this nucleus are obtained for each considered potential.

The ground-state nuclear wave function, which is obtained because of the diagonalization of the ground-state energy matrix, is used to calculate the nuclear root mean-square radius from the well-known formula [16] [20] [21].

$$R = \sqrt{r_p^2 + \langle R_{Nuc}^2 \rangle}, \tag{5.2}$$

where  $r_p = 0.85$  fm is the proton radius and the second term is the mean value of the operator

$$R_{Nuc}^2 = \frac{1}{A^2} \sum_{1=i<j}^A r_{ij}^2. \tag{5.3}$$

The method of calculating the nuclear root mean-square radius by using the basis functions of the TISM together with the two-particle orbital FPCs are given in [16] [20] [21] and the final formula is given by:

$$R = \left[ r_p^2 + \frac{A-1}{A[m\omega/\hbar]} \times \sum_{\Gamma\Gamma\bar{\Gamma}\varepsilon\ell s j t \varepsilon' \ell'} C_{\Gamma}^{J\pi T} C_{\Gamma'}^{J\pi T} \times \langle A\Gamma J T | A-2\bar{\Gamma}; 2\varepsilon\ell s j t \rangle \right. \\ \times \langle A\Gamma J T | A-2\bar{\Gamma}; 2\varepsilon\ell s j t \rangle \left\{ \left( \varepsilon + \frac{3}{2} \right) \delta_{\varepsilon',\varepsilon} - \frac{1}{2} \sqrt{(\varepsilon - \ell + 2)(\varepsilon + \ell + 3)} \delta_{\varepsilon',\varepsilon+2} \right. \\ \left. - \frac{1}{2} \sqrt{(\varepsilon - \ell)(\varepsilon + \ell + 1)} \delta_{\varepsilon',\varepsilon+2} \right\} \delta_{\ell',\ell} \delta_{(\nu'),(\nu)} \delta_{[f'],[f]} \delta_{S',S} \left. \right]^{\frac{1}{2}}, \tag{5.4}$$

The magnetic dipole moment of a nucleus is defined as the mean-value of the operator  $\hat{\mu}$  which is written in the form [12] [24].

$$\hat{\mu} = \hat{\mu}_o + \hat{\mu}_\sigma, \tag{5.5}$$

where the orbital part  $\hat{\mu}_o$  is given by

$$\hat{\mu}_o = \frac{1}{2} \sum_{i=1}^A (1 - 2t_{oi}) \ell_{oi}, \tag{5.6}$$

and the spin-isospin part  $\hat{\mu}_\sigma$  is given by

$$\hat{\mu}_\sigma = \sum_{i=1}^A [(\mu_p + \mu_n) + 2(\mu_p - \mu_n)t_{oi}] s_{oi}, \tag{5.7}$$

calculated in a state with  $M_J = J$ . In Equations (5.6) and (5.7),  $\ell_{oi}$ ,  $s_{oi}$  and

$t_{oi}$  are the z-components of the orbital angular momentum, the spin momentum and the isotopic spin momentum of the  $i$ th-nucleon, respectively. The quantities  $\mu_p$  and  $\mu_n$  are the proton and the neutron magnetic moments, respectively. The method of calculating the nuclear magnetic dipole moment is given in [24].

## 6. Results and Discussions

In **Table 1**, we give the depth and the range parameters for the four-parameters potential, which has been introduced in our previous paper [26]. This potential produced good results for the deuteron characteristics [14] as well as for the ground-state energy, the root-mean square radius, and the magnetic dipole moment of the triton nucleus [26], so that it is important to know if it could be used successfully for the ground-state characteristics of the nucleus  ${}^6\text{Li}$ .

In **Table 2**, we present the calculated values of the binding energy (B.E.), the root-mean-square radius ( $R$ ), and the magnetic dipole moment ( $\mu$ ) of the nucleus  ${}^6\text{Li}$  by using our potential (Kharroube pot). Also, we present in this table the corresponding experimental values [11] [12]. Moreover, we present in **Table 2** the corresponding previous values by using the Gogny, Pires and De Tourreil (GPT)-pot [28] and the Argon (Av8') pot [30] [31]. The values of  $\hbar\omega$ , for which the binding energy of the nucleus  ${}^6\text{Li}$  is in good agreement with the corresponding experimental value, are also given in both cases.

In **Figure 1**, we present the variation of the binding energy of the nucleus  ${}^6\text{Li}$  with respect to the oscillator parameter  $\hbar\omega$ .

**Table 1.** Depth and range parameters for the four-parameters potential [26].

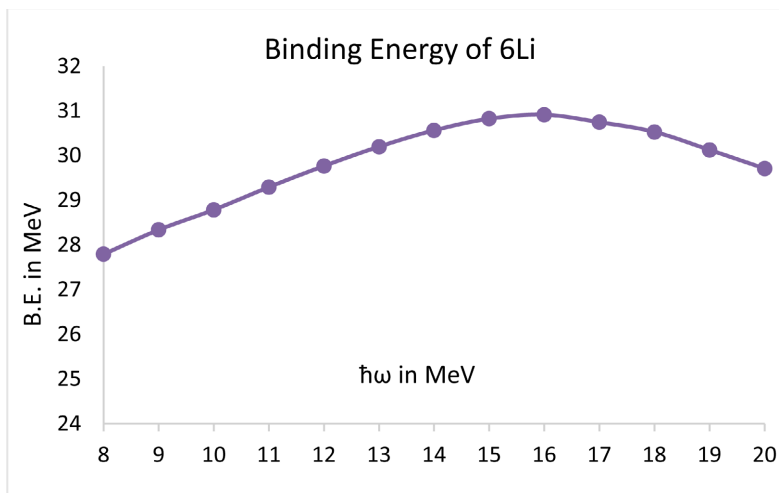
Parameter	$V_{C_1}$	$V_{C_2}$	$V_{C_3}$	$V_{C_4}$	$V_{T_1}$	$V_{T_2}$	$V_{T_3}$	$V_{T_4}$
Value	39.956	-91.468	-30.154	-80.243	-19.698	40.233	-19.574	50.221
Parameter	$r_{C_1}$	$r_{C_2}$	$r_{C_3}$	$r_{C_4}$	$r_{T_1}$	$r_{T_2}$	$r_{T_3}$	$r_{T_4}$
Value	1.356	0.622	0.973	0.497	2.473			
Parameter	$V_{S_1}$	$V_{S_2}$	$V_{S_3}$	$V_{S_4}$	$V_{L_1}$	$V_{L_2}$	$V_{L_3}$	$V_{L_4}$
Value	-25.266	40.213	-22.654	50.212	-10.914	31.554	-18.431	66.221
Parameter	$r_{S_1}$	$r_{S_2}$	$r_{S_3}$	$r_{S_4}$	$r_{L_1}$	$r_{L_2}$		$r_{L_4}$
Value	0.852	0.501	0.774	0.434	1.985	0.228	2.295	0.416

**Table 2.** Binding energy, root-mean-square radius, and magnetic dipole moment of  ${}^6\text{Li}$ .

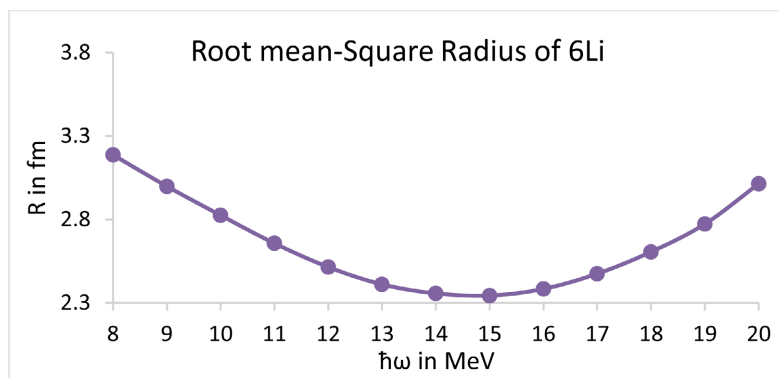
Case	B.E. (MeV)	$R$ (fm)	$\mu$ (N.M.)	$\hbar\omega$ (MeV)
GPT pot [21]	27.634	2.491	0.836	14
Av8' pot [21]	29.744	2.361	0.839	15
Kharroube pot	30.914	2.344	0.827	16
Exp. [30] [31]	32.0	2.32	0.822	---

Also, in **Figure 2** we present the variation of the root-mean square radius of the nucleus  ${}^6\text{Li}$  with respect to the oscillator parameter  $\hbar\omega$ .

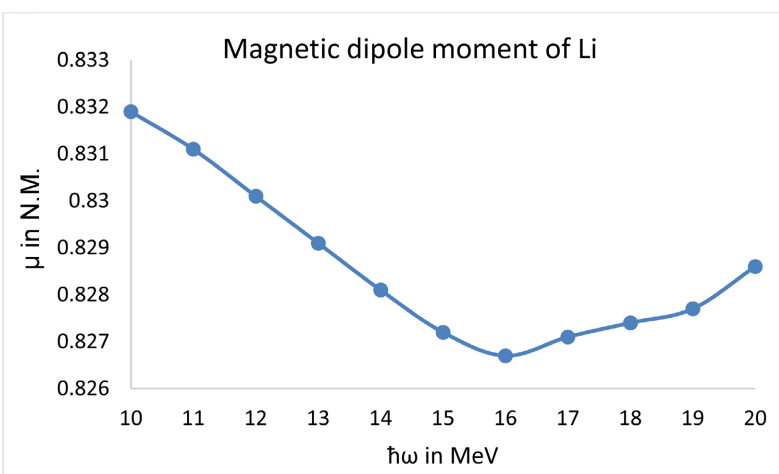
Moreover, in **Figure 3** we present the variation of the magnetic dipole moment of the nucleus  ${}^6\text{Li}$  with respect to the oscillator parameter  $\hbar\omega$ .



**Figure 1.** Variation of the binding energy of  ${}^6\text{Li}$  with  $\hbar\omega$ .



**Figure 2.** Variation of the root-mean-square radius of  ${}^6\text{Li}$  with  $\hbar\omega$ .



**Figure 3.** Variation of the magnetic dipole moment of  ${}^6\text{Li}$  with  $\hbar\omega$ .

It is seen from **Figure 2** and **Figure 3**, that the root-mean square radius and the magnetic dipole moment of  ${}^6\text{Li}$  have minimum values. Furthermore, it is seen from **Figure 1** that the binding-energy curve has maximum value at  $\hbar\omega = 16$  MeV, *i.e.*, the ground-state energy eigenvalue has minimum value at this value. This result shows that the method has been correctly applied to the nucleus  ${}^6\text{Li}$ . The result occurred for the magnetic dipole moment of  ${}^6\text{Li}$  at  $\hbar\omega = 16$  MeV agreed with that of the binding energy case. Concerning the root-mean square radius of  ${}^6\text{Li}$ , the minimum value occurs at a nearing value ( $\hbar\omega = 15$  MeV), a result which always occurred since we are not concerned with the spectrum of this nucleus.

It is seen from **Table 2**, that the calculated values of the binding energy, the root mean-square radius and the magnetic dipole moment of the nucleus  ${}^6\text{Li}$  are in good agreement with the corresponding experimental values.

## 7. Conclusion

The inclusion of states corresponding to number of quanta of excitations  $N = 10$  in our present calculations has mainly improved the previous results obtained by using the GPT-potential with basis functions belonging to  $N \leq 8$  [14]. Furthermore, our four-parameters potential gave results for the ground-state characteristics of the nucleus  ${}^6\text{Li}$  in good agreement with the corresponding experimental values. Accordingly, our four-parameter potential describes the ground-state characteristics of the deuteron, the triton, and the  ${}^6\text{Li}$  nuclei well.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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