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Some Results on Cordial Digraphs

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Abstract

A digraph is a graph in which each edge has an orientation. A linear directed path, P_n, is a path whose all edges have the same orientation. A linear simple graph is called directed cordial if it admits 0 - 1 labeling that satisfies certain condition. In this paper, we study the cordiality of directed paths P_n and their second power P_n^2. Similar studies are done for $P_n \cup P_m$ and the join $P_n + P_m$. We show that P_n, P_n^2 and $P_n \cup P_m$ are directed cordial. Sufficient conditions are given to the join $P_n + P_m$ to be directed cordial.

Keywords

Paths, Second Power of Path, Join of Paths, Cordial Graph

1. Introduction

One of the major problems concerning graph labeling is the cordialities of graphs. It is related to many applications in computer science and communication network. An excellent reference on this subject in the survey is by Gallian [1] and Harary [2]. A path $P_n = a_0 a_1 \cdots a_{n-1}$ is an alternating sequence of distinct vertices and $n-1$ edges. P_n is said to be linearly directed if all its edges have the same direction: clockwise or counterclockwise. The distance $d(x, y)$ between two vertices x, y in V of a graph $G = (V, E)$ is the length of shortest path joining them in *G*. The second power of a path P_n, denoted P_n^2, is the union of P_n and the set of all edges $a_i a_j$ with distance 2 and $i < j$. In Particular $P_2^2 \cong C_2$, $P_3^2 \cong C_3$ [3]. The origin concept of cordial graphs is due to Cahit [4]. In 1990, Cahit [5] proved the following: each tree is cordial; a complete graph K_n is cordial if and only if $n \le 3$ and a complete bipartite graph $K_{n,m}$ is cordial for all positive integers *n* and *m*. In this paper, we only deal with linearly directed paths and their second power together with the union and join of directed paths. Let $G = P_n$ be a digraph and let $f: V \to \{0,1\}$ be a labeling of its vertices and its set of edges, and let P_n be linearly directed (see **Figure 1**).

We define the edge labeling as follows $f^*: E \to \{0,1\}$

$$f^*(v_i v_{i+1}) = 2^{v_i} \pmod 2$$

Let v_0 and v_1 be the numbers of vertices that are labeled by 0 and 1, respectively, in G and let e_0 and e_1 be the corresponding numbers of edges. Such a labeling is called cordial if both $|v_0 - v_1| \le 1$ and $|e_0 - e_1| \le 1$ holds [4]. A graph is called cordial if it admits a cordial labeling [2]. If a linearly directed path is cordial, we call it a directed cordial path. The directed second power of a path is a linearly directed path with the added edges in P_n^2 that are endowed with directions defined as follows: $a_i a_{i+1}, a_{i+1} a_{i+2}$ and $a_{i+2} a_i$ have the same orientation (see **Figure 2**).

Given two disjoint paths P_n and P_m, then their union, $P_n \cup P_m$, is simply the unions of their sets of vertices and edges. If x_i and a_i represent the numbers of vertices and edges that are labeled i in P_n, respectively, and the corresponding quantities for P_m are y_i and b_i. Therefore, it is obvious that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$.

The join (sum) $P_n + P_m$ is obtained from $P_n \cup P_m$ by adding all edges that join each vertex of P_n to all vertices of P_m. Consider P_n and P_m have the same orientation. Then, we define the direction of all new edges that are connecting vertices of P_n and P_m to be from vertices of P_n to vertices of P_m. It follows that $|v_0 - v_1| = |(x_0 - x_1) + (y_0 - y_1)|$, and $|e_0 - e_1| = |(a_0 - a_1) + (b_0 - b_1) - m(x_0 - x_1)|$.

We shall show that P_n, P_n^2 and $P_n \cup P_m$ are directed cordial for all positive integers *n*, *m*. Some sufficient conditions are given to make the join $P_n + P_m$ is directed cordial. It is worth noting that, although $K_4 = P_2 + P_2$ is directed cordial but according to Cahit [5], it is not cordial.

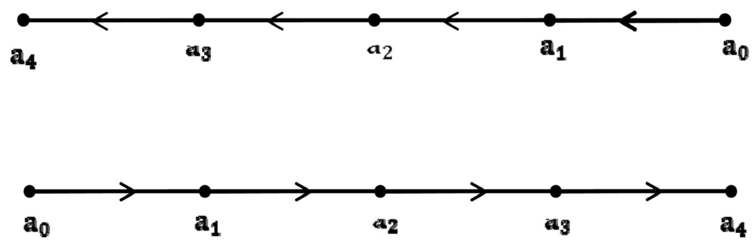

Figure 1. Linearly Directed Path.

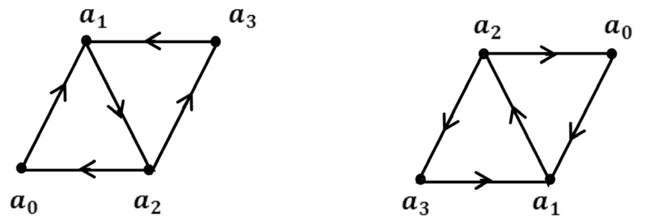

Figure 2. The Directed Second Power of a Path Have The Same Orientation.

2. Terminology and Notation

Let M_{2r} denote the labeling $01\cdots01$ (r times), and the labeling $01\cdots010$ is denoted by M_{2r+1}. We let M'_{2r} denote the labeling $10\cdots10$. Let L_{4r} denote the labeling $0011\cdots0011$ (r-times) and L'_{4r} denote the labeling $1100\cdots1100$ (r-times), where $r \geq 1$. The labeling $0\cdots0$ (r-times) is denoted by 0_r. Similarly, $1\cdots1$ (r-times) is denoted by 1_r. We can modify this by adding symbols at one end or the other (or both), thus $L_{4r}101$ denotes the labeling $0011\cdots0011101$, when $r \geq 1$ and 101 when $r = 0$. Similarly, $0L'_{4r}1$ is the labeling $01100\cdots11001$ when $r \geq 1$ and 01 when $r = 0$.

3. Directed Cordial Paths

Let $G = (V, E)$ be a graph where G is linearly directed path. We show that each linearly directed path is directed cordial.

Lemma 3.1.

The directed path P_m, is directed cordial; $m \geq 2$.

Proof: Let us first examine the particular cases P_2 and P_3.

For P_2, we choose the labeling 01; therefore $x_0 = x_1 = 1, a_0 = 0, a_1 = 1$. Without any loss of generality, one may use the clockwise direction and hence $|v_0 - v_1| = 0$, and, $|e_0 - e_1| = 1$. So P_2 is directed cordial.

For P_3, we choose the labeling 010; therefore $x_0 = 2, x_1 = 1$, and $a_0 = a_1 = 1$, also we may consider the direction of P_3 as done in P_2; hence $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 0$, so P_3 is directed cordial (Figure 3).

To complete the proof, we need to study the following four cases:

Case (1). $P_m; m \equiv 0 (\bmod 4)$.

Suppose that $m = 4r, r \geq 1$. We choose the labeling, L_{4r} for P_{4r}. Therefore $x_0 = x_1 = 2r$, $a_0 = 2r - 1$, and $a_1 = 2r$. Consequently, $|v_0 - v_1| = 0$ and, $|e_0 - e_1| = 1$. Thus P_{4r} is directed cordial. Figure 4 illustrates the directed cordial path P_8.

Case (2). $P_m; m \equiv 1 (\bmod 4)$.

Suppose that $m = 4r + 1, r \geq 1$. Then, one can choose the labeling, $L_{4s}0$ for P_{2r+1}. Therefore $x_0 = 2r + 1, x_1 = 2r$, and $a_0 = a_1 = 2r$. Consequently, $|v_0 - v_1| = 1$ and, $|e_0 - e_1| = 0$. Thus P_{4r+1} is directed cordial. Figure 5 illustrates the directed cordial path P_5.

Case (3). $P_m; m \equiv 2 (\bmod 4)$.

Suppose that $m = 4r + 2; r \geq 1$. Then, one can choose the labeling, M_{2r} for P_{4r+2}. Therefore, $x_0 = x_1 = 2r$, $a_0 = 2r - 1$, and $a_1 = 2r$. Consequently, $|v_0 - v_1| = 0$ and, $|e_0 - e_1| = 1$. Thus P_{4r+2} is directed cordial. Figure 6 illustrates the directed cordial path P_6.

Case (4). $P_m; m \equiv 3 (\bmod 4)$.

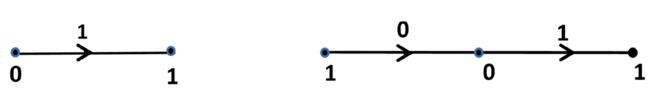

Figure 3. Linearly directed cordial path P_2 and P_3.

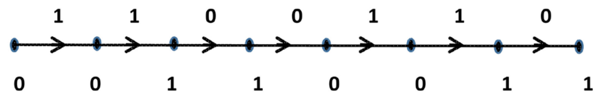

Figure 4. Linearly directed cordial path P_8.

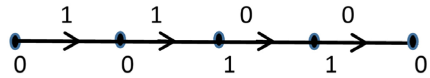

Figure 5. Linearly directed cordial path P_5.

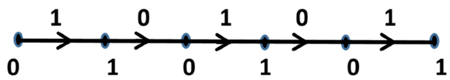

Figure 6. Linearly directed cordial path P_6.

Suppose that $m = 4r + 3; r \geq 1$. Then, one can choose the labeling M_{2r+1} for P_{4r+3}. Therefore, $x_0 = 2r + 1, x_1 = 2r$, and $a_0 = a_1 = 2r$. Consequently, $|v_0 - v_1| = 1$ and, $|e_0 - e_1| = 0$. Thus P_{4r+3} is directed cordial. **Figure 7** illustrates the directed cordial path P_7. Thus the lemma is proved.

4. The Directed Cordiality of P_n^2

It is known that the number of edges in P_n^2 is $2n - 3$. In this section we show that P_n^2 is directed cordial for all positive integers $n \geq 2$.

Theorem 4.1. Each directed second power path, P_n^2, is directed cordial for all $n \geq 2$.

Proof: By previous theorem $P_2^2 \cong P_2$.

Now, we need to study the following four cases:

Case (1). $P_n^2; n \equiv 0 \pmod 4$.

Suppose that $n = 4r; r \geq 1$. Without loss of generality, we may take the anti-clock direction throughout. Then, one can choose the labeling, L_{4r} for P_{4r} .Therefore, $x_0 = x_1 = 2r, a_0 = 4r - 1$, and $a_1 = 4r - 2$. Consequently, $|v_0 - v_1| = 0$ and, $|e_0 - e_1| = 1$. **Figure 8** illustrates the directed cordial path P_8^2.

Case (2). $P_n^2; n \equiv 1 \pmod 4$.

Suppose that $n = 4r + 1; r \geq 1$. Then, one can choose the labeling, M_{2r+1} for P_{4r+1}. Therefore, $x_0 = 2r + 1, x_1 = 2r, a_0 = 4r - 1$, and $a_1 = 4r$. Consequently, $|v_0 - v_1| = |e_0 - e_1| = 1$. **Figure 9** illustrates the directed cordial path P_9^2.

Case (3). $P_n^2; n \equiv 2 \pmod 4$.

Suppose that $n = 4r + 2; r \geq 1$. Then, one can choose the labeling, $M'_{2r}10$ for P_{4r+2}. Therefore, $x_0 = x_1 = 2r + 1, a_0 = 4r + 1$, and $a_1 = 4r$. Consequently, $|v_0 - v_1| = 0$ and, $|e_0 - e_1| = 1$. **Figure 10** illustrates the directed cordial path P_6^2.

Case (4). $P_n^2; n \equiv 3 \pmod 4$.

Suppose that $n = 4r + 3; r \geq 0$. Then, one can choose the labeling, $M_{2r}1$ for P_{4r+3}. Therefore, $x_0 = 2r + 1, x_1 = 2r + 2, a_0 = 4r + 2$ and $a_1 = 4r + 1$. Consequently, $|v_0 - v_1| = |e_0 - e_1| = 1$.

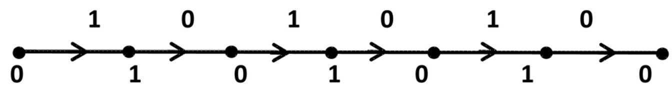

Figure 7. Linearly directed cordial path P_7.

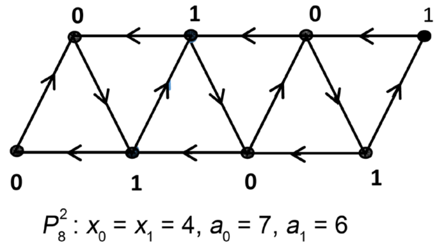

Figure 8. Linearly directed cordial path P_8^2.

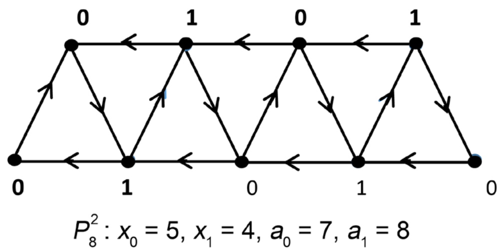

Figure 9. Linearly directed cordial path P_9^2.

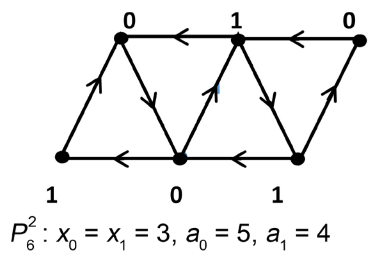

Figure 10. Linearly directed cordial path P_6^2.

Figure 11 illustrates the directed cordial path P_{11}^2. Thus the theorem is proved.

5. The Union of Two Directed Paths

In this section we study the directed cordiality of union of two directed paths P_n and P_m. Throughout, we use the following inequalities to prove the directed cordiality.

$$|v_0 - v_1| = |(x_0 - x_1) + (y_0 - y_1)| \le 1$$

$$|e_0 - e_1| = |(a_0 - a_1) + (b_0 - b_1)| \le 1$$

Lemma 5.1. The union $P_n \cup P_m$ of two directed paths is always directed cordial for all $n, m \ge 2$.

Proof: There are three cases to be examined:

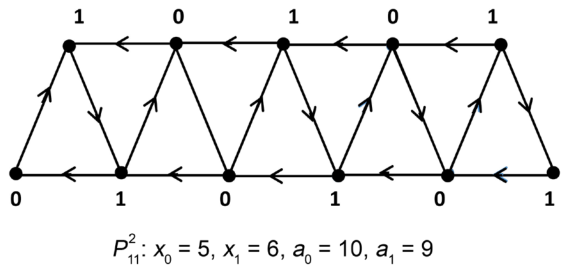

Figure 11. Linearly directed cordial path P_{11}^2.

Case (1). $P_{2r} \cup P_{2s}$

Choose the labeling M'_{2s} for P_{2r} and M_{2r} for P_{2s}. Then

$x_0 = x_1 = r$, $a_0 = r$, $a_1 = r - 1$, $y_0 = y_1 = s$, $b_0 = s - 1$, and $b_1 = s$.

Therefore

$$|v_0 - v_1| = |(x_0 - x_1) + (y_0 - y_1)| = |r - r + s - s| = 0$$

and

$$|e_0 - e_1| = |(a_0 - a_1) + (b_0 - b_1)| = |r - r + 1 + s - 1 - s| = 0.$$

Thus $P_{2r} \cup P_{2s}$ is directed cordial as we wanted to show.

Case (2). $P_{2r+1} \cup P_{2s+1}$

Choose the labeling M_{2r+1} for P_{2r} and M'_{2s+1} for P_{2s}. Then

$x_0 = r + 1$, $x_1 = a_0 = a_1 = r$, $y_0 = s$, $y_1 = s + 1$, and $b_0 = b_1 = s$.

Therefore

$$|v_0 - v_1| = |r + 1 - r + s - s - 1| = 0$$

and

$$|e_0 - e_1| = |r - r + s - s| = 0.$$

Thus $P_{2r+1} \cup P_{2s+1}$ is directed cordial.

Case (3). $P_{2r} \cup P_{2s+1}$

Choose the labeling M_{2r+1} for P_{2r} and M'_{2s+1} for P_{2s}.Then

$x_0 = x_1 = r$, $a_0 = r$, $a_1 = r - 1$, $y_0 = s$, $y_1 = s + 1$, and $b_0 = b_1 = s$.

Therefore

$$|v_0 - v_1| = |r - r + s - s - 1| = 1$$

and

$$|e_0 - e_1| = |r - r + 1 + s - s| = 1.$$

Thus, $P_{2r} \cup P_{2s+1}$ is directed cordial and the lemma is proved.

6. The Union of Two Directed Paths

In this section we give some sufficient conditions for the sum of two linearly directed paths P_n and P_m to be directed cordial. As indicted in the introduction we shall use the following equations to show that $P_n + P_m$ is directed cordial:

$$\left|v_0 - v_1\right| = \left|\left(x_0 - x_1\right) + \left(y_0 - y_1\right)\right|,$$

and

$$\left|e_0 - e_1\right| = \left|\left(a_0 - a_1\right) + \left(b_0 - b_1\right) - m\left(x_0 - x_1\right)\right|.$$

Lemma 6.1. If n is even, and P_n and P_m have the same orientation. Then $P_n + P_m$ is directed cordial for all m.

Proof. Let us first study the following two special cases:

Let $n = m = 2$; then the labeling [01; 10] is sufficient for $P_2 + P_2$. Let $n = 2$ and $m = 3$; then we choose the labeling [10; 010] for $P_2 + P_2$. Therefore, $x_0 = x_1 = 1$, $a_0 = 1$, $a_1 = 0$, $y_0 = 2$, $y_1 = 1$, and $b_0 = b_1 = 1$. Hence, $\left|v_0 - v_1\right| = \left|e_0 - e_1\right| = 1$, so $P_2 + P_3$ is directed cordial. See Figure 12.

To complete the proof, we need to examine the following two cases.

Case (1). m is even.

Let $n = 2r, r > 2$ and $m = 2s, s > 2$. Then one can choose the labeling $\left[M_{2r}; M'_{2s}\right]$ for $P_{2r} + P_{2s}$, where without loss of generality, we consider the given direction to both b_n and b_m is from left to right. It follows that, $x_0 = x_1 = r$, $a_0 = r - 1$, $a_1 = r$, $y_0 = y_1 = s$, $b_0 = s$, and $b_1 = s - 1$. Therefore,

$$\left|v_1 - v_0\right| = \left|\left(x_1 - x_0\right) + \left(y_1 - y_0\right)\right| = 0,$$

and $\left|e_1 - e_0\right| = \left|\left(a_1 - a_0\right) + \left(b_1 - b_0\right) - m\left(x_1 - x_0\right)\right| = 0$. So, $P_{2r} + P_{2s}$ is directed cordial.

Case (2). m is odd.

Let $n = 2r, r > 2$ and $m = 2s + 1, s > 1$. Then one can choose the labeling $\left[M_{2r}; M'_{2s+1}\right]$ for $P_{2r} + P_{2s+1}$. Then

$$x_0 = x_1 = r,\ a_0 = r,\ a_1 = r - 1,\ y_0 = s,\ y_1 = s + 1, \text{ and } b_0 = b_1 = s,$$

Therefore

$$\left|v_0 - v_1\right| = \left|\left(x_0 - x_1\right) + \left(y_0 - y_1\right)\right| = \left|r - r + s - s - 1\right| = 1,$$

and

$$\begin{aligned}\left|e_0 - e_1\right| &= \left|\left(a_0 - a_1\right) + \left(b_0 - b_1\right) - m\left(x_0 - x_1\right)\right| \\ &= \left|r - r + 1 + s - s - (2s+1)\left(r - r\right)\right| = 1\end{aligned}$$

Thus $P_{2r} + P_{2s+1}$ is directed cordial. Hence, the lemma is proved.

It is very important noting that, although K_4 is directed cordial, but according to Cahit, K_4 is not cordial [3]. This shows a difference between cordial graphs and directed cordial graphs.

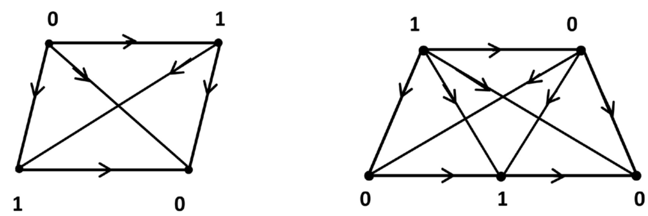

Figure 12. Directed cordial path $P_2 + P_3$.

Lemma 6.2. If n is odd, and P_n and P_2 have the same direction. Then $P_n + P_2$ is directed cordial.

Proof: Let $n = 2r+1, r \geq 1$. Then one can choose the labeling $[M_{2r+1};10]$ for $P_{2r+1} + P_2$. It follows that $x_0 = r+1$, $x_1 = r$, $a_0 = a_1 = r$, $y_0 = y_1 = 1$, $b_0 = 1$, and $b_1 = 0$. Hence $|v_0 - v_1| = |e_0 - e_1| = 1$. See **Figure 13** and **Figure 14**.

Lemma 6.3. If n is odd, and both P_n and P_3 are similarly directed. Then $P_n + P_3$ is directed cordial.

Proof. Suppose that $n = 2r+1$. Then one can choose the labeling $[0_r 1_{r+1};0_2 1]$ for $P_n + P_3$. It follows that $x_0 = r$, $x_1 = r+1$, $a_0 = a_1 = r$, $y_0 = 2$, $y_1 = 1$, $b_0 = 0$, and $b_1 = 2$. It is easy to show that $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. Thus, $P_{2r+1} + P_3$ is directed cordial. See **Figure 15**.

Lemma 6.4. If n is odd and both P_n and P_4 are similarly directed. Then $P_n + P_4$ is directed cordial.

Proof. Suppose that $n = 2r+1$, and P_n and P_4 have the same direction. Then one can choose the labeling $[0_r 1_{r+1};0_3 1]$ for $P_{2r+1} + P_4$. It follows that $x_0 = r$, $x_1 = r+1$, $a_0 = a_1 = r$, $y_0 = 3$, $y_1 = 1$, $b_0 = 0$, and $b_1 = 3$. Hence $|v_0 - v_1| = |e_0 - e_1| = 0$, and so, $P_{2r+1} + P_4$ is directed cordial. See **Figure 16**.

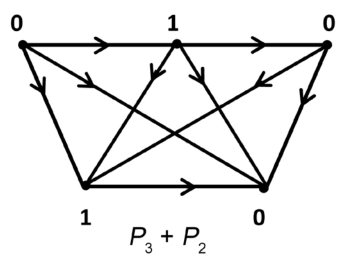

Figure 13. Directed cordial path $P_3 + P_2$.

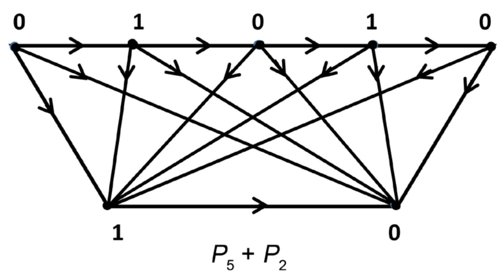

Figure 14. Directed cordial path $P_5 + P_2$.

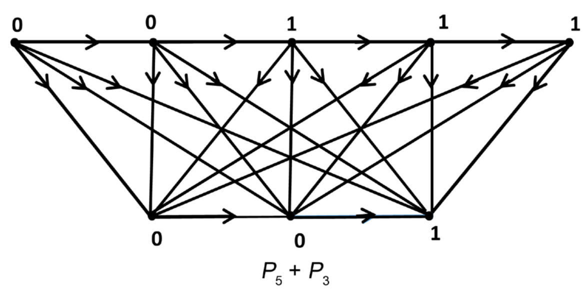

Figure 15. Directed cordial path $P_5 + P_3$.

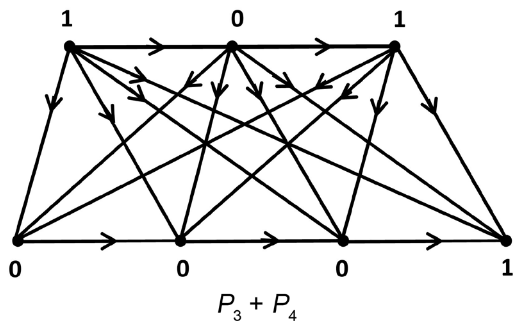

Figure 16. Directed cordial path $P_3 + P_4$.

7. Applications and Conclusions

The water and gas pipelines supply to a building represent examples of diagraphs. We think of edges of a directed path as pipes can flow through it in one way flow. Another example is the waste systems of plumbing in a building. In conclusion, the directed paths, second power directed paths and the union of any two directed paths are all cordial digraphs. If n is even, then the join $P_n + P_m$ is always cordial diagraph. Also, $P_n + P_i$ is cordial digraph for all $i = 2,3,4$, and n is odd.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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