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Investigating the Mathematical Foundations of the Euler and Navier-Stokes Equations

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Abstract

This study examines the mathematical foundations of the Euler and Navier-Stokes equations of fluid dynamics, identifying some inconsistencies in the mathematical definitions of flow velocity and the material derivative. We show that the flow velocity of a fluid parcel, which in the Lagrangian description is traditionally modeled as a bivariate function of the presumed independent variables of initial parcel position and time, is more accurately defined as a parametric function of time, with the initial parcel position treated as a time-dependent parameter. This finding leads to the result that the standard form of the material derivative in the Lagrangian description is mathematically inconsistent. We also show that if the fluid flow is non-unidirectional, then the map from parcel position to flow velocity becomes a one-to-many map, leading to the conclusion that the flow velocity is not a valid mathematical function of position in both the Lagrangian and Eulerian descriptions under such conditions. Therefore, if flow velocity is not a valid mathematical function of position, we conclude that the inability to integrate the Euler and Navier-Stokes differential equations in the spatial domain implies the nonexistence of a mathematical solution of these equations under these conditions. Additionally, through mathematical and theoretical analysis, supported by experimental and numerical simulations, we uncover challenges in the material consistency of the definition of the material derivative in the Eulerian description. This inconsistency leads to a decoupling between the Lagrangian and Eulerian descriptions, especially under complex non-unidirectional flow conditions and multi-directional flows with intersecting pathlines. We also show that the Eulerian description is a quasi-continuum mechanics model that, when applied to certain fluids, especially gases and low-viscosity liquids where intermolecular forces are weak or intermediate, limits the ability to accurately model the bi-directional transmission of deformation and force continuously between neighboring parcels. While the Euler and Navier-Stokes equations remain largely valid and effective for modeling unidirectional flows in viscous fluids, our findings suggest the need to refocus on developing fluid dynamics solutions rooted in the Lagrangian model to more accurately capture complex flow behaviors and improve applicability across fields such as atmospheric sciences, oceanography, and plasma physics. These insights aim to advance our understanding of the limits of existing fluid dynamics models by addressing foundational inconsistencies, the understanding of which can contribute to refining these mathematical models.

Keywords

Euler's Equation, Navier-Stokes Equations, Fluid Dynamics, Fluid Mechanics

1. Introduction

Matter is typically classified into three primary phases: solid, liquid, and gas, with liquids and gases collectively termed fluids. In gases, atoms and molecules are typically far apart, moving randomly and freely relative to one another, whereas liquids have more closely packed molecules that lack the rigid structure of solids.

Gas molecules create internal static pressure through their constant, active collisions with one another within the confined space of the gas. In contrast, liquids develop internal static pressure as a passive reaction of their particles, which resist external forces that attempt to change the liquid's volume [1]-[3]. From a metrological perspective, static pressure can be visualized as the pressure measured by a neutrally buoyant pressure transducer immersed in the fluid, and moving at the same velocity as the fluid.

The study of fluid behavior and properties is a cornerstone of modern science and engineering, with applications in fields such as mechanical engineering, environmental science, oceanography, and astrophysics [2] [4]-[9].

The mathematical modeling of fluid dynamics has traditionally relied on two fundamental equations: Euler's equation for inviscid flow (Equation (1)), and the Navier-Stokes equations (NSE) for viscous flow (Equation (2)), as follows:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right)=-\nabla p+\rho\boldsymbol{g} \tag{1}$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right)=-\nabla p+\mu\nabla^2\boldsymbol{u}+\rho\boldsymbol{g} \tag{2}$$

where ρ is the fluid density, $\boldsymbol{u}$ is the flow velocity vector, p is the static pressure, $\boldsymbol{g}$ is the gravitational acceleration, and μ is the dynamic viscosity. The first term inside the brackets, $\frac{\partial \boldsymbol{u}}{\partial t}$ represents the local acceleration, while the second term, $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, is the convective acceleration; the sum of these two terms is known as the material derivative. Together, these two equations form the cornerstone of our understanding of fluid motion. However, studies suggest that the traditional form of the Navier-Stokes equations shown in Equation (2) may not adequately capture the behavior of complex flows [10]-[12].

Despite their widespread use, the Euler and Navier-Stokes equations rely on some crucial assumptions about the nature of flow velocity and its dependence on position and time. In particular, the representation of parcel velocity as a bivariate function of two presumed mutually independent variables, namely the initial parcel position and time in the Lagrangian description, and the use of the material derivative to link the Lagrangian and Eulerian descriptions of fluid motion have long been accepted without sufficient scrutiny.

In this work, we critically re-examine the mathematical foundations of these equations, focusing on inconsistencies in the traditional definitions of velocity and the material derivative. Through mathematical and theoretical analysis, experimental evidence, and numerical simulations, we aim to address these challenges and contribute to a more robust understanding of the limits of the traditional mathematical models of fluid dynamics.

2. Methods

2.1. Analyzing the Trajectory of a Fluid Parcel along Its Pathline

In fluid dynamics, a pathline represents the actual trajectory that an individual fluid particle follows as it moves through space and time [3]. This concept is associated with the Lagrangian description, which tracks and describes the physical attributes of an isolated fluid particle or a small control volume, collectively referred to as a parcel. This parcel contains a minute quantity of fluid atoms or molecules, facilitating the monitoring of its properties as the parcel travels along its pathline.

In this context, the function $\boldsymbol{x}(t)$, which describes the parcel's position as a function of the independent variable time t, provides sufficient information to determine the instantaneous velocity and kinetic energy of the parcel at any given t along the pathline. Thus, $\boldsymbol{x}(t)$ effectively captures the complete dynamic state of the parcel throughout its movement in the flow field.

Let us define the above using mathematical notation.

1) Let $\boldsymbol{x}:\mathbb{R}\to\mathbb{R}^3$ be a pathline of the center of mass of a fluid parcel in Euclidean space. Then:

$$\begin{aligned}&\boldsymbol{x}=\boldsymbol{x}(t),\ \ t\in\mathbb{R}\\&\boldsymbol{x}(0)=\boldsymbol{0}\end{aligned}\tag{3}$$

where t is time, and $\boldsymbol{0}$ is the origin of an arbitrary inertial reference frame. It is worth emphasizing that in Equation (3) the variable t is an independent variable and $\boldsymbol{x}$ is a dependent variable. Consequently we can conclude that, in the Lagrangian description, the position $\boldsymbol{x}$ of the parcel cannot be varied independently of t. This is a subtle yet important conclusion that we will build on below.

2) The position $\boldsymbol{x}_0$ of the parcel at time t_0 along the pathline can be expressed in terms of the function $\boldsymbol{x}(t)$ as:

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0) \tag{4}$$

2.2. Analyzing Velocity as a Univariate Function $\boldsymbol{u}(t)$ in the Lagrangian Description

We are now ready to derive the functional relationship between parcel velocity $\boldsymbol{u}$ in the domain t. Let $\boldsymbol{x}(t)$ be the position of a fluid parcel as a function of t, as defined in Section 2.1. Then:

1) The first derivative of $\boldsymbol{x}$ with respect to t is a univariate function of t.

From Section 2.1, we know that $\boldsymbol{x} = \boldsymbol{x}(t)$ is a univariate function of t. The first derivative of a univariate function is also a univariate function. Therefore, $\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{x}'(t)$ is a univariate function of t.

2) The instantaneous velocity $\boldsymbol{u}(t)$ of the parcel along its pathline is a univariate function of t.

The instantaneous velocity, $\boldsymbol{u}(t)$, is defined as the first derivative of position with respect to time:

$$\boldsymbol{u}(t) = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{x}'(t) \tag{5}$$

Since $\boldsymbol{x}'(t)$ is a univariate function of t (as established in part (a) above), $\boldsymbol{u}(t)$ is also a univariate function of t.

3) The velocity $\boldsymbol{u}$ is not a bivariate function of t and $\boldsymbol{x}$.

To show that $\boldsymbol{u}$ is not a bivariate function of t and $\boldsymbol{x}$, we consider the following.

- From Section 2.1(a), we know that $\boldsymbol{x} = \boldsymbol{x}(t)$.
- This means $\boldsymbol{x}$ is a dependent variable, fully determined by t.
- For $\boldsymbol{u}$ to be a bivariate function, it would need to depend on two independent variables.
- However, $\boldsymbol{x}$ is not independent of t, as established by Section 2.1(a).
- Therefore, $\boldsymbol{u}$ cannot be a bivariate function of t and $\boldsymbol{x}$.

Thus, $\boldsymbol{u}$ is a univariate function of t, *i.e.*, $\boldsymbol{u} = \boldsymbol{u}(t)$.

2.3. Analyzing the Lagrangian Position Function x_0

In the Lagrangian description, the position $\boldsymbol{X}$ of a fluid parcel relative to an arbitrary non-inertial reference frame, such that the origin of the non-inertial reference frame is at a given position along the parcel's pathline, can be expressed as follows.

1) Let the origin of the non-inertial reference frame be at position $\boldsymbol{x}_0$ along the parcel's pathline. Then:

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}_0, t) \tag{6}$$

where t is time, and $\boldsymbol{x}_0$ is the origin of the non-inertial reference frame.

This part is given as a definition and does not require validation.

2) The position function $\boldsymbol{X}(\boldsymbol{x}_0, t)$ is identical to the univariate parametric

function $\boldsymbol{X}\left(\boldsymbol{x}_0\left(t_0\right),t\right)$ of t, where $\boldsymbol{x}_0\left(t\right)$ is the parameter evaluated at $t=t_0$:

$$\boldsymbol{X}\left(\boldsymbol{x}_0,t\right)\equiv\boldsymbol{X}\left(\boldsymbol{x}_0\left(t_0\right),t\right) \tag{7}$$

To verify Equation (7), we use Section 2.1(b), where we have:

$$\boldsymbol{x}_0=\boldsymbol{x}\left(t_0\right) \tag{8}$$

This means that $\boldsymbol{x}_0$ can be expressed as $\boldsymbol{x}_0\left(t_0\right)$, where t_0 is the time at which the origin of the non-inertial reference frame is defined.

Substituting this into the left-hand side of Equation (7):

$$\boldsymbol{X}\left(\boldsymbol{x}_0,t\right)=\boldsymbol{X}\left(\boldsymbol{x}\left(t_0\right),t\right)\equiv\boldsymbol{X}\left(\boldsymbol{x}_0\left(t_0\right),t\right) \tag{9}$$

This validates the equivalence stated in Part (b) above.

2.4. Analyzing the Lagrangian Velocity Function and Its Material Derivative

In the Lagrangian description:

1) The velocity of a fluid parcel relative to a non-inertial reference frame with origin at position $\boldsymbol{x}_0$ along the parcel's pathline can be expressed as the first derivative of the position function $\boldsymbol{X}\left(\boldsymbol{x}_0\left(t_0\right),t\right)$ with respect to t, which is a univariate parametric function of t with parameter $\boldsymbol{x}_0\left(t_0\right)$.

From Section 2.3(b), we know that the position of a fluid parcel in the Lagrangian description can be expressed as $\boldsymbol{X}\left(\boldsymbol{x}_0,t\right)\equiv\boldsymbol{X}\left(\boldsymbol{x}_0\left(t_0\right),t\right)$.

The instantaneous velocity is defined as the first derivative of position with respect to time:

$$\boldsymbol{u}\left(t\right)=\frac{\partial\boldsymbol{X}\left(\boldsymbol{x}_0\left(t_0\right),t\right)}{\partial t} \tag{10}$$

This is a univariate parametric function because:

- t is the only independent variable being differentiated with respect to.
- $\boldsymbol{x}_0\left(t_0\right)$ is a fixed parameter, that is evaluated at a specific time t_0.

This is consistent with Section 2.2(a), which states that the first derivative of position with respect to time is a univariate function of time.

2) The velocity $\boldsymbol{u}$ is not a bivariate function of t and $\boldsymbol{x}_0\left(t\right)$, because $\boldsymbol{x}_0\left(t_0\right)$ is a dependent variable $\boldsymbol{x}_0\left(t\right)$ evaluated at $t=t_0$, and not an independent variable.

To show that $\boldsymbol{u}$ is not a bivariate function of t and $\boldsymbol{x}_0\left(t\right)$, we consider:

- From Section 2.1(b), we know that $\boldsymbol{x}_0=\boldsymbol{x}_0\left(t\right)$ is a function of t.
- In the expression $\boldsymbol{X}\left(\boldsymbol{x}_0\left(t_0\right),t\right)$, the term $\boldsymbol{x}_0\left(t_0\right)$ is a constant parameter, not an independent variable as established by Section 2.1(b).
- For $\boldsymbol{u}$ to be a bivariate function, it would need to depend on two independent variables.

This is consistent with Section 2.2(c), which states that velocity is not a bivariate function of time and position.

Therefore, $\boldsymbol{u}$ is solely a function of t, *i.e.*, $\boldsymbol{u}=\boldsymbol{u}\left(t\right)$, with $\boldsymbol{x}_0\left(t_0\right)$ serving as a parameter, not an independent variable.

3) The velocity of the fluid parcel in an inertial reference frame is the sum of the velocity in the non-inertial reference frame plus the velocity of the reference frame:

$$\boldsymbol{u}_{\text{inertial}} = \boldsymbol{u}_{\text{non-inertial}} + \frac{\partial \boldsymbol{x}_0}{\partial t} \tag{11}$$

Equation (11) follows directly from the definition of inertial and non-inertial reference frames.

4) Referring to Equation (11), from the additivity property of derivatives, we know that the total derivative of velocity in the inertial reference frame can be expressed as:

$$\frac{D\boldsymbol{u}_{\text{inertial}}}{Dt} = \frac{\partial \boldsymbol{u}_{\text{non-inertial}}}{\partial t} + \frac{\partial \boldsymbol{u}_0(t)}{\partial t} \tag{12}$$

where $\boldsymbol{u}_0(t) = \frac{\partial \boldsymbol{x}_0(t)}{\partial t}$.

5) In the Lagrangian description, the traditional definition of material derivative of the velocity of the parcel can be equated to the total derivative of velocity of the parcel in the inertial reference frame. This total derivative is, in turn, equal to the sum of the derivative of velocity of the parcel in the local non-inertial reference frame, and the derivative of the velocity of the non-inertial reference frame relative to an arbitrary inertial reference frame, as follows:

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \frac{\partial \boldsymbol{u}(t)}{\partial t} + \frac{\partial \boldsymbol{u}_0(t)}{\partial t} \tag{13}$$

where $\frac{\partial \boldsymbol{u}}{\partial t}$ is the local time derivative of velocity, and $(\boldsymbol{u} \cdot \nabla \boldsymbol{u})$ is the advective term representing the change in velocity due to the fluid's motion. By definition, the local velocity $\boldsymbol{u}(t) = \boldsymbol{u}_{\text{non-inertial}}$.

To validate Equation (13) and show that the standard definition of material derivative is identical to total derivative of velocity (the acceleration of the parcel) in the inertial reference frame.

- The standard definition of material derivative is:

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \tag{14}$$

- The acceleration of the parcel in the inertial reference frame is:

$$\boldsymbol{a}_{\text{inertial}} = \frac{\partial \boldsymbol{u}_{\text{inertial}}}{\partial t} = \frac{\partial}{\partial t}\left(\boldsymbol{u}_{\text{non-inertial}} + \boldsymbol{u}_0\right) = \frac{\partial \boldsymbol{u}_{\text{non-inertial}}}{\partial t} + \frac{\partial \boldsymbol{u}_0}{\partial t} \tag{15}$$

- From Newton's second law, we know that the total momentum $\rho \boldsymbol{u}$ in the inertial reference frame must be conserved. Therefore:

$$\frac{D(\rho \boldsymbol{u}_{\text{inertial}})}{Dt} = \rho \frac{D\boldsymbol{u}_{\text{inertial}}}{Dt} = \rho \boldsymbol{a}_{\text{inertial}} \tag{16}$$

- Combining Equations (14), (15), and (16), we can see that:

$$\frac{D\boldsymbol{u}}{Dt} = \frac{D\boldsymbol{u}_{\text{inertial}}}{Dt} \boldsymbol{u} = \boldsymbol{a}_{\text{inertial}} \tag{17}$$

$$\frac{\partial \boldsymbol{u}}{\partial t}+\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=\frac{\partial \boldsymbol{u}_{\text{non-inertial}}}{\partial t}+\frac{\partial \boldsymbol{u}_0}{\partial t} \tag{18}$$

- This proves that the standard definition of material derivative in Equation (14) is indeed identical to Equation (12) which is the acceleration of the parcel in the inertial reference frame. Comparison of the terms in Equation (18) reveals that the convective term of the material derivative corresponds to the acceleration of the non-inertial (local) reference frame, as shown in Equation (19):

$$\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=\frac{\partial \boldsymbol{u}_0}{\partial t} \tag{19}$$

2.5. Relation between Pathlines and Streamlines in Lagrangian and Eulerian Descriptions

We defined the pathline and its relation to the Lagrangian description in Section 2.1 above. We now introduce the Eulerian description, which is used to describe the behavior of fluids relative to specific locations in space without following any specific fluid parcel.

In the Eulerian description, fluid flow is described by observing how flow properties, such as velocity, vary as functions of position and time relative to a fixed reference position that is decoupled from the motion of a specific volume of fluid material. Therefore, we can refer to the Lagrangian description as being material-centric, while the Eulerian description is position-centric.

In the Eulerian description, we can define a streamline as a curve that is tangent to the velocity vector of the flow at every point within the fluid volume. Streamlines provide a visual representation of the direction of fluid flow at a given instant.

In the Eulerian description, the velocity field is traditionally assumed to be a bivariate function of position and time. We will demonstrate below how this assumption can lead to an inconsistency between the mathematical and physical definitions.

For a fluid in motion:

1) A volume that does not contain fluid parcels cannot contain any pathlines by definition as shown in Section 2.1, and therefore, the velocity field inside such a hypothetical volume is everywhere null.

The above statement is self-evident because if there are no fluid particles, then there can be no fluid motion that defines pathlines as per Section 2.1. Consequently, we find that the velocity field is null everywhere.

2) In an arbitrary volume of fluid comprising multiple fluid parcels in a state of flow, given a fluid parcel that occupies a given position at a given time along the parcel's pathline in the Lagrangian description, the velocity vector of the parcel is identical to the value of the velocity vector of the streamline passing through the said given position at the said given time in the Eulerian description.

To validate the above statement, we refer to Section 2.1, where we know that the motion of a fluid parcel is defined by its pathline in the Lagrangian description. Section 2.2(b) establishes that the instantaneous velocity $\boldsymbol{u}(t)$ of the parcel along its pathline is a univariate function of time. By the definition of a streamline, at any given time and position, the velocity vector of the streamline is tangent to the local fluid velocity. Therefore, for a parcel at a given position and time, its velocity vector in the Lagrangian description must be identical to the velocity vector in the Eulerian description at that same position and time.

3) In an arbitrary volume of fluid comprising multiple fluid parcels in a state of flow in the Lagrangian description, a snapshot of the velocity vectors of all the parcels inside the volume is identical to the velocity vectors at every point inside the volume where the velocity is not null, at the given time in the Eulerian description.

This follows as a logical extension of Part (b) above. If the velocity vector of each individual parcel is identical to the velocity vector of the streamline passing through its position at a given time, then the collection of all parcel velocity vectors at that time must be identical to the collection of all velocity vectors inside the volume.

4) A continuous curve in space that connects the velocity directions of contiguous discrete parcels traveling in their individual pathlines as defined in the Lagrangian description, is identical in the Eulerian description to the definition of the streamline as the corresponding curve that is tangent to the velocity field at the given time.

This equivalence can be established by considering the following.

- In the Lagrangian description, the velocity of a fluid parcel is given by $\boldsymbol{u}(t)=\frac{\partial \boldsymbol{X}\left(\boldsymbol{x}_0\left(t_0\right),t\right)}{\partial t}$ as per Section 2.4(a).

- The continuous curve connecting these velocity directions of contiguous parcels represents the instantaneous direction of motion for all parcels at a given time.

- In the Eulerian description, a streamline is defined as a curve that is tangent to the velocity field at every point at a given instant.

- Since both the Lagrangian curve and the Eulerian streamline represent the instantaneous direction of fluid motion, and we have established in parts (b) and (c) that the velocity vectors are identical in both descriptions, the curves must be identical.

Given the above relationship between pathlines and streamlines, we conclude that under multi-directional flow conditions where pathlines can intersect at a given position, provided that the passage of the associated parcels through the intersection position is not contemporaneous. Similarly, we see that when pathlines intersect at a given position, it follows that streamlines also must intersect at the same given position, as shown above. We can visualize intersecting pathlines and streamlines more readily in the case of cross-flowing gases and the confluence of liquid flow streams. Our conclusion is contrary to the traditional assumption that streamlines do not intersect under any circumstances.

2.6. Analyzing the Decoupling between Lagrangian and Eulerian Material Derivatives

The material derivative in the Lagrangian description is fundamentally different from the material derivative in the Eulerian description due to the following:

1) In the Lagrangian description, the velocity $\boldsymbol{u}$ of a given parcel along a pathline is not a function of position $\boldsymbol{x}$.

From Section 2.2(c), we have established that the velocity $\boldsymbol{u}$ is not a bivariate function of t and $\boldsymbol{x}$ in the Lagrangian description. Instead, it is solely a function of time t for a given parcel, *i.e.*, $\boldsymbol{u}=\boldsymbol{u}(t)$.

2) In the Eulerian description, the velocity $\boldsymbol{u}$ and the material derivative are defined as bivariate functions of time t and position $\boldsymbol{x}$.

In the Eulerian description, by traditional definition, we consider the velocity field as a function of both time and position, *i.e.*, $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x},t)$. The material derivative in the Eulerian description is typically expressed as shown in Equation (20). If the Lagrangian and Eulerian descriptions were describing the same fluid parcel, then it would be self evident that the spatial derivative term (known as the convective or advective term) in Equation (20), that implies that $\boldsymbol{u}$ is a function of $\boldsymbol{x}$ would vanish, in which case the Eulerian description of the material derivative would be consistent with that of the Lagrangian description as, we showed in Section 2.4. Therefore, the standard definition of the material derivative in the Eulerian description does not describe the same material as the Lagrangian description.

$$\frac{D\boldsymbol{u}}{Dt}=\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \tag{20}$$

where both $\boldsymbol{u}$ and its derivatives are assumed to be functions of $\boldsymbol{x}$ and t.

3) The Eulerian description inherently describes multiple commingled parcels, unlike the Lagrangian description which follows individual parcels.

The Eulerian description, by considering the velocity field at fixed points in space, inherently describes the behavior of multiple fluid parcels passing through these points over time. This is in contrast to the Lagrangian description, which follows individual parcels along their pathlines, as established in Section 2.1.

4) Because the Lagrangian and Eulerian descriptions describe different parcels of fluid material, the material derivative of the Lagrangian description becomes decoupled from the material derivative of the Eulerian description.

Let us validate the above statement. The decoupling of the material derivatives in the two descriptions follows from the previous points.

- In the Lagrangian description, as shown in Section 2.4(e), the material derivative is:

$$\frac{D\boldsymbol{u}}{Dt}=\frac{\partial\boldsymbol{u}(t)}{\partial t}+\frac{\partial\boldsymbol{u}_0(t)}{\partial t} \tag{21}$$

where $\boldsymbol{u}$ is a function of t only, and $\frac{\partial\boldsymbol{u}_0(t)}{\partial t}$ represents the acceleration of the reference frame.

- In the Eulerian description, the material derivative includes the convective term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, which arises from the spatial variation of the velocity field, which we see from Equation (21) to be identical to the acceleration of the non-inertial (local) reference frame.

- Because the material derivative in the Lagrangian description is a description of the total derivative of a single parcel, whereas the material derivative in the Eulerian description is a description of multiple parcels from the perspective of a fixed position, the two forms of material derivative are not equivalent.

This decoupling demonstrates that while both the Lagrangian and Eulerian descriptions describe the same physical reality, they do so from fundamentally different perspectives, leading to distinct mathematical formulations of the material derivative that are not compatible with one another.

2.7. Analyzing the Velocity-Position Mapping for Non-Unidirectional or Oscillating Velocity

For a fluid parcel moving along a pathline,

1) If the velocity $\boldsymbol{u}(t)$ of the parcel is a non-unidirectional or an oscillating function of time t, then the map from the domain of position $\boldsymbol{x}$ to the co-domain velocity $\boldsymbol{u}$ is a one-to-many map or multivalued.

Let us validate the above statement by assuming that $\boldsymbol{u}(t)$ is a non-unidirectional or an oscillating function of time. By definition, this means that there exist at least two times t_1 and t_2, where $t_1 \neq t_2$, such that:

$$\boldsymbol{u}(t_1) \neq \boldsymbol{u}(t_2) \tag{22}$$

Now, consider the positions at these times:

$$\boldsymbol{x}(t_1) = \boldsymbol{x}(t_2) \tag{23}$$

This equality is true because the parcel has moved between t_1 and t_2 (assuming non-zero velocity) and returned to the same previous position.

Therefore, we have two different velocities mapping to the same position:

$$\boldsymbol{u}(\boldsymbol{x}(t_1)) = \boldsymbol{u}(t_1) \neq \boldsymbol{u}(t_2) = \boldsymbol{u}(\boldsymbol{x}(t_2)) \tag{24}$$

This violates the condition for the mathematical definition of the function to be valid, such that each element in the co-domain (velocity) is paired with at most one element in the domain (position).

2) Consequently, $\boldsymbol{u}$ cannot be expressed as a function of $\boldsymbol{x}$.

Since the mapping from $\boldsymbol{x}$ to $\boldsymbol{u}$ is a one-to-many map, it cannot satisfy the definition of a function. A function requires that each input (in this case, position $\boldsymbol{x}$) corresponds to a unique output (velocity $\boldsymbol{u}$). Also, since we have shown that a given position can map to more than one velocity, a plot of $\boldsymbol{u}$ versus $\boldsymbol{x}$ will fail the vertical line test. Therefore, $\boldsymbol{u}$ cannot be expressed as a function of $\boldsymbol{x}$.

3) If $\boldsymbol{u}$ is not a function of $\boldsymbol{x}$, then the standard definition of the material derivative shown in Equation (20) is not mathematically valid. Finally, if the flow is non-unidirectional and $\boldsymbol{u}$ is not a function of $\boldsymbol{x}$, then the Euler's and Navier-Stokes partial differential equations shown in Equation (1) and Equation (2) cannot be integrated in the position (space) domain, consequently their solution does not exist under these conditions

3. Experimental Data and Numerical Simulation

To further support the analysis presented in Section 2, we conducted a physical experiment to measure the oscillatory velocity of water inside a semicircular tube, designed to resemble the geometry of the human vestibular canal, specifically the anterior semicircular canal, in a vertical gravity field, apart from size differences. The tube, made of clear vinyl with an inside diameter of 1 cm, allowed oscillatory water flow (see **Figure 1** and **Table 1**). A numerical simulation, using a simple pendulum analog to represent the flow of a discrete fluid parcel inside a semicircular tube, was performed with equivalent parameters used in both the experiment and the simulation (see **Table 2**).

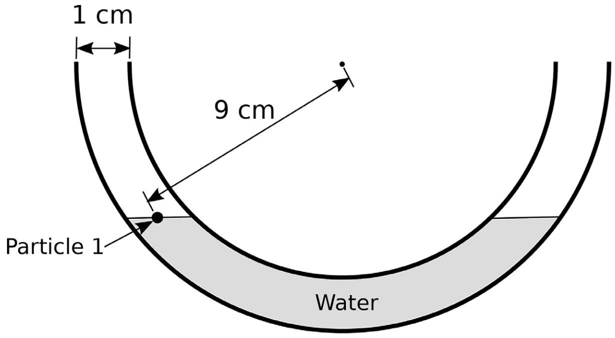

Figure 1. Geometry of semicircular tube.

Table 1. Experimental parameters.

Parameter	Value
Tube material	Clear vinyl
Inside diameter	1 cm
Measurement technique	Particle Image Velocimetry (PIV)

Table 2. Numerical simulation parameters.

Parameter	Value
Gravitational acceleration (g)	9.81 m/s^2
Pendulum length (L_1)	0.09 m
Initial angle (θ_0)	$-\pi/4$ rad
Damping coefficient (b)	0.15

3.1. Experimental Simulation

The experimental data consists of time-series measurements of the positions of Particle 1 (see **Figure 1**), representing a fluid particle located near the edge of the water volume.

The experimental data were processed using a high-order polynomial fit to the raw data, after which we calculated the velocity of the particle position.

3.2. Numerical Simulation

The numerical simulation utilizes a simple pendulum analog to model the oscillatory water flow within a semicircular tube. When a fluid body is placed inside a semicircular tube under a vertical gravity field, and there is an initial difference of elevation between the two ends of the fluid body, this setup generates an oscillatory fluid flow. This example of oscillatory flow is used to highlight the limitations of the Euler and Navier-Stokes equations in modeling such flow behavior because under such conditions, the x-velocity of the flow cannot be expressed as a function of x-position, as shown above in Section 2.7 and discussed below in Section 3.2.1.

Rather than directly solving the Euler and Navier-Stokes equations, our numerical simulation models the motion of the fluid body by reasoning that it can be represented as the motion of a simple pendulum with a fixed arm length. This length corresponds to the radius of curvature of the semicircular tube. The simulation solves numerically the pendulum's equation of motion, including damping effects, using the parameters shown in Table 2.

This approach provides a simplified yet effective method for modeling the complex behavior of oscillatory fluid flow in a semicircular tube, while avoiding some of the challenges associated with attempting to model oscillatory flow using the traditional fluid dynamics equations.

The equation of motion for the damped simple pendulum is solved using a numerical ordinary differential equation (ODE) solver. The simulation calculates the pendulum's angle, angular velocity, and the corresponding x-position and x-velocity over time.

Figure 2 shows a comparison between the experimental velocity data, and the numerical simulation results using the simple pendulum analog. Additionally, Figure 3 presents the results of a numerical simulation using six neighboring particles, each modeled individually using a simple pendulum analog.

This combination of experimental measurements and numerical simulation allows for the analysis of the temporal, spatial and oscillatory water flow behavior in the semicircular tube.

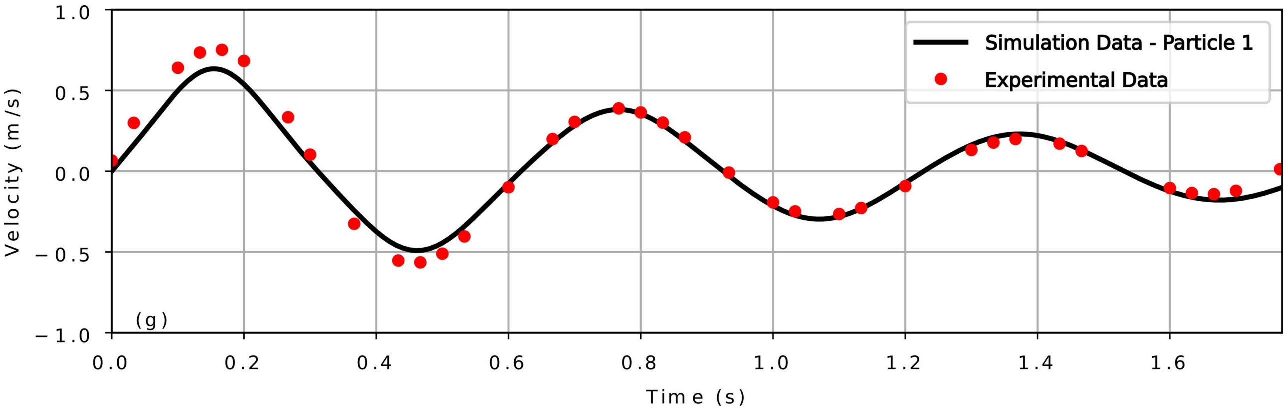

Figure 2. Gravity flow inside a semicircular tube: Validation of simulation data with experimental data.

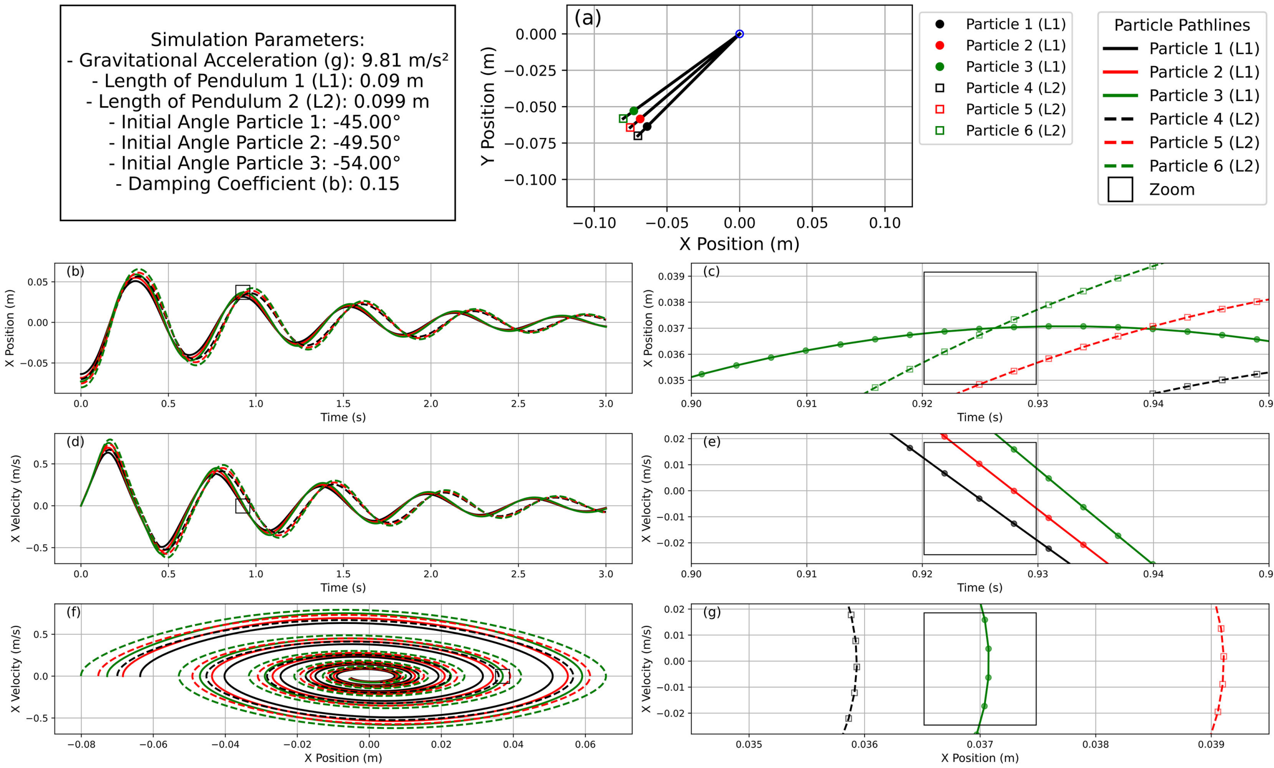

Figure 3. Gravity flow inside a semicircular tube: Multiparticle flow analysis.

Numerical Simulation Results

The experimental results and numerical simulations are presented in Figure 3, which consists of several panels providing a comprehensive view of the fluid motion in the semicircular tube.

Figure 3(a) shows the initial positions of the six simulated particles relative to one another, each attached to a separate arm of the six simple pendulums. In the legend of Figure 3(a) we see that, for example, Particle 1 is attached to a pendulum with an arm length of L1, and an initial position as shown in the figure where the marker for Particle 1 corresponds to the mass attached to the end of the pendulum arm.

In Figure 3(b), we observe the x-position of the simulated particles over time. Figure 3(c) shows a magnified region (small square box overlay) which highlights the relative position of the shown particles relative to one another. This plot effectively demonstrates that pathlines do indeed intersect while the particles remain separated from one another when they pass through the same position at different times.

Figure 3(d) presents the x-velocity of the simulated particle as a function of time. Figure 3(e) is the magnified region showing that particle velocity is a mathematical function in the time domain, which implies that for each particle the map from time to velocity is a one-to-one map.

In Figure 3(f), we see the plot of the x-velocity of the simulated particles as a function of x-position. The magnified region in Figure 3(g) shows that the x-velocity is a multivalued map from the domain of x-position, which implies that for a given particle, in some regions of the plot there is more than one value of x-velocity at a given x-position. When the map is multivalued, the plot of x-velocity versus x-position fails the vertical line test, in which case we conclude that x-velocity is not a function of x-position, as shown above in Section 2.7.

4. Discussion

In our study of fluid flow modeling, we have rigorously examined the foundations and implications of both the Lagrangian and Eulerian descriptions. Through a series of validated arguments, we have demonstrated that the Eulerian description, while widely used, has significant limitations that call into question some of its fundamental assumptions.

Specifically:

1) Section 2.1 and Section 2.2 established that the velocity of a fluid parcel in the Lagrangian description is a univariate function of time, not of position. This is in contrast to the standard assumption that velocity can be expressed as a function of both time and initial parcel position.

2) Section 2.2 and Section 2.4 imply that in the Lagrangian description the traditional expression of the material derivative shown in Equation 20 is not mathematically valid since $\boldsymbol{u}$ is not a bivariate function of t and $\boldsymbol{x}_0\left(t\right)$ because $\boldsymbol{x}_0\left(t\right)$ is not an independent variable, therefore the chain rule of differentiation does not apply.

3) Section 2.7 showed that for non-unidirectional or oscillating flows, which are common in many fluid systems, velocity cannot be expressed as a function of position, directly challenging the validity of the Eulerian description's approach to describing fluid velocity fields.

4) The equivalence between pathlines and streamlines, as shown in Section 2.5, further reinforces the Lagrangian description's ability to capture essential fluid dynamics without the use of the Eulerian description and its associated physical limitations as discussed below.

Other Limitations of the Eulerian Description

1) In traditional continuum mechanics, we know that solids have a rigid internal structure that enables nearby atomic or molecular particles to be subject to a stress and strain field that couples the movement of adjacent particles to one another. This is also true to a lesser extent for high viscosity liquids. In such materials, the Eulerian description is ideally defined to allow the extrapolative prediction of the unknown velocity state of particles at defined positions within the fluid continuum, by using the velocity state of nearby particles to which they are connected by inter-particle atomic and molecular forces that allow for the continuous bi-directional exchange of deformation and force between adjacent particles.

2) In gaseous fluids, gas atoms and molecules (*i.e.*, gas parcels) undergo autonomous random Brownian motion. Under normal conditions, no significant molecular forces act between individual parcels, making it unrealistic to predict the velocity of one parcel based on its neighbors, as these may be far apart and lack

adequate chemical or physical connections.

3) Similarly, in inviscid liquids, parcels are relatively free to undergo sliding motion without significant resistance from adjacent parcels, which makes it challenging to predict the unknown velocity state of a liquid parcel from the velocity states of its neighbors.

4) Therefore, in addition to the mathematical reasons highlighted above, we conclude that the Eulerian description is not ideal for analyzing fluid dynamics in gases or inviscid liquids.

In summary, we highlight limitations of the Eulerian description in modeling complex flows, particularly in gases and inviscid liquids, where intermolecular forces are generally weaker and less bidirectional than those in solids, which adhere more closely to the robust physical coupling between adjacent particles, as implied by the continuum mechanics modeling methodology. Our analysis shows that the Eulerian description is a framework derived from continuum mechanics that models the behavior of materials by describing the internal state of stress and strain as a response to external forces. In continuum mechanics, it is assumed that adjacent particles within a material are bonded, enabling the motion of one control volume to be linked to its neighbors through the constitutive equation, which relates stress and strain. However, when applied to fluids, the Eulerian approach encounters limitations. For gases, there are no strong permanent bonds between molecules, which interact only through collisions, lacking the cohesive forces necessary for transmitting tensile stress and strain. Therefore, in general, it is not possible to predict the motion of a gas parcel from the motion of its adjacent parcels, which may indeed be far removed from one another. Similarly, inviscid liquids do not exhibit rigid bonds like those in solids, further challenging the strict applicability of the continuum model. As a result, the Eulerian description, while useful for modeling viscous fluids, is not a fully robust quasi-continuum framework for gases and inviscid liquids.

These findings conclude that the Lagrangian description provides a more robust framework for modeling fluid flow properties. It avoids the limitations inherent in the Eulerian description, particularly in situations involving complex, time-dependent flows.

5. Conclusions

This study critically examines the mathematical foundations of the Euler and Navier-Stokes equations, revealing significant inconsistencies in the mathematical definitions of flow velocity and the material derivative.

We show that the flow velocity of a fluid parcel, traditionally modeled as a bivariate function of initial position and time in the Lagrangian description, is more accurately defined as a univariate parametric function of time, with parcel position treated as a time-dependent parameter. This leads to the conclusion that the standard form of the material derivative in the Lagrangian description is mathematically invalid because the chain rule of differentiation ceases to apply in this case.

Furthermore, if fluid flow is non-unidirectional, the mapping from parcel position to flow velocity becomes a one-to-many map or multivalued, indicating that flow velocity cannot be treated as a valid mathematical function of position. Under these conditions, the inability to integrate the Euler and Navier-Stokes differential equations in the spatial domain implies a lack of mathematical solutions. However, under unidirectional flow conditions, velocity can indeed be represented as a univariate mathematical function of either time or position in the Lagrangian and Eulerian descriptions. In such scenarios, the Euler and Navier-Stokes equations remain effective models for unidirectional flow, especially in viscous fluids.

Through rigorous mathematical analysis and supported by experimental simulations, we also identify challenges in maintaining material consistency within the Eulerian description, especially under complex flow conditions such as non-unidirectional and oscillatory flow. This material inconsistency results in a decoupling between Lagrangian and Eulerian descriptions under such complex flow conditions. Also, the relationship between pathlines in the Lagrangian description and streamlines in the Eulerian description, as demonstrated in this study, further highlights the limitations of the Eulerian description to accurately capture the physics of fluid dynamics in conditions of complex flow, such as non-unidirectional and intersecting flows.

We showed that the standard definition of the convective term in the material derivative term of the Lagrangian description is equivalent to the sum of the temporal derivatives of a local non-inertial velocity term and the derivative of the velocity of the local non-inertial reference frame relative to an arbitrary inertial reference frame. This underscores the need for re-evaluating the presumed traditional coupling between the Lagrangian and Eulerian frameworks and the interpretation of the definition of the material derivative in each description. Our findings identify mathematical and physical inconsistencies in these foundational assumptions.

The findings of this study aim to help refine existing models by addressing foundational inconsistencies, thereby enhancing their applicability across various scientific fields. Future work should further explore the inconsistencies identified in this study, refine these models, and assess their potential and limitations for fields such as atmospheric sciences, oceanography, and plasma physics. Addressing these foundational inconsistencies can advance fluid dynamics by enabling more robust and comprehensive modeling techniques.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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