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The V_2^{2r-1} of $PG(2r,q)$ as a Representation of $PG(2,q)$: Sections and Partitions

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Abstract

This note investigates the hyperplane sections of a ruled variety V_2^{2r-1} embedded in $PG(2r,q)$, which yield caps associated with specific arcs in $PG(2,q)$. We construct a partition of the affine points of V_2^{2r-1} into caps, corresponding to a partition of the affine plane $PG(2,q)$ into conics.

Keywords

Finite Geometry, Translation Planes, Spreads, Varieties, Caps

1. Introduction

In the André/Bruck and Bose representation of $PG(2,q^2)$ in $PG(4,q)$ (cf. [1] [2]) a non-affine Baer subplane B corresponds to a ruled variety V_2^3 (cf. [3] [4]). In [5] is proved that a non degenerate conic in B is a rational normal curve of V_2^3.

Using that technique, in [6] is studied the representation of the projective plane $PG(2,q^r)$ in $PG(2r,q)$ and of a non-affine subplane $PG(2,q)$ in a variety V_2^{2r-1}.

More precisely, if $\Pi = PG(2,q^r)$ with kernel $F = GF(q)$, then it can be represented in a $2r$-dimensional projective space $\Sigma = PG(2r,q)$, by fixing a hyperplane $\Sigma' = PG(2r-1,q)$ and a spread $\mathcal{S}$ of Σ' consisting of $(r-1)$-dimensional subspaces, with $|\mathcal{S}| = q^r + 1$.

The affine points of Π correspond to the points of $\Sigma \setminus \Sigma'$, the points at infinity correspond to the elements of $\mathcal{S}$, and the affine lines are represented by the r-subspaces S_r of Σ such that $S_r \cap \Sigma' \in \mathcal{S}$. The line at infinity is represented by the spread $\mathcal{S}$ itself. If Π is Desarguesian, then the spread $\mathcal{S}$ is regular (cf.

[1] [2], see also [4], and [3] for $r=2$, [7] for $r=3$, and [6] for the general case).

A subplane of Π is *affine* or *non-affine* (also referred to as *tangent*) depending on whether it intersects the line at infinity in a *subline* or in one point, respectively.

An *affine* subplane of order q is represented by a *transversal* plane to the spread; that is, a plane of Σ intersecting $q+1$ elements of the spread.

A *non-affine* subplane π of $\Pi = PG(2,q^r)$ of order q is represented by a variety V_2^{2r-1}, a ruled variety in $\Sigma = PG(2r,q)$ whose minimum order directrix a rational curve of order $r-1$ and with a maximum order directrix is a rational curve of order r. These two curves lie in two complementary spaces of dimension $r-1$ and r, respectively. The variety V_2^{2r-1} can be obtained by joining corresponding points on the two directrix curves via a projectivity (cf. [8], Cap.13, 8., 9. and [6], Section 4).

Building on the results obtained in [5], Theorems 3.1 and 3.2 for $r=2$, this note studies a generalization for $r>2$. Several properties of the hyperplanes of Σ and their intersections with the variety V_2^{2r-1} are established, an essential step in demonstrating how to represent substructures of the plane π in the space Σ (cf. Subsection 3.1) and viceversa (cf. Subsection 3.2), specifically certain types of arcs and caps.

In Theorem 3.7 is shown how to construct in V_2^{2r-1} a rational normal curve of order $r+1$ that represents a conic in π. The paper concludes with Theorem 3.8 which presents a partition of the affine points of V_2^{2r-1} into caps corresponding to a partition of π into conics.

2. Preliminary Notes and Results

Denote $F = GF(q)$ a finite field, $q = p^s$, p an odd prime, $\overline{F}$ the algebraic closure of the field F, F^{n+1} the $(n+1)$-dimensional vector space over F, $PG(n,q) = PrF^{n+1}$ the n-dimensional projective space contraction of F^{n+1} over F. The geometry $PG(n,q)$ is considered a sub-geometry of $\overline{PG(n,q)}$, the projective geometry over $\overline{F}$. A subspace of $PG(n,q)$ of dimension h (an h-space) is denoted by S_h (cf. [7], Section 2), possibly with an apex used when needed to distinguish between different subspaces.

Definition 2.1. *A k-arc $\mathcal{K}$ in $PG(n,q)$ is a set of $k \geq n+1$ points no $n+1$ of which are lie in a hyperplane.*

A k-cap $\mathcal{K}$ of $PG(n,q)$, $n \geq 3$ is a set of k points no three of which are collinear.

A tangent of $\mathcal{K}$ is a line which has exactly one point in common with $\mathcal{K}$.

See Thas [9].

A curve of order r is denoted $\mathcal{C}^r$, possibly with a subscript used when needed to distinguish between different curves.

Definition 2.2. *A rational normal curve $\mathcal{C}^n$ of $PG(n,q)$ consists of $q+1$ points ($q \geq n$) no $n+1$ of which in a hyperplane S_{n-1} (that is, a hyperplane meets $\mathcal{C}^n$ in at most n points).*

See Hirschfeld [10] p. 229, Theorem 21.1.1, (iv).

Consequence—No set of n points lie in an S_{n-2}; no set of $n-1$ points lies in an S_{n-3}; and so on, down to the fact that no three points lie on a line. That is, an S_{n-2} meets the curve in at most $n-1$ points, an S_{n-3} in $n-2$ points,..., a line in 2 points).

In $PG(3,q)$, q odd, a $(q+1)$-arc is a twisted cubic, that is, a rational normal curve of degree 3 (cf. [10], pp.242-243, Theorem 21.2.3).

Definition 2.3. *A variety* V_u^v *of dimension* u *and of order* v *of* $PG(n,q)$ *is the set of the rational points of a projective variety* $\bar{V}_u^v$ *of* $\overline{PG(n,q)}$ *defined by a finite set of polynomials with coefficients in the field* F.

Definition 2.4. *The ruled variety* V_2^{n-1} *of* $PG(n,q)$, $n \geq 4$ *and* $n \neq 5$, *is generated by the* $q+1$ *lines joining the corresponding points of two birationally (projectively) equivalent curves of order* m *and* $n-1-m$, *respectively, lying in two complementary subspaces of the same dimensions,* m *and* $n-1-m$ *respectively. As such directrix curves have no point in common, then the number of points of* V_2^{n-1} *is* $(q+1)^2$ *and the order is the sum of the orders of the curves.*

The $q+1$ lines are generatrices (or, generatrix lines).

See Bertini [8], Cap.9, n.1–3, Cap.13, n.1–8, p. 290, 7., Vincenti [7], Lemma 2.2, and [6].

From [8], p. 287, 3., follows

RESULT 1—*In* $PG(n,q)$ *a hyperplane* S_{n-1} *meets a ruled variety* V_2^{n-1} *in one of the following ways:*

1) *in a rational normal curve of degree* $n-1$ *(provided* $q \geq n-1$*)*

or,

2) *in a curve of degree* $m < n-1$ *met by all the generatrix lines and in* $n-1-m$ *generatrix lines and not composed of two or more distinct curves.*

3*) Every irreducible curve* $\mathcal{C}^m$ *of degree* $m \leq n-1$ *contained in* V_2^{n-1} *is a rational normal curve, that is, it lies in an* m*-dimensional space* S_m.

Note that since throughout this paper we refer to rational normal curves $\mathcal{C}^h$ of $PG(h,q)$ for some $h \geq 2$, we assume $q \geq h$ (see Definition 2.2).

Let Σ be the projective space $PG(2r,q)$, $r \geq 2$, $\Sigma' = PG(2r-1,q)$ a hyperplane of Σ, $\mathcal{S}$ a *regular spread* of $(r-1)$-spaces of Σ'. It is $|\mathcal{S}| = q^r + 1$. For the definition of spread, regulus and regular spread see [2] and [7], Definition 2.3 and the representation.

The Desarguesian plane $PG(2,q^r)$ is represented by Σ and by the spread $\mathcal{S}$ of Σ' according the André/Bruck and Bose method (cf. [1] [2]).

Let $r=2$. In such a case the projective plane is $PG(2,q^2)$, $\Sigma = PG(4,q)$, $\Sigma' = PG(3,q)$, $\mathcal{S}$ is a regular spread of q^2+1 lines of Σ'. If $B = PG(2,q)$ denotes a non-affine Baer subplane of $PG(2,q^2)$, and $P_\infty \in B$ its the unique point on the line at infinity, then B is represented by a variety V_2^3, which has as its linear directrix a line $r_\infty \in \mathcal{S}$ and as a conic directrix a conic $\mathcal{C}$ lying in a plane that meets Σ' in a line $l_\infty \in \mathcal{S}$, $l_\infty \neq r_\infty$ with $\mathcal{C} \cap l_\infty = \varnothing$ (cf. [3]-[5]).

RESULT 2—*A non degenerate conic* $\mathcal{C}$ *of a non-affine Baer subplane* B *through* P_∞ *is represented on the variety* V_2^3 *by either a twisted cubic curve* *(and viceversa), or by a normal rational curve of order* 4 *depending on whether* P_∞ *belongs to* $\mathcal{C}$ *or not.*

See [5], Theorem 3.1 and Theorem 3.2.

Let $r \geq 2$, $\Pi = PG(2, q^r)$. It is $\Sigma = PG(2r, q)$, $\Sigma' = PG(2r-1, q)$, $\mathcal{S}$ is a regular spread of $(r-1)$-subspaces, $|\mathcal{S}| = q^r + 1$.

RESULT 3—*A non-affine subplane* $\pi = PG(2, q)$ *of* Π *having only one point* P_∞ *at infinity is represented in* Σ *by a ruled variety* V_2^{2r-1}. *Such a variety is the locus of the lines connecting corresponding points (via a projectivity) of a curve* $\mathcal{C}_\infty^{r-1}$ *of a subspace* $S_{r-1}^\infty \in \mathcal{S}$ *and of a curve* $\mathcal{C}_0^r$ *of a subspace* $S_r^0 \subset \Sigma$ *such that* $S_r^0 \cap \Sigma' = S_{r-1}^0 \neq S_{r-1}^\infty$ *and* $\mathcal{C}_0^r \cap S_{r-1}^0 = \varnothing$. *Such lines are the generatrix lines, the curves* $\mathcal{C}^r = S_r \cap V_2^{2r-1}$ *with* $S_r \cap \Sigma' \in \mathcal{S}$ *are directrices of* V_2^{2r-1}. *The* $q+1$ *lines of* π *through* P_∞ *are represented by the generatrix lines, the other lines by the* q^2 *directrix curves of* V_2^{2r-1}.

See [6], Theorems 4.7, 4.8.

RESULT 4—*A subspace* S_i *of* $S_r = PG(r, q)$ *with* $i \geq r-k$, *meets a variety* V_k^n *in a variety* V_{i+k-r}^n.

See [8], p. 191, 3., comma 2.

3. Main Results

3.1. From Σ to Π

Represent $\Pi = PG(2, q^r)$ in $\Sigma = PG(2r, q)$ with $r \geq 2$, $q \geq 2r-1$.

Let $\mathcal{S}$ be a regular spread of $(r-1)$-spaces of a hyperplane $\Sigma' = PG(2r-1, q)$, $|\mathcal{S}| = q^r + 1$. The elements of $\mathcal{S}$ are the points at infinity of Π, the r-spaces S_r such that $S_r \cap \Sigma' \in \mathcal{S}$ are the affine lines of Π, $\mathcal{S}$ is the line at infinity l_∞ of Π.

Note that a *transversal* r*-space*, that is, an S_r with $S_r \cap \Sigma' \notin \mathcal{S}$ can represent a Baer subplane β of Π only if r is even. This is because β would have order $q^{\frac{r}{2}}$, and hence the $(r-1)$-space $S_r \cap \Sigma'$ must intersect exactly $q^{\frac{r}{2}} + 1$ elements of $\mathcal{S}$.

Fix an $(r-1)$-space S_{r-1}^∞ of $\mathcal{S}$, and choose a curve $\mathcal{C}_\infty^{r-1} \subset S_{r-1}^\infty$ of order $r-1$. Let S_r^0 be an r-space such that $S_r^0 \cap \Sigma' = S_{r-1}^0 \neq S_{r-1}^\infty$, and let curve $\mathcal{C}_0^r \subset S_r^0$ be a curve satisfying $\mathcal{C}_0^r \cap S_{r-1}^0 = \varnothing$.

Let $\pi = PG(2, q)$ be a non-affine subplane of Π of order q with $P_\infty \in l_\infty$ as its unique point at infinity, corresponding to the $(r-1)$-space S_{r-1}^∞. Then π is represented by the ruled variety $\mathcal{V} = V_2^{2r-1}$, obtained by connecting corresponding points of $\mathcal{C}_\infty^{r-1}$ and of $\mathcal{C}_0^r$ (see Result 3).

The point P_∞ is the center of a bundle of $q+1$ lines in π. The remaining q^2 lines of π each determine a unique point on the line l_∞. Therefore, there exists a subset $\overline{\mathcal{S}} \subset \mathcal{S}$, with $|\overline{\mathcal{S}}| = q^2$ corresponding to these points.

NOTE 1—For each element $S_{r-1} \in \overline{\mathcal{S}}$, there exists a unique r-space S_r that meets the ruled variety $\mathcal{V}$ in a directrix curve $\mathcal{C}^r$, which corresponds to a line of π. This is the *only line of* π whose *point at infinity* is S_{r-1}. Similarly, for S_{r-1}^0, there exists a unique r-space S_r^0 that intersects $\mathcal{V}$ in the directrix curve

$\mathcal{C}_0^r \subset S_r^0$.

Since $\mathcal{C}_\infty^{r-1}$ is a rational normal curve, it consists of $q+1$ points, no r of which in a hyperplane S_{r-2}. This holds under the assumption $q \ge r-1$, which is satisfied by the hypothesis $q \ge 2r-1$ (cf. Definition 2.2).

Choose a subset $\mathcal{P} = \{P_1, \cdots, P_{r-1}\} \subset \mathcal{C}_\infty^{r-1}$ consisting of $r-1$ independent points. Let $S_{r-2}^{\mathcal{P}} = \langle \mathcal{P} \rangle$ denote the $(r-2)$-space of S_{r-1}^∞ generated by the points of $\mathcal{P}$. For $r=2$ the set $\mathcal{P}$ is a singleton.

Let $\mathcal{G}_{\mathcal{P}} = \{g_1, \cdots, g_{r-1}\}$ be the set of the $r-1$ generatrix lines joining the points of $\mathcal{P}$ and the corresponding $r-1$ points of $\mathcal{C}_0^r$. Let $\mathcal{G}' = \{g_r', \cdots, g_{q+1}'\}$ be the set of the remaining $q+2-r$ generatrix lines.

The hyperplane $H = S_r^0 + S_{r-2}^{\mathcal{P}}$ intersects the ruled variety $\mathcal{V}$ in the union of the curve $\mathcal{C}_0^r$ and the set of generatrix lines $\mathcal{G}_{\mathcal{P}}$ (cf. Result 1, 2)).

Consider the subspace $S_{2r-2} = S_{r-1}^0 + S_{r-2}^{\mathcal{P}}$, the direct sum of the two subspaces.

The bundle $\mathcal{B}_{2r-2}$ of hyperplanes with axes S_{2r-2} contains $q+1$ hyperplanes. Among them are the hyperplane Σ' and the hyperplane $H = S_r^0 + S_{r-2}^{\mathcal{P}}$.

Each hyperplane $H' \in \mathcal{B}_{2r-2} \setminus \Sigma'$ intersects all $q+1$ generatrix lines $\mathcal{G}_{\mathcal{P}} \cup \mathcal{G}'$. If $H' \ne H$ and contains no generatrix line, then the $r-1$ lines of $\mathcal{G}_{\mathcal{P}}$ meet H' in the points of $\mathcal{P} \subset \mathcal{C}_\infty^{r-1}$. The remaining $q+2-r$ lines in $\mathcal{G}'$ meet H' in affine points. Denote this set of affine points by $\mathcal{Q} = \{Q_r, \cdots, Q_{q+1}\}$ such a set.

Lemma 3.1. *If* $r > 2$*, then the set* $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ *contains both hyperplanes that include one generatrix line from* $\mathcal{G}_{\mathcal{P}}$ *and hyperplanes that contain no generatrix line from* $\mathcal{G}_{\mathcal{P}}$.

If $r=2$, there exists exactly one hyperplane in $\mathcal{B}_{2r-2} \setminus \Sigma'$ that contains a generatrix line.

Proof. From Result 1, it follows that a hyperplane $H' \in \mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ intersects the ruled variety $\mathcal{V}$ either in a rational normal curve $\mathcal{C}^{2r-1}$ of degree $2r-1$, or in a curve of order $m < n-1$, which is met by all the generatrix lines and in $n-1-m$ generatrix lines.

Let $r > 2$. Assume that $H' \cap \mathcal{V}$ contains at least two generatrix lines, $g_i, g_j \in \mathcal{G}_{\mathcal{P}}$. Since all generatrix lines intersect each directrix curve, denote A_i and A_j the points on g_i and g_j, respectively, that lie on the directrix curve $\mathcal{C}_0^r \subset S_r^0$. Note that $A_i, A_j \notin \mathcal{Q}$. Because $S_{r-1}^0 \subset H'$ and the line A_iA_j meets S_{r-1}^0, it follows that H' contains the entire S_0^r and $H' = H$, which contradicts the assumption.

Hence $H' \cap \mathcal{V}$ contains at most one generatrix line.

Now, assume that each of the $q-1$ hyperplanes of $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ contains exactly one generatrix line from $\mathcal{G}_{\mathcal{P}}$. Let H' and H'' be two distinct hyperplanes from $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$, and let $g_1 \subset H'$, $g_2 \subset H''$ be two generatrix lines such that $g_1, g_2 \in \mathcal{G}_{\mathcal{P}}$.

Two distinct hyperplanes in $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ can intersect only in the space S_{2r-2}. Therefore such generatrix lines they contain must be distinct. Since $q \ge 2r-1 > r$, we have $q-1 = |\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}| > r-1 = |\mathcal{P}|$, which implies that there more generatrix lines than elements in $\mathcal{P}$, contradicting the definition of $\mathcal{G}_{\mathcal{P}}$. This contradiction completes the argument.

Hence in $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ there are both hyperplanes containing a generatrix of $\mathcal{G}_{\mathcal{P}}$ and hyperplanes that contain none.

If $r = 2$, $\mathcal{P} = \{P\}$, then there is only one generatrix line g through $\mathcal{P}$. Hence, there exists a unique hyperplane in $\mathcal{B}_{2r-2} \setminus \Sigma'$ that contains g.

The following cases should be considered.

Theorem 3.2. i) *If* $r \geq 2$ *and* $H' \cap \mathcal{V}$ *contains no generatrix line, then* $H' \cap \mathcal{V}$ *is a rational normal curve* $\mathcal{C}^{2r-1}$*.*

ii$_1$) If $r > 2$ and $H' \cap \mathcal{V}$ contains a generatrix line $g \in \mathcal{G}_{\mathcal{P}}$, then
$H' \cap \mathcal{V} = g \cup \mathcal{C}^{2r-2}$.

In both cases, the curve consists of the $r-1$ points of $\mathcal{P}$ and of $q+2-r$ affine points, each lying on one of the $q+2-r$ generatrix lines of $\mathcal{G}'$ with no two of them lying on the same directrix.

ii$_2$) If $r = 2$ and $H' \cap \mathcal{V}$ contains a generatrix line g, then
$H' \cap \mathcal{V} = g \cup \mathcal{C}$ where $\mathcal{C}$ is a conic directrix.

Proof. i) Assume that H' contains no generatrix line from $\mathcal{G}_{\mathcal{P}}$.

The hyperplane H' intersects all $q+1$ generatrix lines of $\mathcal{G}_{\mathcal{P}} \cup \mathcal{G}'$. The $r-1$ lines of $\mathcal{G}_{\mathcal{P}}$ are intersected at the points of $\mathcal{P} \subset \mathcal{C}_{\infty}^{r-1}$, while the remaining $q+2-r$ lines of $\mathcal{G}'$ are intersected at affine points. Denote $\mathcal{Q} = \{Q_r, \cdots, Q_{q+1}\}$ the set of these affine points.

Then, from Result 1, 1), it follows that $H \cap \mathcal{V} = \mathcal{C}^{2r-1}$, where $\mathcal{C}^{2r-1} \supset \mathcal{P} \cup \mathcal{Q}$ is a rational normal curve. The condition $q \geq 2r-1$, although assumed as hypothesis, can be proved although it was taken as a hypothesis, can be proven (cf. [10], Theorem 21.1.1, (i)).

Assume that a point $Q \in \mathcal{Q}$ lies on a generatrix line $g \in \mathcal{G}_{\mathcal{P}}$. Then the line g, containing two points in H', must lie entirely in H', which contradicts the assumption. A similar contradiction arises if either a point Q and a point $P_k \in \mathcal{P}$, or two points $Q_j, Q_h \in \mathcal{Q}$, lie on the same generatrix g.

Hence, the $q+2-r$ affine points of $\mathcal{Q}$ are distributed one per line over the $q+1-(r-1) = q+2-r$ generatrix lines $\{g'_r, \cdots, g'_{q+1}\}$.

Assume that H' contains two points A, B on a directrix curve. Since
$H' \supset S_{2r-2} \supset S_{r-1}^{0}$ this directrix must be $\mathcal{C}_0^r$, because the line AB intersects
S_{r-1}^{0}. Therefore H' would contain the entire S_0^r, implying that $H' = H$, which is a contradiction.

ii$_1$) If H' contains one generatrix $g \in \mathcal{G}_{\mathcal{P}}$, then from Result 1, 2), it follows that $2r-1-m = 1$ so that $m = 2r-2$. Therefore $H' \cap \mathcal{V} = g \cup \mathcal{C}^{2r-2}$. From Result 3, since $\mathcal{C}^{2r-2}$ is irreducible, it is a rational normal curve and hence lies in a subspace S'_{2r-2} of H'. Note that $\mathcal{C}^{2r-2}$, in addition to $\mathcal{Q}$, contains the entire set $\mathcal{P}$, including the point $g \cap \mathcal{C}_{\infty}^{r-1} \in \mathcal{P}$, otherwise it would have only q points. Moreover, it meets all the generatrix lines.

There are no affine point of $\mathcal{C}^{2r-2}$ on g; otherwise, $\mathcal{C}^{2r-2}$ would have
$q+2$ points. Therefore, the line g intersects $\mathcal{C}^{2r-2}$ in exactly one point and

does not lie in S_{2r-2}. The proof of the first property of $\mathcal{C}^{2r-2}$ is analogous to the proof in i), where contradictions arise from the possibility that H' contains more than one generatrix. The proof of the second one is similar.

ii$_2$) Let $r = 2$. Then $\mathcal{P} = \{P\}$ and $H' = S_3$. The line $g \subset H'$ is the unique generatrix line, so the residual curve of $H' \cap \mathcal{V}$ is a conic $\mathcal{C}$. Let α denote the plane of $\mathcal{C}$; clearly, $\alpha \subset H'$.

Assume that the line $l = \alpha \cap \Sigma'$ contains the point P. If l coincides with the line $\mathcal{C}_\infty^1$, then α would contain q skew generatrix lines of V_2^3, which is a contradiction. If l is a line of Σ' that meets $q+1$ lines of the spread, $\mathcal{C}_\infty^1$ included, then α is a transversal plane and thus represents an affine Baer subplane of Π. Since it would share q points with the non-affine Baer subplane represented by V_2^3, the two would have to coincide, again leading to a contradiction. Therefore l is a line of $\mathcal{S} \setminus \mathcal{C}_\infty^1$, and we conclude that $H' \cap V_2^3 = g \cup \mathcal{C}$, where $\mathcal{C}$ is a directrix curve, that, as required, intersects all the generatrix lines.

Let H' be a hyperplane in the bundle $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ that contains no generatrix line. Denote by $\mathcal{Q} = \{H' \cap g_i' = Q_i \mid i = r, \cdots, q+1\}$ the set of the $q+2-r$ affine points where H' intersects the generatrix lines g_i'.

Let $\mathcal{Q}' = \{Q_r', \cdots, Q_{q+1}'\}$ be the subset of points of the projective plane $\pi \subset \Pi$, represented in Σ by the points of $\mathcal{Q}$, to which we add the point P_∞, represented by S_{r-1}^∞. Denote by $\mathcal{K} = \mathcal{Q}' \cup \{P_\infty\}$ this subset of points of π.

Proposition 3.3. *The set* $\mathcal{K}$ *is a* $(q+3-r)$ *-arc of* π. *It is maximal, that is, a* $(q+1)$ *-arc, when* $r = 2$. *In this case, if* q *is odd,* $\mathcal{K}$ *is a conic.*

Proof. First, note that $\mathcal{K}$ has cardinality $q+1-(r-1)+1 = q+3-r$.

If three affine points of $\mathcal{K}$ were collinear, then the corresponding points in $\mathcal{Q}$ would lie on a directrix curve, contradicting Theorem 3.2. Similarly, if the points Q_i', Q_j', P_∞ were collinear, their line would pass through P_∞, implying that the two affine points Q_i and Q_j of $\mathcal{C}^{2r-1}$ lie on a generatrix line of $\mathcal{V}$, again contradicting both our assumption and Theorem 3.2.

The arc $\mathcal{K}$ is maximal when $q+3-r = q+1$, that is, when $r = 2$, In this case, $\Pi = PG(2, q^2)$ and π is a Baer subplane. If q is odd, $\mathcal{K}$ is a conic.

Denote by t the tangent line of $\mathcal{K}$ at the point P_∞, and let g_t be the corresponding line of the variety $\mathcal{V}$ in Σ.

Corollary 3.4. *The line* t *is represented in* Σ *by a generatrix line* g_t *such that* $g_t \notin H'$.

Proof. The tangent line t to $\mathcal{K}$ at the point P_∞ is a line through P_∞ and, clearly, it lies in the subplane π. Since all lines through P_∞ in π are represented in Σ by the generatrix lines of $\mathcal{V}$, it follows that g_t is a generatrix line of $\mathcal{V}$. By hypothesis, H' contains no generatrix lines; therefore $g_t \notin H'$.

3.2. From Π to Σ

Let $\mathcal{C} \subset \pi$ be a non degenerate conic containing the unique point at infinity P_∞ of π. Denote by $\{P_1', \cdots, P_q'\}$ the q affine points of $\mathcal{C}$, and by $\{g_1', \cdots, g_q'\}$ the q affine lines of π joining P_∞ with the points $\{P_1', \cdots, P_q'\}$. Let t be the tangent line to $\mathcal{C}$ at P_∞.

Let $\mathcal{K}_\mathcal{C}$ be the subset of $\mathcal{V}$ corresponding in Σ to the points of $\mathcal{C}$, $\mathcal{K} = \{P_1, \cdots, P_q\}$ is the set of the q affine points of $\mathcal{K}_\mathcal{C}$ representing $\{P'_1, \cdots, P'_q\}$, $\mathcal{G} = \{g_1, \cdots, g_q, g_t\}$ the set of the $q+1$ generatrix lines of $\mathcal{V}$ corresponding to $\{g'_1, \cdots, g'_q, g_t\}$ where g_t is the generatrix representing t, T the point $g_t \cap \mathcal{C}_\infty^{r-1}$.

Theorem 3.5. i) *The set* $\mathcal{K}_\mathcal{C}$ *consists of* $q+1$ *points of* $\mathcal{V}$. *Among them, the* q *affine points of* $\mathcal{K}$ *each lie on a distinct generatrix in the set* $\{g_1, \cdots, g_q\}$. *The point* T, *which lies on the curve* $\mathcal{C}_\infty^{r-1}$, *is the unique point at infinity of* $\mathcal{K}_\mathcal{C}$.

ii) The set $\mathcal{K}_\mathcal{C}$ forms a $(q+1)$-cap, with the line g_t being the tangent to $\mathcal{K}_\mathcal{C}$ at the point T.

Proof. i) Obviously no point of $\mathcal{K}$ belongs to the generatrix g_t as no affine point of the conic $\mathcal{C}$ belongs to the tangent t.

Assume two points of $\mathcal{K}$ lie on the same generatrix $g \in \{g_1, \cdots, g_q\}$. That would imply that the corresponding two points of the conic $\mathcal{C}$ are collinear with P_∞, contradicting the fact that $\mathcal{C}$ is non-degenerate. Hence, the q affine points of $\mathcal{K}$ lie each on a distinct generatrix among $g_1, \cdots, g_q$, all different from the tangent line g_t; that is, $P_i \in g_i$ for $i = 1, \cdots, q$.

Assume that $\mathcal{K}_\mathcal{C}$ contains a point $T' \in \mathcal{C}_\infty^{r-1}$, with $T' \neq T$. Then the generatrix containing T' must be a line $g_i \in \mathcal{G}$ for some $i = 1, \cdots, q$. Since $P_i \in g_i$, the entire generatrix $g_i = \overline{T'P_i}$ would lie in the configuration. This would introduce to $\mathcal{K}$ the remaining $q-1$ affine points of the generatrix g_i, whose corresponding points in π would lie on a line through P_∞. That line would then be added to the conic $\mathcal{C}$, contradicting its non-degeneracy. Therefore $T' = T$, and T is the unique point at infinity of $\mathcal{K}_\mathcal{C}$.

ii) First, note that no two points of $\mathcal{K}$ lie on the same generatrix, since no two affine points of the conic $\mathcal{C}$ corresponding to them are collinear with P_∞.

Assume that $P_i, P_j, P_h \in \mathcal{K}$ are collinear, and let l be the line passing through them. Let P'_i, P'_j, P'_h be the corresponding points on the conic $\mathcal{C}$. The line l is not a generatrix line, and therefore it determines, via its intersection point $l \cap \Sigma'$ an element $S'_{r-1} \in \mathcal{S}$, and subsequently a space S'_r such that $S'_r \cap \Sigma' = S'_{r-1}$. This space S'_r contains a directrix curve $\mathcal{C}'^r$.

The line in π through P_i and P_j is represented in Σ by the curve $\mathcal{C}'^r$, that is the same line in π through P_i, P_h and P_j, P_h. Hence, the three points $P'_i, P'_j, P'_h \in \mathcal{C}$ must lie on a common line in π, a contradiction to the fact that $\mathcal{C}$ is a non-degenerate conic.

The line $g_t \in \mathcal{V}$, corresponding of the tangent t to the conic $\mathcal{C}$ at the point P_∞, has only the point T in common with $\mathcal{K}_\mathcal{C}$. Therefore, g_t is the tangent line to $\mathcal{K}_\mathcal{C}$ at T.

To complete the characterization of $\mathcal{K}_\mathcal{C} \subset \mathcal{V}$, it remains to understand the nature of $\mathcal{K}_\mathcal{C}$ in the case where it is contained in the intersection of $\mathcal{V}$ with a hyperplane.

Let $S' = S_{2r-1}$ be a hyperplane such that $\mathcal{K}_\mathcal{C} \subset S'$. Necessarily $S' \neq \Sigma'$.

The subspace $S'_{2r-2} = S' \cap \Sigma'$ may either intersect each element of the spread $\mathcal{S}$ in subspaces of the same dimension, or contain one element of $\mathcal{S}$ entirely.

Note that the number of points in S'_{2r-2} is $q^{2r-2} + q^{2r-3} + \cdots + q + 1$.

Since this number is not divisible by $q^r + 1 = |\mathcal{S}|$, the first case, where S'_{2r-2} intersects each element of the spread $\mathcal{S}$ in subspaces of equal dimension, cannot occur. Hence S'_{2r-2} must contain one element of the spread.

1) Assume that $S'_{2r-2} \supset S^{\infty}_{r-1}$, so that $S' \supset S^{\infty}_{r-1}$.

Then S' contains the curve $\mathcal{C}^{r-1}_{\infty}$ as well as the q generatrix lines of $\mathcal{G}$ (excluding g_t) that connect the q affine points of $\mathcal{K}_{\mathcal{C}}$ to the corresponding q points of $\mathcal{C}^{r-1}_{\infty} \setminus \{T\}$. Let g denote the set of the affine points on the line g_t. Then $S' \cap \mathcal{V} = \mathcal{V} \setminus \{g\}$, which consists of the q generatrix lines together with $\mathcal{C}^{r-1}_{\infty}$, that is, a reducible curve of order $q + r + 1 > 2r - 1$, contradicting Result 1.

2) Assume S'_{2r-2} contains an element of $S'_{r-1} \in \mathcal{S} \setminus \{S^{\infty}_{r-1}\}$, so that $S' \supset S'_{r-1}$.

Since $S' \neq \Sigma'$, it follows that $S' \cap S^{\infty}_{r-1}$ is a subspace $\overline{S}_{r-2}$, which intersects the curve $\mathcal{C}^{r-1}_{\infty}$ in $r-1$ points, including the point T (cf. Definition 2.2).

Define $S'_{2r-2} = S'_{r-1} + \overline{S}_{r-2}$.

There are $q+1$ hyperplanes containing S'_{2r-2}, one of which one is Σ' (which contains S^{∞}_{r-1}). Another hyperplane contains an r-dimensional subspace S'_r such that $S'_r \cap \Sigma' = S'_{r-1}$. This subspace S'_r represents a line d of Π, and its intersection with $\mathcal{V}$ is a directrix curve $\mathcal{C}^r_d$, which corresponds to the unique line of π through the point at infinity represented by S'_{r-1}.

Since S' contains the $r-1$ generatrix lines, say $g_1, \cdots, g_{r-1}$, through the points of $\overline{S}_{r-2}$, as well as the corresponding points of the directrix curve $\mathcal{C}^r_d$, it follows that

$$S' \cap \mathcal{V} = \mathcal{C}^r_d \cup \{g_1, \cdots, g_{r-1}\},$$

in accordance with Result 1.

Let $A, B \in \mathcal{K}$ be any two affine points of $\mathcal{K}_{\mathcal{C}}$, and let A_π, B_π be the corresponding points in π represented by A and B, respectively; note that $A_\pi, B_\pi \in \mathcal{C}$. By hypothesis A and B lie in the hyperplane S', since $S' \supset \mathcal{K}_{\mathcal{C}}$. Then the point at infinity R_∞ of the line $r = AB$ lies either in $\overline{S}_{r-2}$ or in S'_{r-1}; that is, $R_\infty \in S^{\infty}_{r-1} \cup S'_{r-1}$.

The line $r_\pi = A_\pi B_\pi$ is a secant of the conic $\mathcal{C}$, and therefore it cannot pass through P_∞. This implies that the point at infinity R_∞ of the line $r = AB$ cannot lie in S^{∞}_{r-1}.

If, on the other hand, $R_\infty \in S'_{r-1}$, then $A, B \in \mathcal{C}^r_d$. Since this would hold for any pair of affine points of $A, B \in \mathcal{K}$, it would follow that $\mathcal{K} = \mathcal{C}^r_d$. However, in π, the points of $\mathcal{C}^r_d$ are represented by the line d, and thus $\mathcal{C}$ would have to coincide with the line d, contradicting the assumption that $\mathcal{C}$ is a non-degenerate conic.

Therefore, $\mathcal{K}_{\mathcal{C}}$ cannot be contained in such a hyperplane S' representing a line of π.

Based on cases of 1) and 2), we now make the following choices. Fix a subspapce

$S'_{r-1} \in \bar{\mathcal{S}} = \mathcal{S} \setminus S^{\infty}_{r-1}$, and choose a subspace $\bar{S}_{r-2} \subset S^{\infty}_{r-1}$ such that $\bar{S}_{r-2}$ intersects the curve $\mathcal{C}^{r-1}_{\infty}$ in exactly $r-1$ points. Define

$$S'_{2r-2} = S'_{r-1} + \bar{S}_{r-2}.$$

In the bundle $\mathcal{B}_{2r-2}$ of hyperplanes with axes S'_{2r-2}, there are $q+1$ hyperplanes: one is Σ', and another is $H = S'_r + \bar{S}_{r-2}$, where S'_r represents a line d of π. Thus,

$$\left|\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}\right| = q-1.$$

Choose a hyperplane $S' \in \mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$.

NOTE 2—From NOTE 1, it follows that for each choice of $S'_{r-1} \in \bar{\mathcal{S}}$, there are q^2 possibilities, and for each of these, there are $q-1$ hyperplanes like S'. Moreover, one must count the possible choices of $r-1$ independent points on $\mathcal{C}^{r-1}_{\infty}$, each determining a distinct subspace $\bar{S}_{r-2}$, and hence giving rise to different hyperplanes.

Assume that S' contains $\mathcal{K}_{\mathcal{C}}$. In this case, the point $T = g_t \cap \mathcal{C}^{r-1}_{\infty}$ is one of the $r-1$ points of the intersection $\bar{S}_{r-2} \cap \mathcal{C}^{r-1}_{\infty}$, and the line g_t cannot contain any affine points. Denote by $\{g_1, \cdots, g_{r-2}, g_t\}$ the $r-1$ generatrix lines passing through the points of $\bar{S}_{r-2} \cap \mathcal{C}^{r-1}_{\infty}$.

Theorem 3.6. i) $r > 2$. *Then* $S' \cap \mathcal{V}$ *consists of* $r-2$ *generatrix lines and a residual curve* $\mathcal{C}^{r+1}$ *representing* $\mathcal{K}_{\mathcal{C}}$. *The curve* $\mathcal{C}^{r+1}$ *meets all the generatrix lines; it is a rational normal curve lying in a subspace* $S_{r+1} \subset S'$, *with* $\mathcal{C}^{r+1} \cap \bar{S}_{r-2} = \{T\}$.

ii) $r = 2$. Then $\mathcal{K}_{\mathcal{C}} = S' \cap \mathcal{V}$. If q is odd, $\mathcal{K}_{\mathcal{C}}$ is a twisted cubic containing in S'.

In no case does g_t, tangent to $\mathcal{K}_{\mathcal{C}}$ at the point at infinity, belong to S'.

Proof. i) $r > 2$. The intersection $S' \cap \mathcal{V}$ must either be a curve of order $2r-1$ consisting of a total of $q+1$ points, or consist of $2r-1-m$ generatrix lines and of a residual curve of order $m < 2r-1$ (cf. Result 1). Since S' contains $\mathcal{K}_{\mathcal{C}}$, which includes q affine points, it follows that $S' \cap \mathcal{V}$ must consist of the $r-2 = 2r-1-m$ generatrix lines $\{g_1, \cdots, g_{r-2}\}$, and of a residual curve of order m satisfying $2r-1-m = r-2$, hence $m = r+1 < 2r-1$. Such a curve $\mathcal{C}^{r+1}$ must intersect all the generatrix lines (cf. Result 1).

More precisely, the lines in $\{g_1, \cdots, g_{r-2}\}$ intersect $\mathcal{C}^{r+1}$ at the corresponding points of $\bar{S}_{r-2}$. The generatrix g_t, which represents the tangent t to the conic $\mathcal{C}$ at its infinite point, does not belong to S' (by hypothesis). Since g_t contains no affine points of $\mathcal{K}_{\mathcal{C}}$, it intersects $\mathcal{C}^{r+1}$ only at the point T.

Therefore $\mathcal{K}_{\mathcal{C}}$ is represented by $\mathcal{C}^{r+1}$, a rational normal curve of $\mathcal{V}$, lying in a subspace S_{r+1} of S' (cf. Result 1, 3)). Indeed, since $dim\left(S_{r+1} \cap \bar{S}_{r-2}\right) = 0$, which is equivalent to $\mathcal{C}^{r+1} \cap \mathcal{C}^{r-1}_{\infty} = \{T\}$, the remaining q affine points are distributed one on each of the remaining q generatrix lines.

ii) For $r = 2$, case i) can still be applied since $2r-1 = r+1$. Specifically, note that $\mathcal{V} = V^3_2$ is embedded in $PG(4, q)$, where the hyperplane S' is a 3-dimensional subspace. The curve at infinity, $\mathcal{C}^1_{\infty} = \mathcal{C}^{2-1}_{\infty}$, is a line l_{∞}. The hyperplane S'

meets l_∞ in exactly one point, T, and intersects each generatrix line in exactly one of the q affine point of $\mathcal{K}_\mathcal{C}$.

From Theorem 3.5, it follows that no three points of $\mathcal{K}_\mathcal{C}$ are collinear. Therefore, in this case and for q odd, $\mathcal{K}_\mathcal{C}$ is a twisted cubic curve of S', that is, a normal rational curve of S' (cf. [10], Theorem 21.2.3, [5], Theorem 3.1).

In neither case i) nor ii), by construction, does g_t belong to S'.

This result shows that for $r > 2$ (that is, for $2r-1 > r+1$), no hyperplane strictly defines a substructure of the variety $\mathcal{V}$ capable of representing a cap of $q+1$ points, unless one considers a subspace of dimension $r+1$ within a hyperplane S', which can be constructed as follows.

First, remind that the curve C_∞^{r-1} of S_{r-1}^∞ consists of $q+1$ points, no r of which lie in a hyperplane of S_{r-1}^∞, with $q \geq r-1$ (since by hypothesis $q \geq 2r-1$; cf. Definition 2.2).

Choose a subset $\mathcal{P} = \{P_1, \cdots, P_{r-2}, T\}$ of $r-1$ points from C_∞^{r-1}. Let $S'_{r-2} = \langle \mathcal{P} \rangle \subset S_{r-1}^\infty$ be the subspace generated by $\mathcal{P}$. Choose a subspace $S_{r-1} \in \mathcal{S} \setminus S_{r-1}^\infty$. Denote by $\mathcal{G} = \{g_1, \cdots, g_{r-2}, g_t\}$ the set of the generatrix lines through the points of $\mathcal{P}$ with $T = g_t \cap \mathcal{P}$.

Theorem 3.7. *There exists a hyperplane* H' *containing a subspace* S_{r+1}*, in which lies a rational normal curve* C^{r+1}*. This curve* C^{r+1} *is a cap of* $\mathcal{V}$*, consisting of* $q+1$ *points, exactly one of which is at infinity. The curve* C^{r+1} *corresponds in the plane* π*, to a conic passing through the point at infinity* P_∞*.*

Proof. Set $r > 2$. Consider the set $\mathcal{G} \setminus \{g_t\}$ consisting of the $r-2$ generatrices $\{g_1, \cdots, g_{r-2}\}$, and denote by S_{r-1}^* the subspace they generate. Let $H' = S_{r-1}^* + S_{r-1}$ be the hyperplane defined as the span of S_{r-1}^* and a chosen element $S_{r-1} \in \mathcal{S} \setminus S_{r-1}^\infty$. Since H' contains $2r-1-m = r-2$ generatrix lines, it follows by Result 1 that $H' \cap \mathcal{V}$ includes a residual curve $\mathcal{C}^{r+1}$ of order $m = r+1$, which meets all the generatrix lines.

Since $H' \cap S_{r-1}^\infty$ is a subspace of dimension $r-2$, it follows that H' contains the entire subspace $S'_{r-2} = \langle \mathcal{P} \rangle$, and in particular, the point T. Therefore, H' is a hyperplane of the bundle $\mathcal{B}_{2r-2}$ of hyperplanes.

The curve $\mathcal{C}^{r+1}$ is a rational normal curve, so it lies in a subspace $S_{r+1} \subset H'$ that intersects S'_{r-2} in a single point. If $\mathcal{C}^{r+1} \subset S_{r+1}$ met each generatrix in an affine point, it would be a directrix. However, the maximum possible order of a directrix is $r < r+1$, which leads to a contradiction. Therefore, $\mathcal{C}^{r+1}$ must intersect one of the generatrices in $\mathcal{G}$ at the point $P = S_{r+1} \cap S'_{r-2}$ which lies in $\mathcal{P}$.

Assume $P \neq T$. Then $\mathcal{C}^{r+1}$ meets g_t in an affine point, implying that g_t must lie in H'. This leads to a contradiction, as including g_t would increase the dimension to S_{r-1}^*. Therefore $P = T$.

The remaining q generatrices are each intersected by $\mathcal{C}^{r+1}$ at exactly one affine point, so that $\mathcal{C}^{r+1}$ contains q affine points and one point at infinity. This configuration does not contradict the presence of the lines $\{g_1, \cdots, g_{r-2}\}$ in H', as they are already contained within in. Since $\mathcal{C}^{r+1}$ is a normal rational curve with $q+1$ points, it clearly forms a $(q+1)$-cap.

Moreover, $H' \neq \Sigma'$ since Σ' contains no generatrices, and $H' \neq H$, as the hyperplane H contains $r-1$ generatrices and the unique directrix curve of order r.

If $r=2$, then $\mathcal{P}=\{T\}$ and $\mathcal{G}=\{g_t\}$. Although the previous procedure can still be applied, to clarify this case, let $H'=S_3$ be one of the $q-1$ hyperplanes around the plane $S_{r-1}+T=S_1+T$ distinct from both Σ' and H, so that H' contains no conic directrix.

If $g_t \in H'$, then $2r-1-m=3-m=1$, so that the residual curve C would have order $m=2r-2=2$. That is, C would be a conic meeting all the generatrix lines (cf. Result 1). Therefore C would be a directrix and $H'=H$, which is a contradiction.

Hence H' does not contain g_t; it meets the line $C_\infty^{r-1}=C^1$ in single point, namely, T. Consequently, $H' \cap \mathcal{V}$ is an irreducible rational normal curve of order $2r-1=3$ (cf. [5], Lemma 2.1) having T as its unique point at infinity.

By construction, in both cases, it is immediate to verify that $\mathcal{C}^{r+1}$ represents a conic of π passing through the point P_∞.

Theorem 3.8. *There exists a partition of the affine points of the variety* $\mathcal{V}=V_2^{2r-1}$ *consisting of* q *rational normal curves of order* $r+1$*, together one generatrix line.*

Proof. In the non-affine subplane π, let $\mathcal{F}$ be the bundle of hyperosculating conics at the point P_∞, all sharing the common tangent line t through P_∞. It is straightforward to verify that $|\mathcal{F}|=q$. Consequently, the affine points of $\mathcal{F} \cup t$ provide a partition of the affine points of π.

Denote by $\mathcal{L}_{\mathcal{F}}$ the set of q curves of order $r+1$ in the variety $\mathcal{V}$ corresponding to the conics of the bundle $\mathcal{F}$. Let g_t be the generatrix line representing the tangent line t at P_∞ (cf. Theorem 3.5).

Since two distinct conics in $\mathcal{F}$ meet only at the point P_∞, the corresponding curves in $\mathcal{L}_{\mathcal{F}}$ have no affine points in common. The total number of affine points of $\mathcal{V}$ are q^2+q, each curve in $\mathcal{L}_{\mathcal{F}}$ contains exactly q affine points, so the union of all these curves accounts for $q \cdot q = q^2$. Adding the q affine points of the generatrix line g_t, we cover all q^2+q affine points of $\mathcal{V}$.

4. Conclusion

The representation of a non-affine subplane $PG(2,q)$ of the projective plane $PG(2,q^r)$ in the variety V_2^{2r-1} of $PG(2r,q)$ is a generalization introduced in a earlier work. In this paper, we studied the connection between the caps obtained from certain hyperplane sections of V_2^{2r-1} and specific arcs in $PG(2,q)$. This connection enabled us to establish a partition of the affine points of V_2^{2r-1} into caps, corresponding to a partition of the affine points $PG(2,q)$ into conics.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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