Open Journal of Discrete Mathematics, 2024, 14, 43-53
https://www.scirp.org/journal/ojdm
ISSN Online: 2161-7643
ISSN Print: 2161-7635

Perfect 1-*k* Matchings of Bipartite Graphs

Wenduan Dai, Yan Liu*, Yanfang Wu

School of Mathematical Sciences, South China Normal University, Guangzhou, China
Email: *liuyan@scnu.edu.cn

How to cite this paper: Dai, W.D., Liu, Y. and Wu, Y.F. (2024) Perfect 1-*k* Matchings of Bipartite Graphs. *Open Journal of Discrete Mathematics*, **14**, 43-53.
https://doi.org/10.4236/ojdm.2024.144005

Received: July 29, 2024
Accepted: September 22, 2024
Published: September 25, 2024

Abstract

Let k be a positive integer and G a bipartite graph with bipartition (X,Y). A perfect 1-k matching is an edge subset M of G such that each vertex in Y is incident with exactly one edge in M and each vertex in X is incident with exactly k edges in M. A perfect 1-k matching is an optimal semi-matching related to the load-balancing problem, where a semi-matching is an edge subset M such that each vertex in Y is incident with exactly one edge in M, and a vertex in X can be incident with an arbitrary number of edges in M. In this paper, we give three sufficient and necessary conditions for the existence of perfect 1-k matchings and for the existence of 1-k matchings covering $|X|-d$ vertices in X, respectively, and characterize k-elementary bipartite graph which is a graph such that the subgraph induced by all k-allowed edges is connected, where an edge is k-allowed if it is contained in a perfect 1-k matching.

Keywords

Bipartite Graph, Semi-Matching, Perfect 1-k Matching, k-Elementary Graph

1. Introduction and Preliminaries

All graphs considered are simple, finite and undirected. Let G be a graph. The degree of a vertex v of G is denoted by $d_G(v)$. The *neighbour set* of a vertex subset S of G is the set of vertices adjacent to a vertex in S, denoted by $N_G(S)$. For two subsets E_1 and E_2 of $E(G)$, the *symmetric difference* of E_1 and E_2 is denoted by $E_1 \Delta E_2$, that is, $E_1 \Delta E_2 = (E_1 - E_2) \cup (E_2 - E_1)$. For two subsets S,T of $V(G)$, let $E_G(S,T) = \{uv \in E(G) : u \in S, v \in T\}$. The complete bipartite graph with bipartition (X,Y) with that $|X| = s$ and $|Y| = t$ is denoted by $K_{s,t}$. Specially, we denote by $K_{1,0}$ a single vertex. We refer to the book [1] for graph theoretical notations and terminology that are not defined here.

Let $G = (X,Y)$ be a bipartite graph with bipartition (X,Y). A *semi-matching* of G is defined as a set of edges $M \subseteq E(G)$ such that each vertex in Y is incident with exactly one edge in M, and a vertex in X can be incident with an arbitrary number of edges in M. In general, valid semi-matchings of a connected graph G can be easily obtained by matching each vertex $y \in Y$ with an arbitrary vertex $x \in X$ for which $xy \in E(G)$. In the computer science, the problem about the semi-matchings in a bipartite graph $G = (X, Y)$ is well known (see [2]), where Y corresponds to the clients (or tasks), X to the servers (or machines) and the number of edges in a semi-matching incident with a server (machine) in X is seen as the load on the server (machine). So semi-matchings are widely considered on different optimization objectives (see [3]-[6]). Many optimization objectives are to find the fairest semi-matching related to the load-balancing problem in a system, where a set of tasks need to be assigned to a set of machines in the fairest way. An example of the fairest semi-matching according to Jain's index is shown in **Figure 1** [5] and M_4 is the fairest one with the highest index of 0.86. Clearly, if $G = (X, Y)$ has a semi-matching M such that every vertex in X has the same load k, then M is a fairest semi-matching of G, which is said to be a perfect 1-k matching. Therefore, It is significant to study the new problem about 1-k matchings of a bipartite graph.

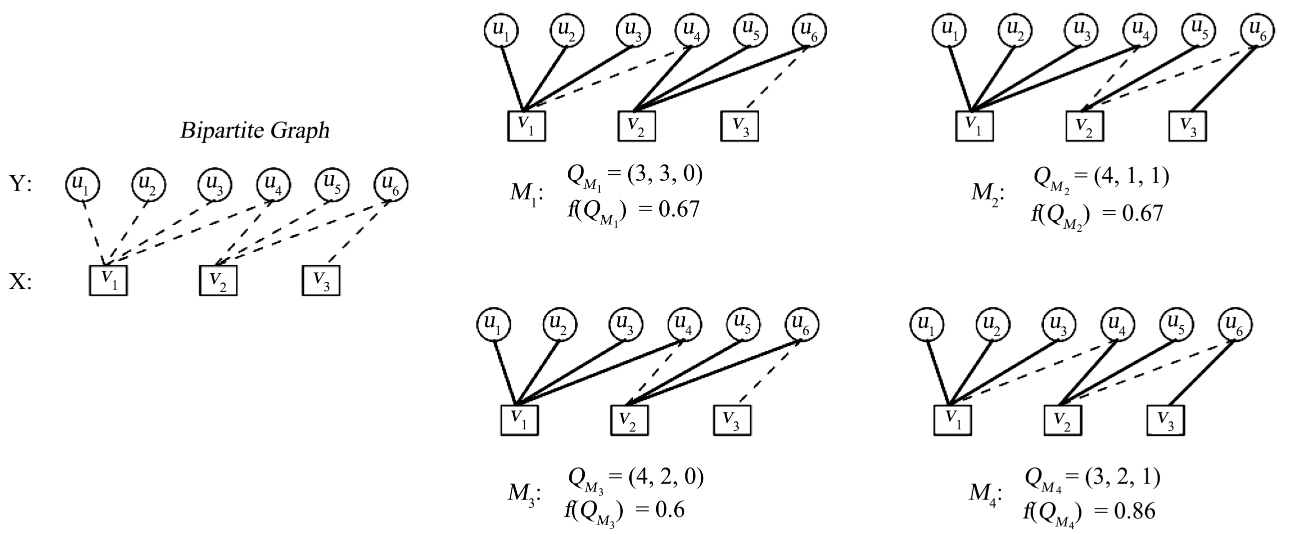

Figure 1. Semi-matching.

Now we give the definition of 1-k matchings. Let k be a positive integer. A *1-k matching with respect to X* is an edge subset M of G such that each vertex in Y is incident with at most one edge in M and each vertex in X either is incident with exactly k edges in M or is not incident with any edge in M. In the following, 1-k matchings refer to ones with respect to X. A vertex is called *M-saturated* if it is incident with an edge in M, otherwise, it is *M-unsaturated*. A 1-k matching M is *perfect* if every vertex is M-saturated, that is, M covers every vertex of G. Then a perfect 1-k matching is an ideal state of semi-matchings which optimize some objectives.

When $k = 1$, a 1-k matching is a matching. When $k = 2$, Izumi and Watanabe

[7] studied maximum 1 - 2 matching by giving augmenting trail, where augmenting trail is a generalization of the augmenting path for matchings and presented an algorithm for finding a maximum 1 - 2 matching in bipartite graphs.

Our work is to extend the matching theory on bipartite graphs to 1-*k* matching, motivated by the classical matching problem. Hall's theorem is well known for judging the existence of perfect matchings [8] (see Theorem 1.1) and the deficient form of Hall's theorem is given in [9] (see Corollary 1.1). Moreover, in the matching theory [8], the elementary bipartite graphs about matching are well characterized (see Proposition 1.1). An edge of a graph *G* is *allowed* if it lies in some perfect matching of *G*, otherwise, it is *forbidden*. A graph *G* is said to be *elementary* if its allowed edges form a connected spanning subgraph of *G*, that is, the subgraph obtained from *G* by deleting all forbidden edges is connected.

Theorem 1.1. (Hall's Theorem [8]) *A bipartite graph* $G=(X,Y)$ *has a matching which covers every vertex in X if and only if*

$$|N_G(S)|\ge|S|$$

for any $S\subseteq X$.

Corollary 1.1. [9] *A bipartite graph* $G=(X,Y)$ *has a matching which covers* $|X|-d$ *vertices in X if and only if*

$$|N_G(S)|\ge|S|-d$$

for any $S\subseteq X$.

Proposition 1.1. [8] *Let G be a bipartite graph with bipartition* (X,Y). *Then the following statements are equivalent*:

1) *G is elementary*;

2) *G has exactly two minimum vertex covers, named X and Y*;

3) $|X|=|Y|$ *and for every non-empty proper subset S of X,* $|N_G(S)|\ge|S|+1$;

4) $G=K_2$, *or* $|V(G)|\ge 4$ *and for any* $u\in X$ *and* $v\in Y$, $G-u-v$ *has a perfect matching*;

5) *G is connected and every edge of G is allowed.*

In this paper, we give three sufficient and necessary conditions for the existence of perfect 1-*k* matchings and for the existence of 1-*k* matchings covering $|X|-d$ vertices in *X*, respectively. Moreover, we characterize *k*-elementary bipartite graph, which is a connected graph such that the subgraph induced by all *k*-allowed edges is connected.

2. Perfect 1-*k* Matching

We give the following theorem about 1-*k* matching similar to Hall's theorem.

Theorem 2.1. *A bipartite graph* $G=(X,Y)$ *has a 1-k matching, which covers every vertex in X if and only if*

$$|N_G(S)|\ge k|S| \quad \text{for any } S\subseteq X. \tag{1}$$

Proof. It is obvious that if there exists a 1-*k* matching covering *X*, then the condition (1) holds. Now we prove "sufficiency" by induction on $|X|$. Suppose that

$|N_G(S)| \ge k|S|$ for any $S \subseteq X$. When $|X| = 1$, we have that $|N_G(X)| \ge k$. Then G has a 1-k matching covering X. Suppose that $|X| \ge 2$. We consider the following three cases.

Case 1 There exists a non-empty proper subset Z of X such that $|N_G(Z)| = k|Z|$.

Let H_1 be the subgraph induced by $Z \cup N_G(Z)$. Then H_1 satisfies the condition (1). By the induction hypothesis, H_1 contains a 1-k matching, say M_1, which covers every vertex in Z. Let H_2 be a subgraph induced by $(X - Z) \cup (Y - N_G(Z))$.

Claim 1 H_2 satisfies the condition (1).

Suppose, to the contrary, that there exists $S \subseteq X - Z$ such that $|N_{H_2}(S)| < k|S|$. Then

$$|N_G(S \cup Z)| = |N_G(Z)| + |N_{H_2}(S)| < k|Z| + k|S| = k|Z \cup S|,$$

which contradicts with the condition (1).

By Claim 1 and the induction hypothesis, there exists a 1-k matching of H_2, say M_2, which covers every vertex in $X - Z$. Then $M_1 \cup M_2$ is a 1-k matching of G covering every vertex in X.

Case 2 For any non-empty proper subset S of X, $|N_G(S)| \ge k|S| + 1$.

We choose a non-empty proper subset S_0 of X such that $|N_G(S_0)| - k|S_0|$ is smallest. Let $j = |N_G(S_0)| - k|S_0|$. Then $j \ge 1$ and for any non-empty proper subset S of X, we have that $|N_G(S)| \ge k|S| + j$. It is clear that $N_G(S_0) \subseteq N_G(X)$ and $N_G(X) - N_G(S_0) \subseteq N_G(X - S_0)$. By the condition (1), $|N_G(X)| \ge k|X|$. Then

$$|N_G(X) - N_G(S_0)| = |N_G(X)| - |N_G(S_0)| \ge k|X - S_0| - j \tag{2}$$

Subcase 1 $|N_G(X) - N_G(S_0)| \ge k|X - S_0|$.

Let H_1 be the subgraph induced by $S_0 \cup N_G(S_0)$ and H_2 the subgraph induced by $(X - S_0) \cup (N_G(X) - N_G(S_0))$. It is clear that H_1 satisfies the condition (1). By the induction hypothesis, H_1 has a 1-k matching M_1 covering every vertex in S_0.

Claim 2 H_2 satisfies the condition (1).

Clearly, $N_{H_2}(X - S_0) \subseteq N_G(X) - N_G(S_0)$. Let $y \in N_G(X) - N_G(S_0)$. Then there exists a vertex $x \in X - S_0$ such that $xy \in E(G)$. Then $xy \in E(H_2)$. Hence $y \in N_{H_2}(X - S_0)$. Thus $N_G(X) - N_G(S_0) = N_{H_2}(X - S_0)$. So we have that

$$|N_{H_2}(X - S_0)| = |N_G(X) - N_G(S_0)| \ge k|X - S_0|.$$

Suppose, to the contrary, that there exists a non-empty proper subset S_1 of $X - S_0$ such that $|N_{H_2}(S_1)| < k|S_1|$. Let $S = S_0 \cup S_1$. Then

$$|N_G(S)| = |N_G(S_0)| + |N_{H_2}(S_1)| < k|S_0| + j + k|S_1| = k|S| + j,$$

which contradicts with that j is smallest.

By Claim 2 and the induction hypothesis, H_2 has a 1-k matching M_2 covering every vertex in $X - S_0$. Hence $M_1 \cup M_2$ is a 1-k matching of G covering every vertex in X.

Subcase 2 $\left|N_G(X)-N_G(S_0)\right|<k\left|X-S_0\right|$.

Let $T_1=N_G(X-S_0)\cap N_G(S_0)$ and $T_2=N_G(X-S_0)-T_1$. Then $T_2=N_G(X)-N_G(S_0)$. According to (2) and the condition in the case, we have that

$$k\left|X-S_0\right|-j\le\left|T_2\right|<k\left|X-S_0\right|.$$

Thus $1\le k\left|X-S_0\right|-\left|T_2\right|\le j$. Since

$$\left|T_1\right|+\left|T_2\right|=\left|N_G(X-S_0)\right|\ge k\left|X-S_0\right|+j>\left|T_2\right|+j,$$

we have that $\left|T_1\right|\ge j+1$. Then we can choose a set $Z\subseteq T_1$ such that $\left|Z\right|=k\left|X-S_0\right|-\left|T_2\right|$. Hence $1\le\left|Z\right|\le j$. Let H_1 be the subgraph induced by $S_0\cup(N_G(S_0)-Z)$ and H_2 the subgraph induced by $(X-S_0)\cup(T_2\cup Z)$.

Claim 3 H_1 and H_2 satisfy the condition (1).

Let $\varnothing\ne S'\subseteq S_0$. Then

$$\left|N_{H_1}(S')\right|\ge\left|N_G(S')\right|-\left|Z\right|\ge k\left|S'\right|+j-\left|Z\right|\ge k\left|S'\right|.$$

Hence H_1 satisfies the condition (1). Now we prove that H_2 also satisfies the condition (1). Clearly, $\left|N_{H_2}(X-S_0)\right|=\left|N_G(X)-N_G(S_0)\right|+\left|Z\right|=\left|T_2\right|+\left|Z\right|=k\left|X-S_0\right|$. Suppose, to the contrary, that there exists a non-empty proper subset S_1 of $X-S_0$ such that $\left|N_{H_2}(S_1)\right|<k\left|S_1\right|$. Let $S=S_0\cup S_1$. Then

$$\left|N_G(S)\right|\le\left|N_G(S_0)\right|+\left|N_{H_2}(S_1)\right|<k\left|S_0\right|+j+k\left|S_1\right|=k\left|S\right|+j,$$

which contradicts with that *j* is smallest.

Hence by Claim 3 and the induction hypothesis, H_1 has a 1-*k* matching M_1 covering S_0 and H_2 has a 1-*k* matching M_2 covering every vertex in $X-S_0$. Hence $M_1\cup M_2$ is a 1-*k* matching of *G* covering every vertex in *X*.

A bipartite graph $G=(X,Y)$ is called *k*-balanced with respect to *X* if $\left|Y\right|=k\left|X\right|$. Now, we give the sufficiency and necessary conditions for the existence of perfect 1-*k* matchings.

Theorem 2.2. *Let* $G=(X,Y)$ *be a k-balanced* (*with respect to X*) *bipartite graph. Then the following statements are equivalent*:

1) *G has a perfect 1-k matching*;

2) $\left|N_G(S)\right|\ge k\left|S\right|$ *for any* $S\subseteq X$;

3) $\left|N_G(T)\right|\ge\frac{1}{k}\left|T\right|$ *for any* $T\subseteq Y$;

4) $\left|N_G(T)\right|\ge\frac{1}{k}\left|T\right|$ *for any* $T\subseteq Y$ *such that* $\left|T\right|\equiv1(\bmod k)$.

Proof. "(1) $\Leftrightarrow$ (2)". By Theorem 2.1, it is obvious.

"(2) $\Rightarrow$ (3)". Let *T* be a subset of *Y*. If $T=\varnothing$, then $\left|N_G(T)\right|=0=\frac{1}{k}\left|T\right|$. Suppose that $T\ne\varnothing$. Let $S=X-N_G(T)$. Then

$$k\left|X\right|-k\left|N_G(T)\right|=k\left|S\right|\le\left|N_G(S)\right|\le\left|Y\right|-\left|T\right|.$$

Since $\left|Y\right|=k\left|X\right|$, we have that $\left|N_G(T)\right|\ge\frac{1}{k}\left|T\right|$.

"(3) $\Rightarrow$ (2)". Let *S* be a subset of *X*. If $S=\varnothing$, then $|N_G(S)|=0=k|S|$. Suppose that $S\neq\varnothing$. Let $T=Y-N_G(S)$. Then

$$\frac{1}{k}|Y|-\frac{1}{k}|N_G(S)|=\frac{1}{k}|T|\le|N_G(T)|\le|X|-|S|.$$

Since $|Y|=k|X|$, we have that $|N_G(S)|\ge k|S|$.

"(3) $\Rightarrow$ (4)". It is obvious.

"(4) $\Rightarrow$ (3)". Let *T* be a subset of *Y*. If $T=\varnothing$, then $|N_G(T)|=0=\frac{1}{k}|T|$. Suppose $T\neq\varnothing$. Let $|T|=km+c$, where $m\ge 0$ and $0\le c\le k-1$. We distinguish the following cases.

Case 1 $c=0$.

Then $m\ge 1$. Hence we can choose a subset $Z\subseteq T$ such that $|Z|=k(m-1)+1$. Then $|Z|\equiv 1(\bmod k)$. Hence

$$|N_G(T)|\ge|N_G(Z)|\ge\frac{1}{k}|Z|=\frac{1}{k}(km-k+1).$$

Thus $|N_G(T)|\ge m=\frac{1}{k}|T|$.

Case 2 $c\neq 0$.

Without loss of generality, suppose that $2\le c\le k-1$. Then we can choose a set $Z\subseteq T$ such that $|Z|=km+1$. Hence

$$|N_G(T)|\ge|N_G(Z)|\ge\frac{1}{k}|Z|=\frac{1}{k}(km+1).$$

Thus $|N_G(T)|\ge m+1>\frac{1}{k}|T|$.

In the following, we consider the deficient form of Theorem 2.2, similar to the deficient form of Hall's theorem. If every component of a graph is $K_{s,t}$ and the number of components is *m*, then the graph is denoted by $m\cdot K_{s,t}$. Let $G=(X,Y)$ be a bipartite graph and $\mathcal{H}=\{K_{1,j}:0\le j\le k\}$. An $\mathcal{H}$*-quasi factor* with respect to *X* of *G* is a subgraph *H* of *G* such that $V(H)\cap X=X$ and every component of *H* is isomorphic to a member $K_{1,j}$ in $\mathcal{H}$ such that the vertex with degree *j* in $K_{1,j}$ is in *X*. Then we can assume that $H=\bigcup_{0\le j\le k}t_j\cdot K_{1,j}$, where $t_j\ge 0$. Thus $\sum_{j=0}^{k}t_j=|X|$.

Corollary 2.1. *Let* $G=(X,Y)$ *be a k-balanced* (*with respect to X*) *bipartite graph. Then the following statements are equivalent*:

1) *G has an* $\mathcal{H}$ *-quasi factor* $H=\bigcup_{0\le j\le k}t_j\cdot K_{1,j}$ *with respect to X such that* $t_k\ge|X|-d$ *and* $\sum_{j=0}^{k}jt_j\ge|Y|-d$, *which implies that G has a 1-k matching covering at least* $|X|-d$ *vertices in X*;

2) $|N_G(S)|\ge k|S|-d$ *for any* $S\subseteq X$;

3) $|N_G(T)|\ge\frac{1}{k}(|T|-d)$ *for any* $T\subseteq Y$;

4) $\left|N_G(T)\right| \geq \frac{1}{k}\left(|T|-d\right)$ *for any* $T \subseteq Y$ *such that* $|T| \equiv d+1 (\bmod k)$.

Proof. "(1) $\Rightarrow$ (2)". According to the definition of *H*, $\sum_{j=0}^{k} t_j = |X|$. For any $S \subseteq X$, let $S = S_0 \cup S_1 \cup \cdots \cup S_k$, where $S_j \subseteq V\left[t_j \cdot K_{1,j}\right]$. Then $\left|S_j\right| \leq t_j$. Hence we have that

$$\begin{aligned}
\left|N_G(S)\right| \geq \left|N_H(S)\right| &= \sum_{i=0}^{k} j\left|S_j\right| = \sum_{i=0}^{k}\left(k-(k-j)\right)\left|S_j\right| \\
&= k|S| - \sum_{i=0}^{k}(k-j)\left|S_j\right| \geq k|S| - \sum_{i=0}^{k}(k-j)t_j \\
&= k|S| - k\sum_{j=0}^{k} t_j + \sum_{j=0}^{k} jt_j \\
&\geq k|S| - k|X| + |Y| - d = k|S| - d.
\end{aligned}$$

"(2) $\Rightarrow$ (1)". Adding *d* new vertices $y_1, \cdots, y_d$ to *Y* of *G* and joining them to each vertex in *X*, the resulting graph is denoted by G_1. It is easy to check that G_1 satisfies the condition (1) of the Theorem 2.1. Hence G_1 has a 1-*k* matching, say *M*, which covers every vertex in *X*. Let G_2 be the subgraph induced by *M*. Then $G_2 = |X| \cdot K_{1,k}$. Let $H = G_2 - \{y_1, \cdots, y_d\}$. Then *H* is an $\mathcal{H}$-quasi factor with respect to *X* of *G*, *H* has at least $|X| - d$ components which are $K_{1,k}$, $V(H) \cap X = X$ and $\left|V(H) \cap Y\right| \geq k|X| - d = |Y| - d$. Hence we can assume that $H = \bigcup_{j=0}^{k} t_j \cdot K_{1,j}$ and $\sum_{j=0}^{k} jt_j = \left|V(H) \cap Y\right| \geq |Y| - d$ and $t_k \geq |X| - d$.

"(2) $\Rightarrow$ (3)". Let *T* be a subset of *Y*. If $T = \varnothing$, then $\left|N_G(T)\right| \geq 0 \geq \frac{1}{k}\left(|T|-d\right)$. Suppose that $T \neq \varnothing$. Let $S = X - N_G(T)$. Then

$$k|X| - k\left|N_G(T)\right| - d = k|S| - d \leq \left|N_G(S)\right| \leq |Y| - |T|.$$

Since $|Y| = k|X|$, we have that $\left|N_G(T)\right| \geq \frac{1}{k}\left(|T|-d\right)$.

"(3) $\Rightarrow$ (2)". Let *S* be a subset of *X*. If $S = \varnothing$, then $\left|N_G(S)\right| \geq 0 \geq k|S|$. Suppose that $S \neq \varnothing$. Let $T = Y - N_G(S)$. Then

$$\frac{1}{k}\left(|Y| - \left|N_G(S)\right| - d\right) = \frac{1}{k}\left(|T|-d\right) \leq \left|N_G(T)\right| \leq |X| - |S|.$$

Since $|Y| = k|X|$, we have that $\left|N_G(S)\right| \geq k|S| - d$.

"(3) $\Rightarrow$ (4)". It is obvious.

"(4) $\Rightarrow$ (3)". Let *T* be a subset of *Y*. If $|T| \leq d$, then $\left|N_G(T)\right| \geq 0 \geq \frac{1}{k}\left(|T|-d\right)$. Suppose that $|T| > d$. Let $|T| - d = km + c$, where $m \geq 0$ and $0 \leq c \leq k-1$. We distinguish the following cases.

Case 1 $c = 0$.

Then $m \geq 1$. Hence we can choose a subset $Z \subseteq T$ such that $|Z| - d = k(m-1) + 1$. Hence

$$\left|N_G(T)\right| \geq \left|N_G(Z)\right| \geq \frac{1}{k}\left(|Z|-d\right) = \frac{1}{k}(km-k+1).$$

Thus $\left|N_G(T)\right| \geq m = \frac{1}{k}\left(|T|-d\right)$.

Case 2 $c \neq 0$.

Without loss of generality, suppose that $2 \leq c \leq k-1$. Then we can choose a set $Z \subseteq T$ such that $|Z|-d = km+1$. Hence

$$\left|N_G(T)\right| \geq \left|N_G(Z)\right| \geq \frac{1}{k}\left(|Z|-d\right) = \frac{1}{k}(km+1).$$

Thus $\left|N_G(T)\right| \geq m+1 > \frac{1}{k}(km+c) = \frac{1}{k}\left(|T|-d\right)$.

3. Elementary Bipartite Graph about 1-*k* Matching

A edge of a graph *G* is *k-allowed* if it lies in some perfect 1-*k* matching of *G*, otherwise, it is *k-forbidden*. A bipartite graph is said to be a *k-elementary graph* if its *k*-allowed edges form a connected spanning subgraph of *G*. Now we characterize *k*-elementary bipartite graphs.

Theorem 3.1. *Let* $G=(X,Y)$ *be a bipartite graph. Then the following statements are equivalent*:

1) *G is k-elementary*;

2) *G is connected, k-balanced (with respect to X) and* $\left|N_G(S)\right| > k|S|$ *for every non-empty proper subset S of X*;

3) *G is connected, k-balanced (with respect to X) and* $\left|N_G(T)\right| > \frac{1}{k}|T|$ *for every non-empty proper subset T of Y*;

4) *G is connected, k-balanced (with respect to X) and* $\left|N_G(T)\right| > \frac{1}{k}|T|$ *for every non-empty proper subset T of Y such that* $|T| \equiv 0,1 (\bmod k)$;

5) *G is connected and every edge of G is k-allowed*;

6) $G = K_{1,k}$, *or* $\left|V(G)\right| \geq 2k+2$ *and for any* $x \in X$ *and* $y \in Y$, *there exist* $y_1, y_2, \cdots, y_{k-1} \in N_G(x)-\{y\}$ *such that* $G-\{x, y, y_1, y_2, \cdots, y_{k-1}\}$ *has a perfect 1-k matching.*

Proof. "(1) $\Rightarrow$ (2)". Since *G* is *k*-elementary, then *G* is connected and has a perfect 1-*k* matching. Then $|Y| = k|X|$ and by Theorem 2.1, $\left|N_G(S)\right| \geq k|S|$ for any $S \subseteq X$. Then *G* is *k*-balanced with respect to *X*. Suppose, to the contrary, that there exists a non-empty proper subset S_0 of *X* such that $\left|N(S_0)\right| = k|S_0|$. Since *G* is *k*-elementary, there exists a *k*-allowed edge of *G*, say *uv*, such that $u \in X - S_0$ and $v \in N_G(S_0)$. Hence we can assume that there exists a perfect 1-*k* matching *M* of *G* containing *uv*. Let *H* be the subgraph of *G* induced by $S_0 \cup \left(N_G(S_0)-\{v\}\right)$. Then $M \cap E(H)$ is a 1-*k* matching of *H* covering every vertex in S_0. But $\left|N_H(S_0)\right| = k|S_0|-1$, which contradicts with Theorem 2.1.

"(2) $\Rightarrow$ (3)". Let *T* be a non-empty proper subset of *Y* and $S = X - N_G(T)$. If $N_G(T) = X$, then $\left|N_G(T)\right| = |X| = \frac{1}{k}|Y| > \frac{1}{k}|T|$. Suppose that $N_G(T) \subset X$. Since *G* is connected, $N_G(T) \neq \varnothing$. Then $S \neq \varnothing$ and $S \subset X$. Hence

$$k|X| - k\left|N_G(T)\right| = k|S| < \left|N_G(S)\right| \leq |Y| - |T|.$$

Since $|Y| = k|X|$, we have that $|N_G(T)| > \frac{1}{k}|T|$.

"(3) $\Rightarrow$ (2)". Let S be a non-empty proper subset of X and $T = Y - N_G(S)$. If $N_G(S) = Y$, then $|N_G(S)| = |Y| = k|X| > k|S|$. Suppose that $N_G(S) \subset Y$. Since G is connected, $N_G(S) \neq \varnothing$. Then $T \neq \varnothing$ and $T \subset Y$. Hence

$$\frac{1}{k}|Y| - \frac{1}{k}|N_G(S)| = \frac{1}{k}|T| < |N_G(T)| \leq |X| - |S|.$$

Since $|Y| = k|X|$, we have that $|N_G(S)| > k|S|$.

"(3) $\Rightarrow$ (4)". It is obvious.

"(4) $\Rightarrow$ (3)". Let T be a non-empty proper subset of Y and $|T| = km + c$, where $m \geq 0$ and $0 \leq c \leq k-1$. Without loss of generality, suppose that $2 \leq c \leq k-1$. Then we can choose a set $Z \subseteq T$ such that $|Z| = km+1$. Hence

$$|N_G(T)| \geq |N_G(Z)| > \frac{1}{k}|Z| = \frac{1}{k}(km+1).$$

Thus $|N_G(T)| \geq m+1 > \frac{1}{k}|T|$.

"(2) $\Rightarrow$ (5)". We prove that every edge is k-allowed. Let $v \in Y$, $uv \in E(G)$ and $G_0 = G - E_G(\{v\}, X - \{u\})$. Then $d_{G_0}(v) = 1$. According to the condition (2), $d_G(x) \geq k+1$ for any $x \in X$. Since G is connected, $N_{G_0}(X) = N_G(X) = Y$. Since G is k-balanced, $|N_{G_0}(X)| = |Y| = k|X|$. Suppose, to the contrary, that there exists a non-empty proper subset S of X such that $|N_{G_0}(S)| \leq k|S| - 1$. Then

$$|N_G(S)| \leq |N_{G_0}(S)| + 1 \leq k|S| - 1 + 1 = k|S|,$$

a contradiction. By Theorem 2.2, there exists a perfect 1-k matching M of G_0. Then $uv \in M$. Clearly, M is a perfect 1-k matching M of G. So every edge of G is k-allowed.

"(5) $\Rightarrow$ (1)". It is obvious.

"(6) $\Rightarrow$ (2)". It implies that G is k-balanced with respect to X and has a perfect 1-k matching. First, we prove that G is connected. Suppose, to the contrary, that G is disconnected. Let $G_1 = (X_1, Y_2), \cdots, G_s = (X_s, Y_s)$ be all components of G. Then every G_i has a perfect 1-k matching. Hence $|Y_i| = k|X_i|$ for any $1 \leq i \leq s$. Let $x \in X_1$ and $y \in Y_2$. It is clear that for any $y_1, \cdots, y_{k-1} \in N_G(x) - \{y\}$, $G - \{x, y, y_1, \cdots, y_{k-1}\}$ has no perfect 1-k matchings, a contradiction. So G is connected. Secondly, we prove that $|N_G(S)| > k|S|$ for every non-empty proper subset S of X. By Theorem 2.2, $|N_G(S)| \geq k|S|$. Suppose, to the contrary, that there exists a non-empty proper subset S_0 of X such that $|N_G(S_0)| = k|S_0|$. Let $x \in X - S_0$ and $y \in N_G(S_0)$. Then there exist $y_1, \cdots, y_{k-1} \in N_G(x) - \{y\}$ such that $G - \{x, y, y_1, \cdots, y_{k-1}\}$ has a perfect 1-k matching. Let $H = G - \{x, y, y_1, \cdots, y_{k-1}\}$. Then

$$|N_H(S_0)| \leq k|S_0| - 1,$$

which contradicts with Theorem 2.2.

"(3) $\Rightarrow$ (6)". Since G is connected and k-balanced with respect to X, we have that $|N_G(Y)| = |X| = \frac{1}{k}|Y|$. By Theorem 2.2, G has a perfect 1-k matching, say M.

When $|X|=1$, $G=K_{1,k}$. Suppose that $|X|\ge 2$. Let H be the subgraph induced by M. Let $x\in X$ and $y\in Y$.

Case 1 $xy\in E(H)$

Let $y_1,\cdots,y_{k-1}\in N_H(x)-\{y\}$. Then $H-\{x,y,y_1,\cdots,y_{k-1}\}$ has a perfect 1-k matching. Hence $G-\{x,y,y_1,\cdots,y_{k-1}\}$ has a perfect 1-k matching.

Case 2 $xy\notin E(H)$

Let $X=\{x_1,\cdots,x_m\}$ and $Y_i=N_H(x_i)=\{y_{i1},\cdots,y_{ik}\},1\le i\le m$. Then $|Y_i|=k$. Let G^* be the resulting graph obtained from G by shrinking every Y_i to a single vertex y_i^* and deleting the multiple edges. Then G^* is a bipartite graph with bipartition $\left(X,\{y_1^*,\cdots,y_m^*\}\right)$. Then $M^*=\{x_1y_1^*,\cdots,x_my_m^*\}$ is a perfect matching of G^*. Since $|N_G(T)|>\frac{1}{k}|T|$ for any non-empty proper subset T of Y, we have that

$$\left|N_{G^*}(T^*)\right|=\left|N_G\left(\bigcup_{y_i^*\in T^*}Y_i\right)\right|>\frac{1}{k}\left|\bigcup_{y_i^*\in T^*}Y_i\right|=\left|T^*\right|$$

for any non-empty proper subset T^* of Y^*. Without loss of generality, suppose that $x=x_1$ and $y=y_{mk}\in Y_m$. By Proposition 1.1, $G^*-x_1-y_m^*$ has a perfect matching. Let $M_1^*=M^*-\{x_1y_1^*,x_my_m^*\}$. Then M_1^* is not a maximum matching of $G^*-x_1-y_m^*$ and x_m, y_1^* are M_1^*-unsaturated vertices. So there exists an M_1^*-augmenting path P^* of $G^*-x_1-y_m^*$ joining y_1^* and x_m. Without loss of generality, suppose that $P^*=y_1^*x_2y_2^*x_3y_3^*,\cdots,x_qy_q^*x_m$. Then we can assume that $P=y_{1k}x_2y_{2k}x_3y_{3k},\cdots,x_qy_{qk}x_m$ is a path of $G-x_1-y_{mk}$ joining y_{1k} and x_m. Let $M'=\left(M-\{x_1y_{11},\cdots,x_1y_{1k},x_my_{mk}\}\right)\Delta E(P)$. Then M' is a perfect 1-k matching of $G-\left\{x,y,y_{11},y_{12},\cdots,y_{1(k-1)}\right\}$.

Supported

This work is supported by the National Natural Science Foundation of China (No.12161073).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Bondy, J.A. and Murty, U.S.R. (2008) Graph Theory. Springer-Verlag, Berlin.

[2] Lawler, E. (2001) Combinatorial Optimization: Networks and Matroids. Dover Publications, New York.

[3] Harvey, N.J.A., Ladner, R.E., Lovász, L. and Tamir, T. (2006) Semi-Matchings for Bipartite Graphs and Load Balancing. *Journal of Algorithms*, **59**, 53-78. https://doi.org/10.1016/j.jalgor.2005.01.003

[4] Czygrinow, A., Hanćkowiak, M., Szymańska, E. and Wawrzyniak, W. (2016) On the Distributed Complexity of the Semi-Matching Problem. *Journal of Computer and System Sciences*, **82**, 1251-1267. https://doi.org/10.1016/j.jcss.2016.05.001

[5] Xu, J., Banerjee, S. and Rao, W. (2020) The Existence of Universally Agreed Fairest

Semi-Matchings in Any Given Bipartite Graph. *Theoretical Computer Science*, **818**, 83-91. https://doi.org/10.1016/j.tcs.2018.03.020

[6] Low, C.P. (2006) An Approximation Algorithm for the Load-Balanced Semi-Matching Problem in Weighted Bipartite Graphs. *Information Processing Letters*, **100**, 154-161. https://doi.org/10.1016/j.ipl.2006.06.004

[7] Izumi, H., Watanabe, S. and Watanabe, Y. (2017) Augmenting Trail Theorem for the Maximum 1-2 Matching Problem. *Discrete Mathematics*, *Algorithms and Applications*, **9**, 1750056. https://doi.org/10.1142/s1793830917500562

[8] Lovasz, L. and Plummer, M. (1986) Matching Theory. North-Holland, New York

[9] Bollobás, B. (1998) Modern Graph Theory. Springer, New York.