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Business Ecosystems and Differential Equations: Computer Simulations of Competition and Competitiveness Scenarios

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Abstract

Understanding the interactions that occur within a business niche and their future predictions can play an important role in decision-making in situations of competition, competition and/or cooperation. The similarities of interactions between species and the corporate world make it possible to use systems of ordinary differential equations, containing two companies and their interconnections. The purposes of this work were to mathematically model business ecosystems via systems of differential equations, using own codes written in computer language *R*; to investigate scenarios of competition and concurrence between companies under the effect of noise that simulates market variations, the impacts of advertising actions, spurious interference via random variables and the effects of external adversities. For this, the Euler's and Euler-Maruyama numerical methods were used in the discretizations of the equations. Finally, several scenarios were developed, altering, jointly and/or separately, each factor of the equations, indicating what they represent and the impact they have on the companies involved, associating them with different types of noise and defining a time course for the companies' performance.

Keywords

Differential Equations, Numerical Methods, Competition of Companies

1. Introduction

From ancient to contemporary societies, different events have sparked the need and fascination to try to predict events and anticipate facts. Understanding why something happened in a certain way has driven the development of techniques capable of assisting investigations of different natures.

One of the great events or discoveries was Mathematics and its capacity for representation in the form of numbers and symbols. The art of applying it to problem situations has been standing out throughout the technological and computational development of the 20^{th} and 21^{st} centuries, demanding an increasingly multidisciplinary and transdisciplinary stance, both in Exact Sciences and in Applied Social Sciences.

After defining the variables involved in the topic under study, the need arises to find relationships between such quantities, a process that often involves rates of variation. In mathematical terms, these are ordinary or partial differential equations. Differential equations are currently widely used for the mathematical modeling of real-world problems. Observable phenomena are often governed by such equations, which can be solved analytically using manual techniques or numerically using discretizations and computational implementations.

The corporate world is not different. Understanding the dynamics behind a business niche and its interconnections can help in adopting strategies for situations of competition, competition and/or cooperation (Chiavenato, 2004) and (Chalikias, Lalou, Skordoulis, Papadopoulos, & Fatouros, 2020). Business variables fit into the context of differential equations and research aimed at modeling organizational niches and their relationships have been standing out (Jha, Sahani, Jha, Sahani, & Poudel, 2024).

The works (Bajari, 2001) and (Konstantinov & Polovinkin, 2004) present results of studies of equilibrium points of differential equations, considering competition between companies. Bifurcations in dynamics between competitions and cooperations are analyzed in (Liao, Xu, & Tang, 2014) and (Xu, Liao, & Li, 2019).

A study on control returns, that is, a check on external influences was made by (Xu, Li, Xiao, & Yuan, 2019). A qualitative analysis involving three companies was prepared in (Kirjanen, Malafeyev, Zaitseva, Kovshov, & Kolesov, 2020).

Business niches in a competitive business environment are presented by (Chen, 2015), along with possible equilibrium points. It is important to highlight that, within the scope of the global economic scenario, the prediction of future scenarios contributes to decision-making aimed at maintaining the survival of companies (Hubert, 2018).

Based on (Chen, 2015), this work aimed to add disturbances to the growth of companies via periodic and/or random oscillations, as well as to present the possible trajectories of these companies, until they asymptotically approach the equilibrium point. For this purpose, four systems of ordinary differential equations (ODEs), one of which is composed of stochastic differential equations (SDEs), were described to see interactions between companies that compete within a given niche, over time.

Furthermore, a study on classical models of interactions via systems of ODEs was carried out, aiming at both a quantitative and numerical analysis of possible solution profiles. Then, an investigation on the coefficients of the differential equations was implemented to know which values represent characteristics of coexistence or extinction. Own codes were developed in *R language*, using the Euler and Euler-Maruyama methods, to obtain numerical solutions that represent business scenarios.

2. Mathematical Modeling

In this section, the models used to describe the interactions between two companies, contained in the same business niche, are described.

2.1. Model 1: Two Companies

The ordinary differential equations that represent a mathematical model of competition between two companies are given by:

$$\begin{cases} \dfrac{dx_1}{dt} = r_1 x_1 \left(1 - \dfrac{x_1}{N_1} - \delta_1 \dfrac{x_2}{N_2}\right) \\ \dfrac{dx_2}{dt} = r_2 x_2 \left(1 - \dfrac{x_2}{N_2} - \delta_2 \dfrac{x_1}{N_1}\right) \end{cases}, \quad x_1(0) = x_{10} \text{ and } x_2(0) = x_{20}, . \tag{1}$$

where $x_1(t)$ and $x_2(t)$ represent the companies A_1 and A_2, whose growth rates are respectively r_1 and r_2. The magnitudes N_1 and N_2 symbolize the maximum capacity that such companies have. The coexistence of companies leads to competition between them and how this competition affects each company is portrayed in the factors δ_1 (the company's impact A_2 on A_1) and δ_2 (the company's impact A_1 on A_1). These representations are based on the ecosystem model of organizations described by Chen (2015).

It is noted that the model (1) has similarities with mathematical models of interactions between species, such as that of Lotka-Volterra with two species (Bassanezi, 2012), interactions between three species (Garcia & Silveira, 2018), Holling-Tanner model (Silveira & Garcia, 2020) and May-Holling-Tanner (Silveira & Garcia, 2024).

If δ_1 and δ_2 were equal to zero, the companies would have independent logistical growth, affected only by the limitations of each, represented by N_1 and N_2. Adopting $\delta_1, \delta_2 > 0$, there is the interference that one exerts on the growth of the other. The negative sign in these terms means that the "encounter" between the companies is harmful to both.

Some questions that arise are: what are the possible solution profiles resulting from interactions between the two companies? What are the short, medium and long-term behaviors? Is it possible for both companies to survive in the competitive environment? Will one of the companies be able to cause the other to go bankrupt? An alternative to reaching answers is to develop a qualitative analysis of the system of differential equations and then make computational implementations to obtain numerical solutions.

2.2. Qualitative Theory

The equilibrium solutions of the system (1) are calculated by finding the critical points. Such points are also called equilibrium or stable solutions. Therefore:

$$\begin{cases} r_1 x_1\left(1-\frac{x_1}{N_1}-\delta_1\frac{x_2}{N_2}\right)=0 \\ r_2 x_2\left(1-\frac{x_2}{N_2}-\delta_2\frac{x_1}{N_1}\right)=0 \end{cases}, \tag{2}$$

we have $(0,0)$, $(N_1,0)$, $(0,N_2)$ and $\left(\frac{N_1(1-\delta_1)}{1-\delta_1\delta_2},\frac{N_2(1-\delta_2)}{1-\delta_1\delta_2}\right)$.

In the phase plane, these points represent the intersections between the lines $x_1=0$, $x_2=0$, $1-\frac{x_1}{N_1}-\delta_1\frac{x_2}{N_2}=0$ and $1-\frac{x_2}{N_2}-\delta_2\frac{x_1}{N_1}=0$.

The only possibility for the two companies to coexist is when the two lines $1-\frac{x_1}{N_1}-\delta_1\frac{x_2}{N_2}=0$ and $1-\frac{x_2}{N_2}-\delta_2\frac{x_1}{N_1}=0$ intersect at the point $\left(\frac{N_1(1-\delta_1)}{1-\delta_1\delta_2},\frac{N_2(1-\delta_2)}{1-\delta_1\delta_2}\right)$.

Typically, in mathematical modeling applied to business dynamics, it is necessary that $x_1(t)\ge 0$ and $x_2(t)\ge 0$. Therefore, the coexistence of companies in a competitive environment occurs, as long as the intersection point between the lines is in the first quadrant of the phase plane, in addition to the characteristics of the critical points obtained.

A system $\begin{cases} x'(t)=F(x,y) \\ y'(t)=G(x,y) \end{cases}$ is said to be quasi-linear in the neighborhood of the critical point (x^*,y^*), that satisfies $((F(x^*),F(y^*))=0$, if

$$\begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}=\begin{bmatrix} F_x(x^*,y^*) & F_y(x^*,y^*) \\ G_x(x^*,y^*) & G_y(x^*,y^*) \end{bmatrix}\begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix}+\begin{bmatrix} F_1(x,y) \\ G_1(x,y) \end{bmatrix},$$

with

$$\lim_{(x,y)\to(x^*,y^*)}\frac{F_1(x,y)}{\sqrt{(x-x^*)^2+(y-y^*)^2}}=\lim_{(x,y)\to(x^*,y^*)}\frac{G_1(x,y)}{\sqrt{(x-x^*)^2+(y-y^*)^2}}=(0,0)$$

where F_1 and G_1 are the terms of order two or more, of the Taylor series expansion of the functions F and G around the point (x^*,y^*).

The change of variables $u=x-x^*$ and $v=y-y^*$ makes all the theory developed for critical point analysis $(x^*,y^*)=(0,0)$, is also applicable to critical points $(x^*,y^*)\ne(0,0)$.

The critical point $P_1=(0,0)$, it is an unstable equilibrium point, because the quasi-linear autonomous system obtained by the Lyapunov-Poincaré Linearization Theorem (Boyce & DiPrima, 2017) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \text{ with } u=x \text{ and } v=y.$$

The eigenvalues of the Jacobian matrix of the linearized system are $\lambda_1=r_1$ and

$\lambda_2 = r_2$. As in business competition problems r_1 and r_2 are positive (business growth rates), then $\lambda_1 = r_1 > 0$ and $\lambda_2 = r_2 > 0$, which implies that the equilibrium point $(0,0)$ is unstable.

In practice, this means that for any initial values sufficiently close to $(0,0)$, the solution will move away from $(0,0)$, then both companies will grow.

The critical point $P_2 = (0, N_2)$ is it an unstable equilibrium point or an asymptotically stable point, since the quasi-linear autonomous system obtained is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} r_1(1-\delta_1) & 0 \\ -r_2\dfrac{N_2}{N_1}\delta_2 & -r_2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \quad \text{with } u = x \text{ and } v = y - N_2.$$

The eigenvalues are $\lambda_2 = -r_2 < 0$ and $\lambda_1 = r_1(1-\delta_1)$. Like again r_1 and r_2 are positive, then $\lambda_2 = -r_2 < 0$ (always negative), $\lambda_2 > 0$, if $\delta_1 < 1$ and $\lambda_2 < 0$ (unstable), if $\delta_1 > 1$ (asymptotically stable).

The critical point $P_3 = (N_1, 0)$ is an unstable equilibrium point or an asymptotically stable point, because in an analogous way, the obtained quasi-linear autonomous system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -r_1 & -r_1\dfrac{N_1}{N_2}\delta_1 \\ 0 & r_2(1-\delta_2) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \quad \text{where } u = x - N_1 \text{ and } v = y.$$

The eigenvalues of the Jacobian matrix are $\lambda_1 = -r_1 0$ and $\lambda_2 = r_2(1-\delta_2)$. As seen previously, r_1 and r_2 are positive, therefore $\lambda_1 = -r_1 < 0$ (always negative), $\lambda_2 > 0$, if $\delta_2 < 1$ and $\lambda_2 < 0$ (unstable), ir $\delta_2 > 1$ (asymptotically stable).

For the critical point $P_4 = \left(\dfrac{N_1(1-\delta_1)}{1-\delta_1\delta_2}, \dfrac{N_2(1-\delta_2)}{1-\delta_1\delta_2}\right)$, associated with the intersection of two lines, to be in the first quadrant it is necessary that $x_1^* > 0$ and $x_2^* > 0$, that is,

$$\frac{N_1(1-\delta_1)}{1-\delta_1\delta_2} > 0 \text{ and } \frac{N_2(1-\delta_2)}{1-\delta_1\delta_2} > 0, \text{ with } 1-\delta_1\delta_2 \neq 0.$$

If $1-\delta_1\delta_2 > 0$, we have necessarily that $N_1(1-\delta_1) > 0$ and $N_2(1-\delta_2) > 0$, so that the critical point is in the first quadrant. Therefore, $\delta_1 < 1$ and $\delta_2 < 1$ (situation (i)).

If $1-\delta_1\delta_2 < 0$, we have necessarily that $N_1(1-\delta_1) < 0$ and $N_2(1-\delta_2) < 0$, so that the critical point is in the first quadrant. Thus, $\delta_1 > 1$ and $\delta_2 > 1$ (situation (ii)).

Let's analyze the behavior of the solution in the neighborhood of the point P_4. In this case, we have the following linearized system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{1-\delta_1\delta_2}\begin{bmatrix} -r_1(1-\delta_1) & -\dfrac{r_1\delta_1 N_1}{N_2}(1-\delta_1) \\ -\dfrac{r_2\delta_2 N_2}{N_1}(1-\delta_2) & -r_2(1-\delta_2) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix},$$

where $u = x_1 - x_1^*$ and $v = x_2 - x_2^*$.

Thus, the characteristic polynomial

$$\begin{vmatrix} -r_1(1-\delta_1)-\lambda & -\dfrac{r_1\delta_1 N_1}{N_2}(1-\delta_1) \\ -\dfrac{r_2\delta_2 N_2}{N_1}(1-\delta_2) & -r_2(1-\delta_2)-\lambda \end{vmatrix} = 0$$

has

$$\Delta = \frac{1}{(1-\delta_1\delta_2)^2}\left\{\left[r_1\delta_1 - r_2(1-\delta_2)\right]^2 + 4r_1\delta_1 r_2\delta_2(1-\delta_1)(1-\delta_2)\right\} > 0,$$

because in situations (i) and (ii), $(1-\delta_1)(1-\delta_2) > 0$ and the other terms are all positive.

In this way, the eigenvalues are real and distinct, given by:

$$\lambda_{1,2} = \frac{1}{2(1-\delta_1\delta_2)}\left\{(-1)\left[r_1(1-\delta_1) + r_2(1-\delta_2)\right] \pm \sqrt{\left[r_1(1-\delta_1) - r_2(1-\delta_2)\right]^2 + 4r_1\delta_1 r_2\delta_2(1-\delta_1)(1-\delta_2)}\right\}$$

Note that,

$$\begin{aligned} &\left[r_1(1-\delta_1) + r_2(1-\delta_2)\right] \\ &> \left[r_1(1-\delta_1) - r_2(1-\delta_2)\right]^2 + 4r_1\delta_1 r_2\delta_2(1-\delta_1)(1-\delta_2) \\ &\Leftrightarrow (1-\delta_1\delta_2) > 0. \end{aligned}$$

Therefore, if $1-\delta_1\delta_2 > 0$, then $\lambda_{1,2} < 0$. Thus, the critical point P_4 will be an asymptotically stable node (situation (i)).

Furthermore,

$$\begin{aligned} &\left[r_1(1-\delta_1) + r_2(1-\delta_2)\right] \\ &< \left[r_1(1-\delta_1) - r_2(1-\delta_2)\right]^2 + 4r_1\delta_1 r_2\delta_2(1-\delta_1)(1-\delta_2) \\ &\Leftrightarrow (1-\delta_1\delta_2) < 0, \end{aligned}$$

which indicates that the signal $\pm$ of the root is predominant, resulting in two distinct eigenvalues, different from zero and with opposite signs, for example, $\lambda_1 < 0 < \lambda_2$. Therefore, is a saddle point, unstable node, situation (ii).

When

$$\frac{r_1\delta_1}{N_2}\frac{r_2\delta_2}{N_1} < \frac{r_1}{N_1}\frac{r_2}{N_2},$$

there is a predominance of logistical terms. In addition,

$$\frac{r_1\delta_1}{N_2}\frac{r_2\delta_2}{N_1} < \frac{r_1}{N_1}\frac{r_2}{N_2} \Leftrightarrow (1-\delta_1\delta_2) > 0.$$

Thus, we have the situation (i), an asymptotically stable equilibrium point. In this way, the two companies will be able to coexist, even when competing with each other.

If

$$\frac{r_1\delta_1}{N_2}\frac{r_2\delta_2}{N_1} > \frac{r_1}{N_1}\frac{r_2}{N_2},$$

the predominance is of the terms of competition. Additionally,

$$\frac{r_1\delta_1}{N_2}\frac{r_2\delta_2}{N_1} > \frac{r_1}{N_1}\frac{r_2}{N_2} \Leftrightarrow (1-\delta_1\delta_2) < 0.$$

Therefore, we have situation (ii), unstable node. Consequently, the two companies will not be able to coexist, that is, the competition will result in the extinction of one of the companies, depending on the initial conditions. This condition is known as *Competitive Exclusion Principle (Gause's Law)*, which in the case of companies says that two companies in a state of strong competition cannot coexist.

2.3. Two Companies with Oscillations

The ordinary differential equations that represent the competition model between two companies, with oscillatory influence on the growth rate, of the type *cosine*, are:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \left(r_1 \pm a_1\cos(\omega_1)t\right)x_1\left(1-\dfrac{x_1}{N_1}-\delta_1\dfrac{x_2}{N_2}\right) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \left(r_2 \mp a_2\cos(\omega_2)t\right)x_2\left(1-\dfrac{x_2}{N_2}-\delta_2\dfrac{x_1}{N_1}\right) \end{cases}, \tag{3}$$

with $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$.

2.4. Two Companies with White Noise

The competition model between two companies, with the influence of white noise on the growth rate, is given by:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = (r_1 \pm \varepsilon_1)x_1\left(1-\dfrac{x_1}{N_1}-\delta_1\dfrac{x_2}{N_2}\right) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = (r_2 \mp \varepsilon_2)x_2\left(1-\dfrac{x_2}{N_2}-\delta_2\dfrac{x_1}{N_1}\right) \end{cases}, \quad x_1(0) = x_{10} \text{ e } x_2(0) = x_{20}. \tag{4}$$

2.5. Two Companies as Random Variables

The stochastic differential equations (SDE) that represent the competition model between two companies, taking as a basis the deterministic system: system (1) as the average value, are:

$$\begin{cases} \mathrm{d}X_t^{(1)} = \left[r_1X_t^{(1)}\left(1-\dfrac{X_t^{(1)}}{N_1}-\delta_1\dfrac{X_t^{(2)}}{N_2}\right)\right]\mathrm{d}t + b_1\sigma_1\mathrm{d}W_t^{(1)} \\ \mathrm{d}X_t^{(2)} = \left[r_2X_t^{(2)}\left(1-\dfrac{X_t^{(2)}}{N_2}-\delta_2\dfrac{X_t^{(1)}}{N_1}\right)\right]\mathrm{d}t + b_2\sigma_2\mathrm{d}W_t^{(2)} \end{cases}, \tag{5}$$

where σ_1 and σ_2 are related to the deviation from the average evolution, $W_t^{(1)}$ and $W_t^{(2)}$ are Wiener's processes associated with some interval $[0,T]$, where we want to solve the SDE (Braumann, 2019). In this model, the companies are represented by random variables $X_t^{(1)}$ and $X_t^{(2)}$ and the initial conditions are, respectively, $X_0^{(1)} = x_{10}$ and $X_0^{(2)} = x_{20}$.

3. Numerical Methods

In this section, the numerical methods used to obtain approximate solutions to differential equations are presented. For deterministic differential equations, models (1), (3) and (4), we use the Euler's method and for stochastic differential equations, we implement the Euler-Maruyama method.

3.1. Euler Method

Consider the Initial Value Problem (IVP):

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t,x), & t \in I \subset \mathbb{R} \\ x(0) = x_0 \end{cases}, \tag{6}$$

in which $f : I \times \mathbb{R} \to \mathbb{R}$ and $x = x(t)$, $x : I \to \mathbb{R}$.

The numerical scheme chosen was Euler's method, adopting for $t \in [0, t_f] \subset \mathbb{R}$, a regular partition

$$\Pi : 0 = t_0 < t_1 < \cdots < t_N = t_f \tag{7}$$

with N subintervals and spacing $h = (t_f - t_0)/N$.

Denoting by ξ_i the approximate solution in time $t = t_i$, whose exact solution is $x_i = x(t_i)$, the set $\{\xi_0 = x_0, \xi_1, \cdots, \xi_N\}$ obtained by the numerical method is an approximation of the IVP solution.

In the first-order single-step explicit Euler's method, the approximate solution ξ_{i+1} is defined by

$$\xi_{i+1} = \xi_i + hf(t_i, \xi_i), \ i = 0, \cdots, N-1. \tag{8}$$

Details about the method are in (Buchanan & Turner 1992); a numerical and comparative study of Euler's method with other methods are in (Garcia & Silveira, 2018).

The absolute stability interval J of a numerical method can be achieved by applying it to an IVP (6), where the function f is $f(t,x) = \lambda x(t)$, with $\lambda < 0$. If the numerical solution obtained satisfies $\lim \xi_n = 0$ for $\lambda h \in J$, then J is the stability interval of method (Buchanan & Turner, 1992). For Euler's method we have $J = (-2, 0)$.

3.2. Euler-Maruyama Method

Given an Initial Value Problem for a Stochastic Differential Equation:

$$\begin{cases} \mathrm{d}X_t = a(t, X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}W_t, & t \in I \subset \mathbb{R} \\ X_0 = x_0 \in \mathbb{R} \end{cases}, \tag{9}$$

in which W_t is a Wiener's process, defined within the range where the SDE is intended to be resolved, that is, $I = [0, t_f] \subset \mathbb{R}$ and X_t is a stochastic process that satisfies the IVP (9) (Braumann, 2019).

The Euler-Maruyama method consists of calculating an approximate Ξ_t for a stochastic process X_t, via Markov Chains, as follows: let Π, a regular partition identical to the Equation (7), whose spacing h will be denoted by Δt, then the approximate solution is a Markov Chain of order one,

$$\Xi_{i+1} = \Xi_i + a\left(t_i, \Xi_{t_i}\right)\Delta t + b\left(t_i, \Xi_{t_i}\right)\Delta W_{t_i}, \quad i = 0, \cdots, N, \tag{10}$$

where the random variables $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$ are independent and identical to determinates values by a normal distribution of expected value zero and variance Δt, that is, $\Delta W_{t_i} = \mathcal{N}\left(0, \Delta t\right)$ (da Silva, Dunczmal, & Dunczmal, 2016).

4. Simulations and Results

For the simulations, the company A_1 has been in the market longer than the company A_2, which is new to the market and, therefore, it has a higher initial value. In this way, $x_1 = x_1(t)$ represents the company A_1 already consolidated and $x_2 = x_2(t)$ the company A_2 that is new in the same business niche as A_1.

Both companies have the maximum capacity (support capacity) to achieve 1000 units, i.e. $N_1 = N_2 = 1000$. The company A_1 initially is at the level of $x_1(0) = 400$ (40% of the support capacity) and the company A_2 at the level of $x_2(0) = 10$ (1% of the support capacity).

Companies have the same growth rate $r_1 = r_2 = 0.2$ and what differs is how much one company impacts the other. A_2 affect the growth of A_1 at a rate of δ_1 and A_1 affects the growth of A_2, with a rate of δ_2.

As described in Section 0, asymptotic behavior is predicted when the values of δ_1 and δ_2. However, in most cases, the course of temporal evolution will only be known by numerically solving the ODE systems.

In all implemented scenarios, we employed $h = 0.05$. All codes were written by the authors themselves in *R* programming language and the discretizations used satisfy the stability criteria described in (Buchanan & Turner, 1992) for Euler's method.

4.1. Case $\delta_1 = \delta_2 = 0$

In this scenario, the companies have independent logistics growths, without one interfering with the other, and such growths are affected only by the limitations of each company, represented by parameters N_1 and N_2. **Figures 1-6** show the simulations, for different situations. The temporal discretization in this example was $t \in [0, 100]$, with 2000 subintervals. **Figure 1(a)** illustrates the classical logistic model, whose solution can also be found analytically, for each company (Garcia & Silveira, 2018).

For the Model 2, **Figure 1(b)** contains the temporal evolution with the oscillations created by the trigonometric function included in the growth rate, following the trend of logistic model solution (**Figure 1(a)**).

The Models 3 and 4, **Figure 2(a)** and **Figure 2(b)**, present random oscillations

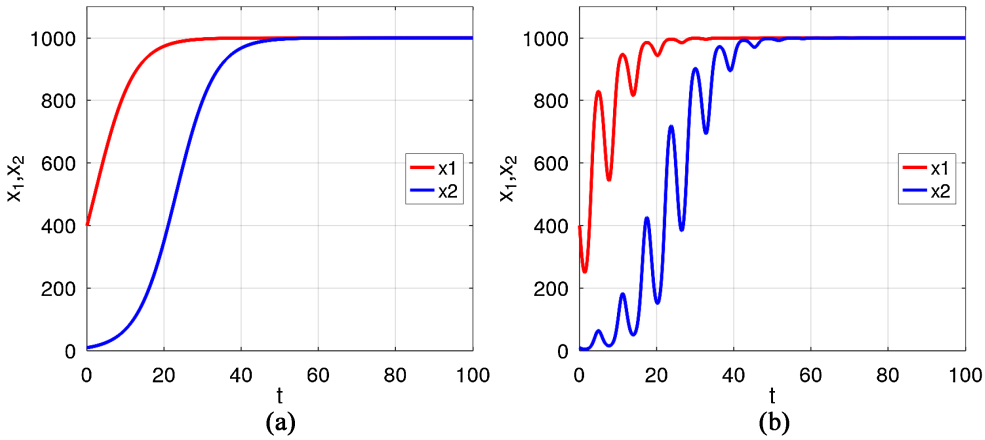

Figure 1. Numerical solutions of models 1 and 2. (a) Model 1; (b) Model 2.

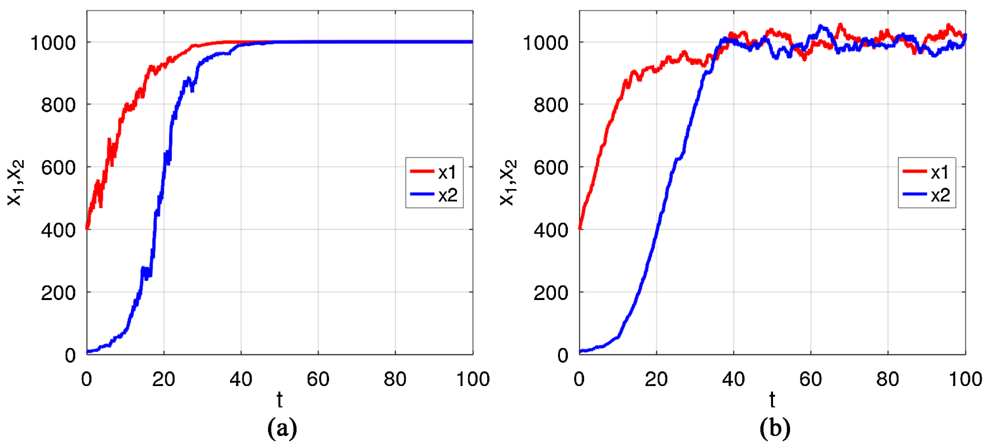

Figure 2. Time evolution by models 3 and 4. (a) Model 3; (b) Model 4.

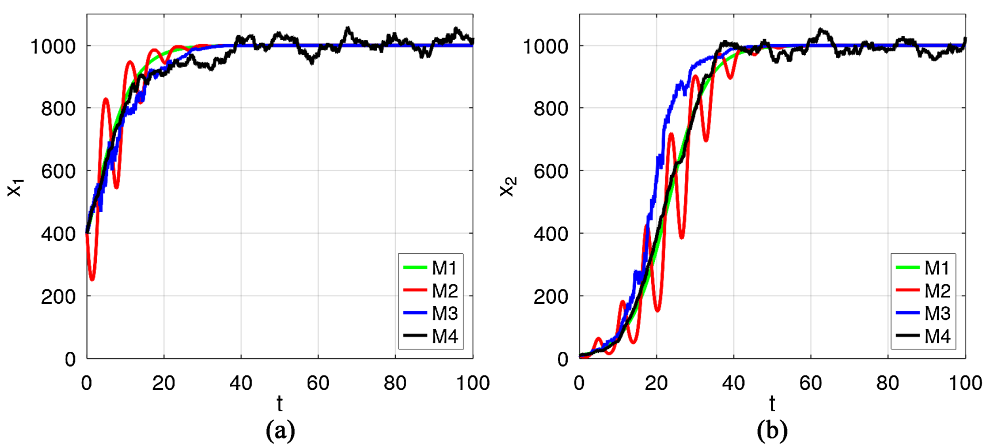

Figure 3. Comparison between numerical solutions of each company. (a) Company A_1; (b) Company A_2.

following the trend of logistic growth that are highlighted at the beginning and in the middle of the temporal evolution, before stabilization. In Model 4, the oscillations were more evident in the stabilization part, however, both models, on

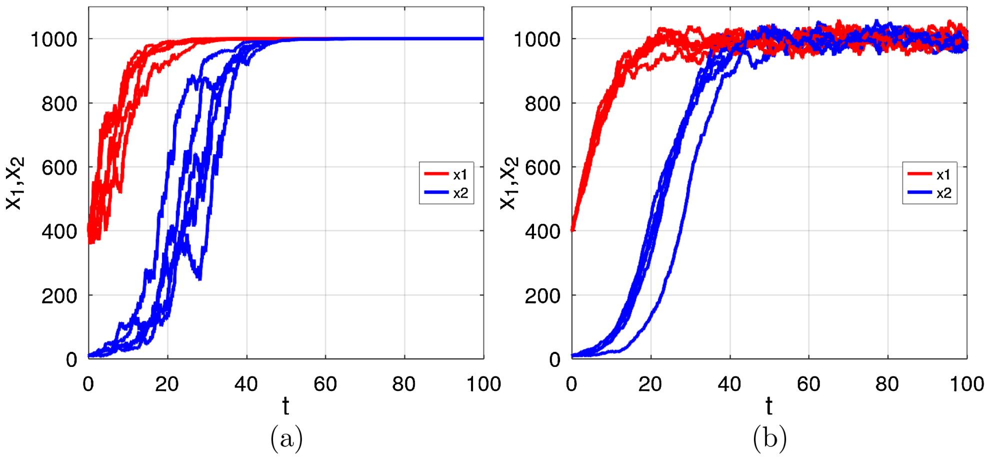

Figure 4. Models 3 and 4 with $N = 5$ simulations. (a) Model 3; (b) Model 4.

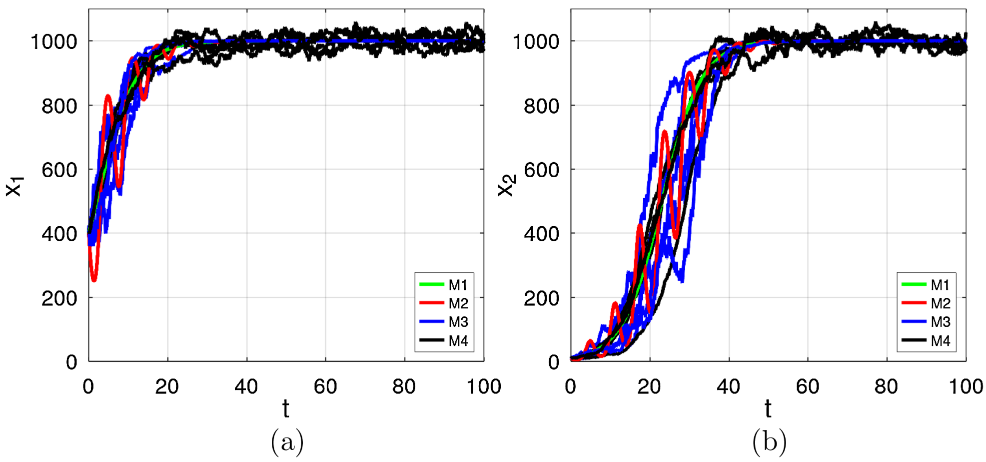

Figure 5. Comparison between numerical solutions with $N = 5$ simulations. (a) Company A_1; (b) Company A_2.

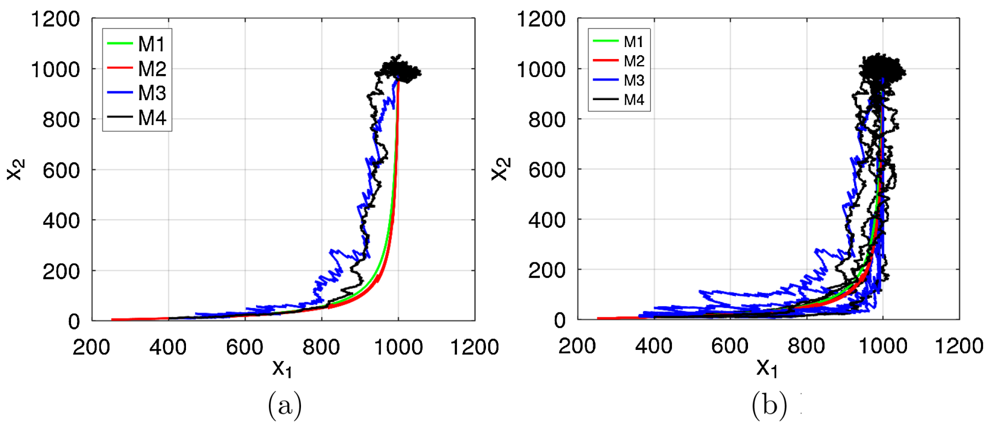

Figure 6. Trajectories in the phase plane. (a) Phase Plane 1; (b) Phase Plane 2.

average, follow the logistic behavior of model 1 (Figure 1(a)).

Figure 3(a) and Figure 3(b) explicit, respectively, the comparison between the models for the companies A_1 and A_2.

Since models (3) and (4) have random characteristics, Figure 4(a) and Figure 4(b) display five distinct simulations for each model. Even though the trajectory is influenced by each draw of the random variable, the logistic trend prevails, and all simulations follow it.

Figure 5(a) and Figure 5(b) show the evolution of each company, together with the five simulations of models (3) and (4). The phase plans, Figure 6(a) and Figure 6(b) expose the evolutions going towards the same equilibrium point.

4.2. Cases $\delta_1 > 1$ and $\delta_2 > 1$; $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$

Assuming $\delta_1 > 1$ and $\delta_2 > 1$, regardless of whether $\delta_1 \geq \delta_2$ or $\delta_1 < \delta_2$, the new company will not be able to grow and sustain itself in the market (results similar to those that will be seen in Subsection 4.3). Thus, in this example, the initially dominant company prevailed.

Considering now that $\delta_1 = 0.8$ and $\delta_2 = 0.6$, there is the possibility of coexistence between the two competing companies. As $\delta_1 > \delta_2$, the competition strategies adopted by the company A_2 harm more A_1, than the opposite. At first, it seems that the company A_1, that dominates the market, will maintain its rapid growth compared to the company A_2 and reach the top of the sector, represented by N_1, Figure 7(a).

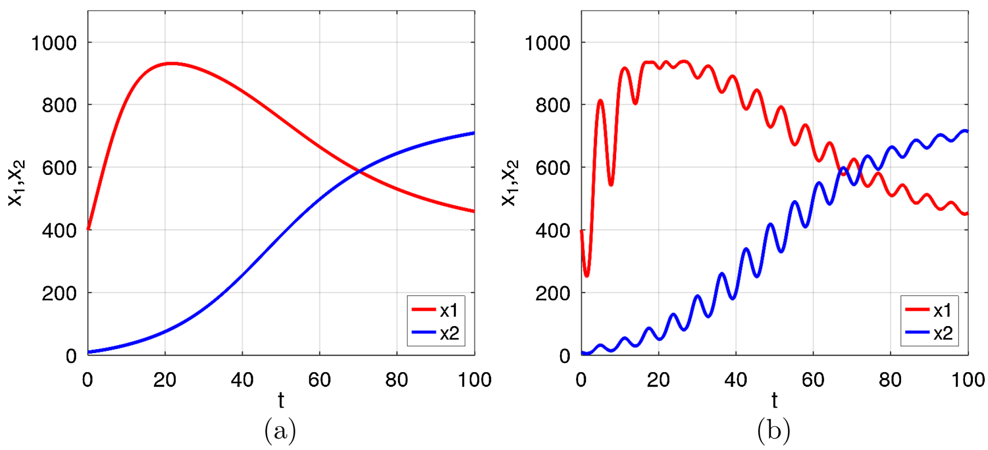

Figure 7. Numerical solutions of models 1 and 2. (a) Model 1; (b) Model 2.

However, the company A_1 had a fall and stabilized close to the value of 400, indicating that $\lim_{t\to\infty} x_1(t) \cong 400$. On the other hand, the company A_2 maintained a growth similar to the logistics trend, but its stabilization was close to 800, i.e. $\lim_{t\to\infty} x_2(t) \cong 800$. In this situation, we have $\lim_{t\to\infty} x_1(t) < \lim_{t\to\infty} x_2(t)$, when $\delta_1 > \delta_2$.

Even with models that include periodic oscillations, Figure 7(b) and random, Figure 8(a) and Figure 8(b), on average, all models follow the trend of model 1, see Figure 9(a) and Figure 9(b). Furthermore, the stabilizations were evidenced in extra simulations, Figure 10 and Figure 11 with five simulations involving random variables, and by the phase plans, Figure 12.

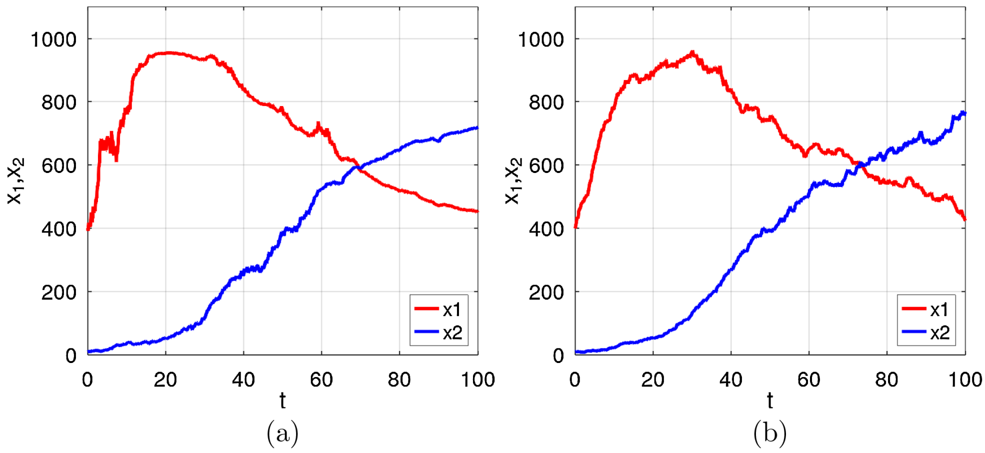

Figure 8. Time evolution by models 3 and 4. (a) Model 3; (b) Model 4.

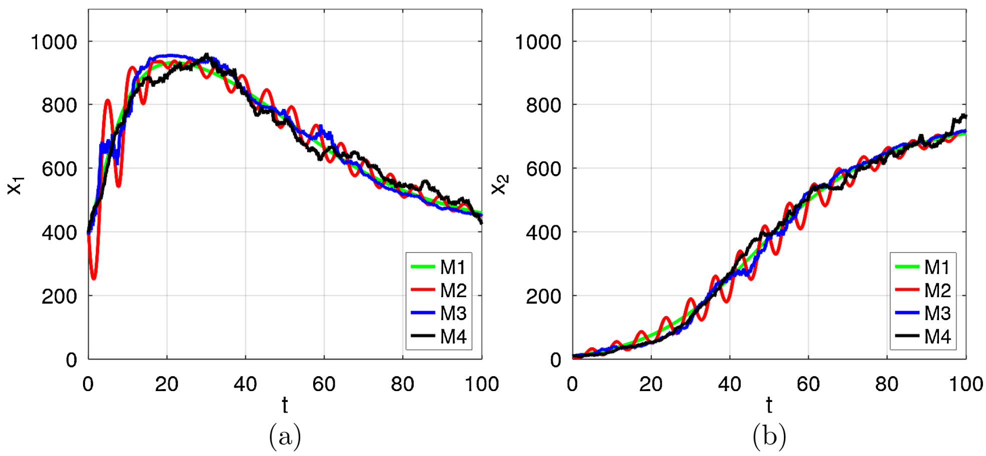

Figure 9. Comparison between numerical solutions of each company. (a) Company A_1; (b) Company A_2.

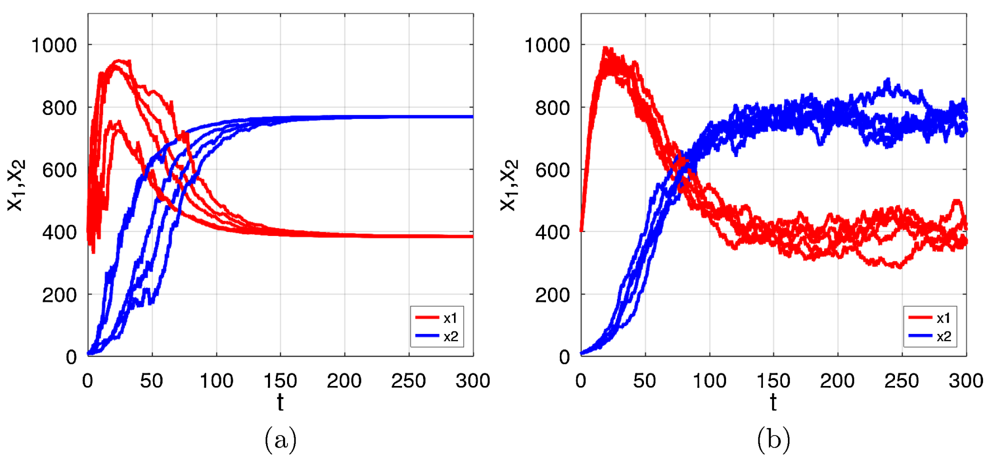

Figure 10. Models 3 and 4 with $N = 5$ simulations. (a) Model 3; (b) Model 4.

The stabilizations are visible in **Figures 10-12**. Note that the values are less than $N_1 = N_2 = 1000$, which indicates that interactions between companies in a competitive manner do not allow them to reach the entire market. However, with

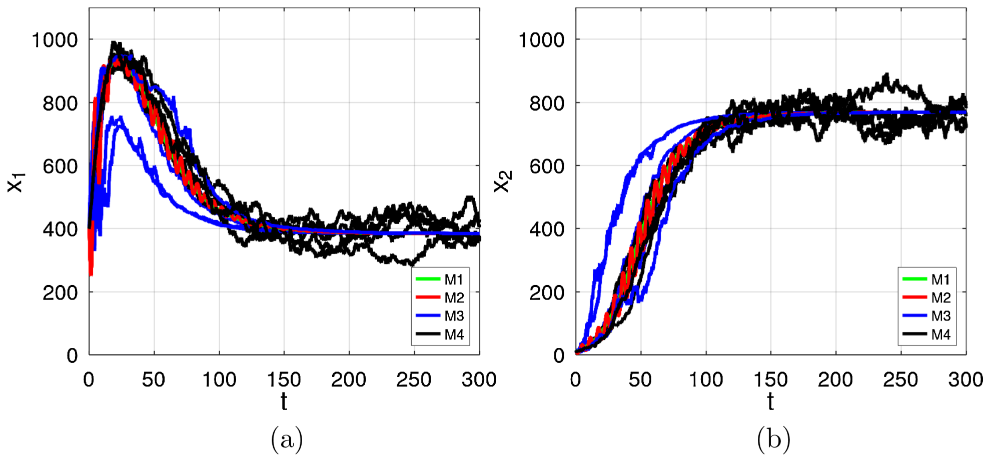

Figure 11. Comparison between numerical solutions with $N = 5$ simulations. (a) Company A_1; (b) Company A_2.

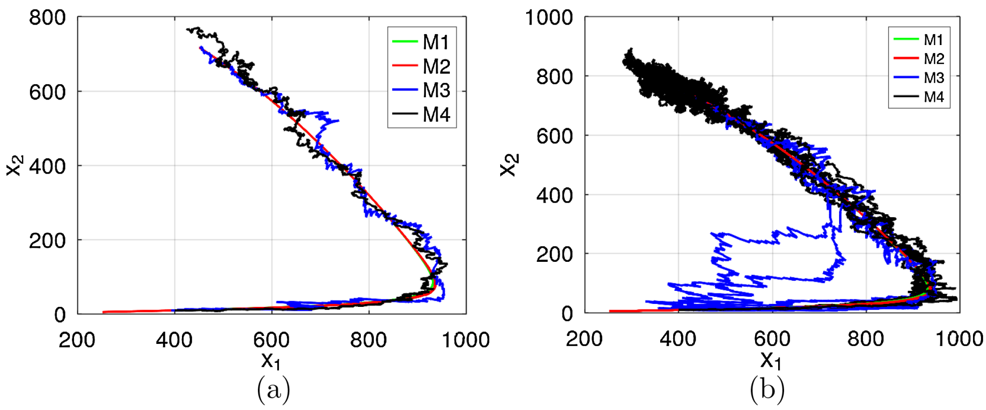

Figure 12. Trajetórias no plano de fase de x_1 e x_2. (a) Phase Plane 1; (b) Phase Plane 2.

$0 < \delta_1, \delta_2 < 1$ their coexistence occurs.

The simulations shown in Figures 7-9 and Figure 12(a) emphasize more the initial behavior of companies, before stabilization and Figures 10-12 provide a focus on stabilization by extending the end time of $t_f = 100$ to $t_f = 300$, maintaining the same spacing $h = 0.05$.

When considering $\delta_2 > \delta_1$, the competition strategies practiced by the company A_1 harm the company more A_2 and then $\lim_{t\to\infty} x_1(t) > \lim_{t\to\infty} x_2(t)$.

4.3. Case $0 < \delta_1 < 1$ and $\delta_2 > 1$

In this scenario (Figures 13-18), the new company A_2 that enters the niche market of the consolidated company A_1, cannot exert a greater impact on the corporation A_1, when compared to what company A_1 does in A_2, i.e. $\delta_2 > \delta_1$.

Moreover, the fact that $\delta_2 > 1$ makes the growth of company A_2 not be enough to outperform the competition with A_1, to the point of being able to sustain itself at some level. Consequently, the company A_2, who tried to enter the market, goes to extinction, that is, $\lim_{t\to\infty} x_2(t) \cong 0$, and therefore, $\lim_{t\to\infty} x_1(t) \cong 1000$, in any of the models considered. The values used were $\delta_1 = 0.8$ and $\delta_2 = 1.1$.

4.4. Case $\delta_1 > 1$ and $0 < \delta_2 < 1$

For this situation, we choose $\delta_1 = 1.1$ and $\delta_2 = 0.2$. In this case, the company

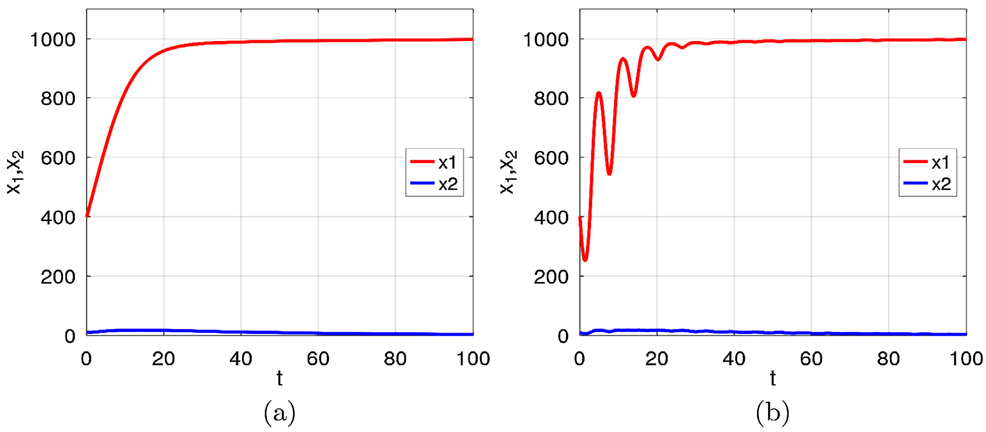

Figure 13. Numerical solutions of models 1 and 2. (a) Model 1; (b) Model 2.

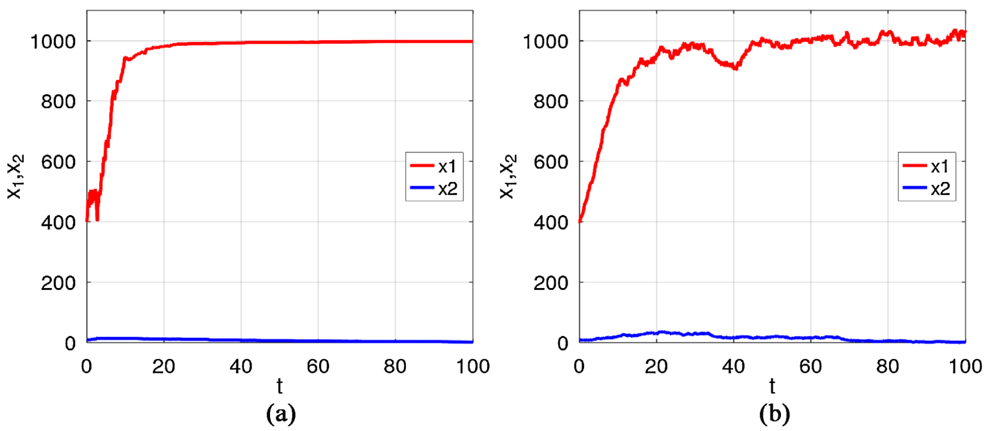

Figure 14. Time evolution by models 3 and 4. (a) Model 3; (b) Model 4.

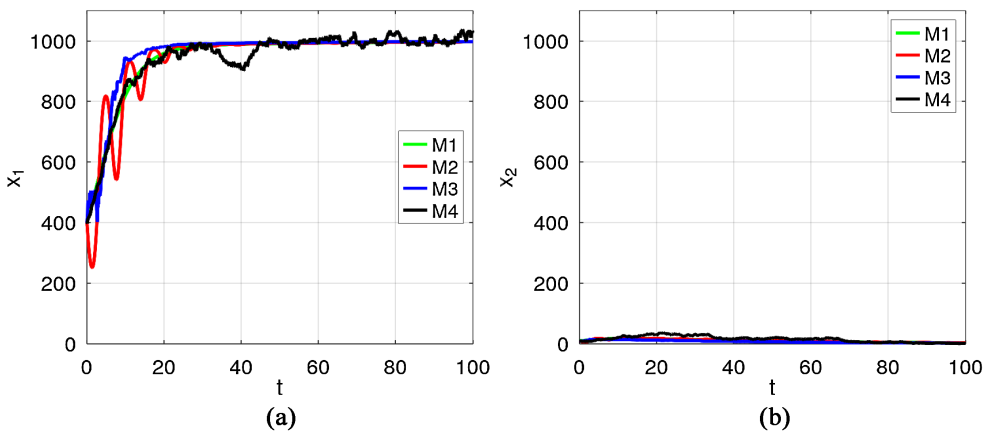

Figure 15. Comparison between numerical solutions of each company. (a) Company A_1; (b) Company A_2.

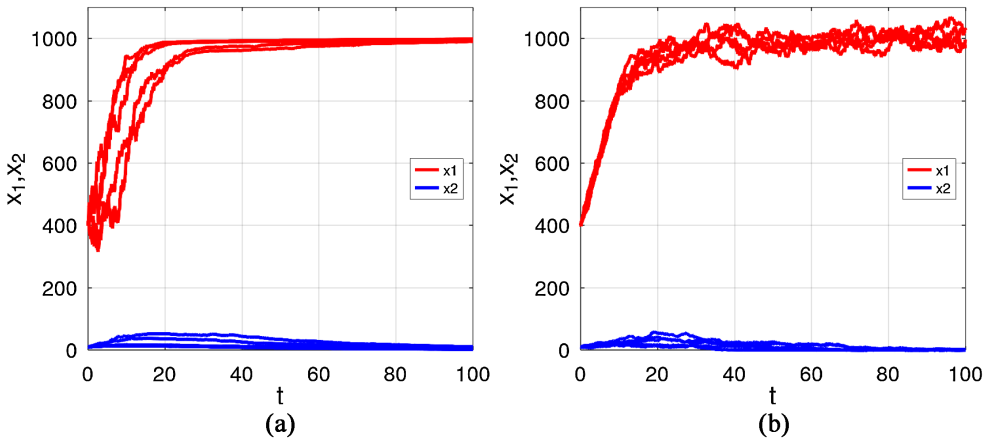

Figure 16. Models 3 and 4 with $N = 5$ simulations. (a) Model 3; (b) Model 4.

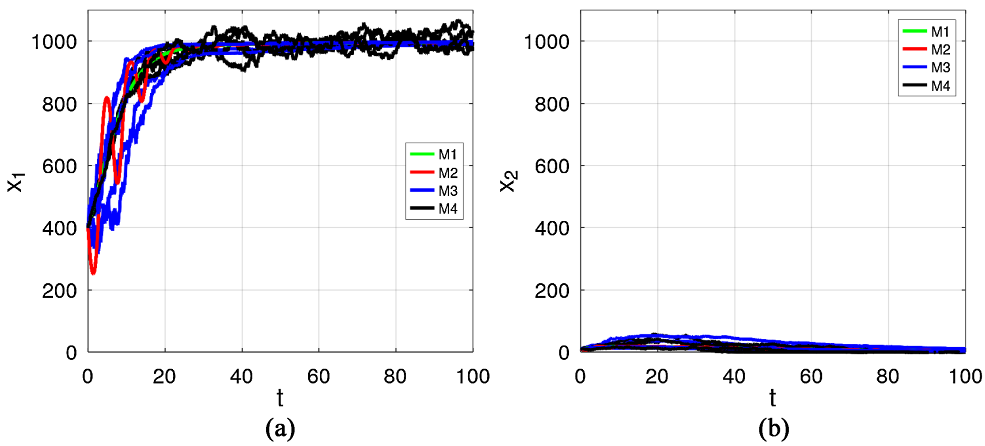

Figure 17. Comparison between numerical solutions with $N = 5$ simulations. (a) Company A_1; (b) Company A_2.

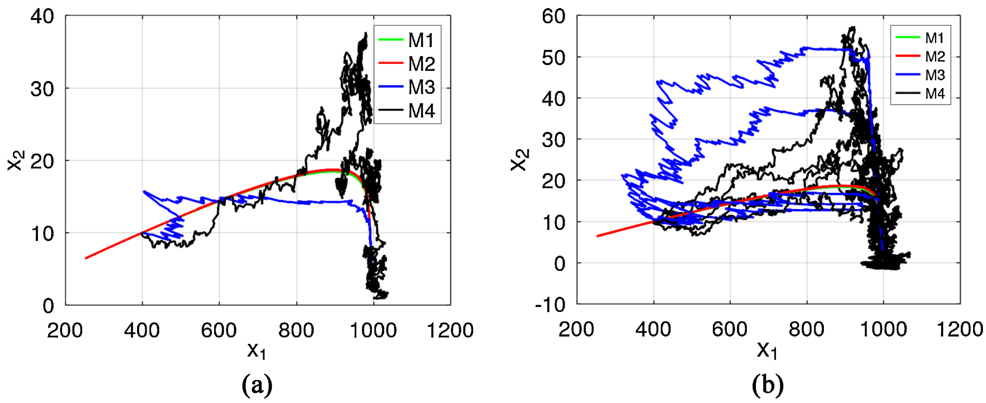

Figure 18. Trajectories in the phase plane. (a) Phase Plane 1; (b) Phase Plane 2.

A_2 new to the market affects the company's growth much more A_1, than the company A_1 (consolidated in the market) harms the new corporation A_2. At the beginning of temporal evolution, A_1 grows while maintaining its dominance in the market, reaches almost the level of total dominance, close to 1000, and then begins its decline. At the same time, the company A_2 always maintains its growth until it reaches the level of approximately 1000 and can therefore impact A_1, to the point of extinguishing it, Figures 19-24.

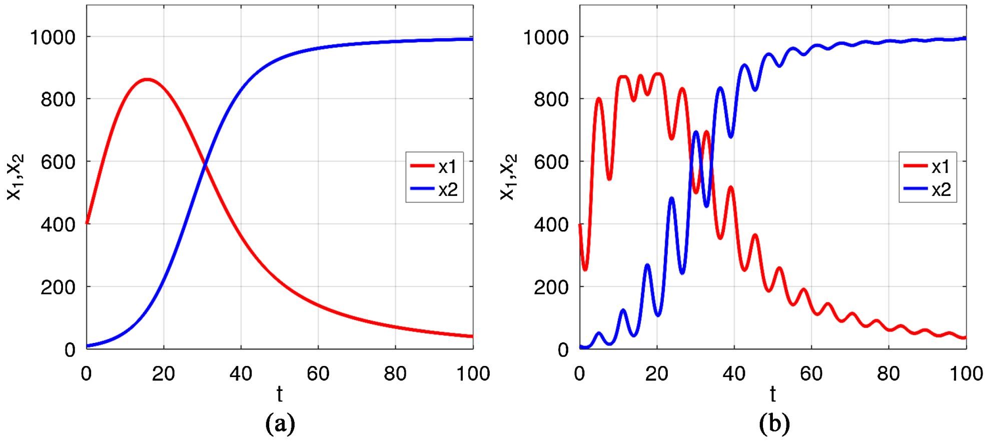

Figure 19. Numerical solutions of models 1 and 2. (a) Model 1; (b) Model 2.

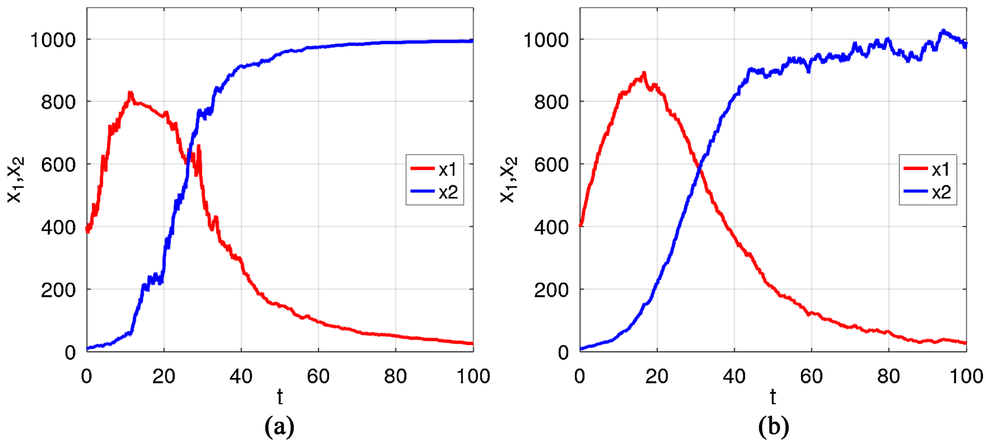

Figure 20. Time evolution by models 3 and 4. (a) Model 3; (b) Model 4.

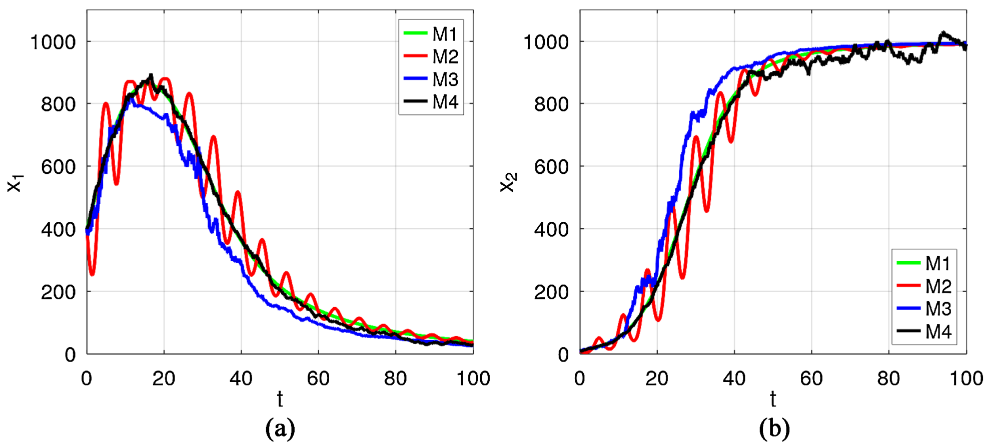

Figure 21. Comparison between numerical solutions of each company. (a) Company A_1; (b) Company A_2.

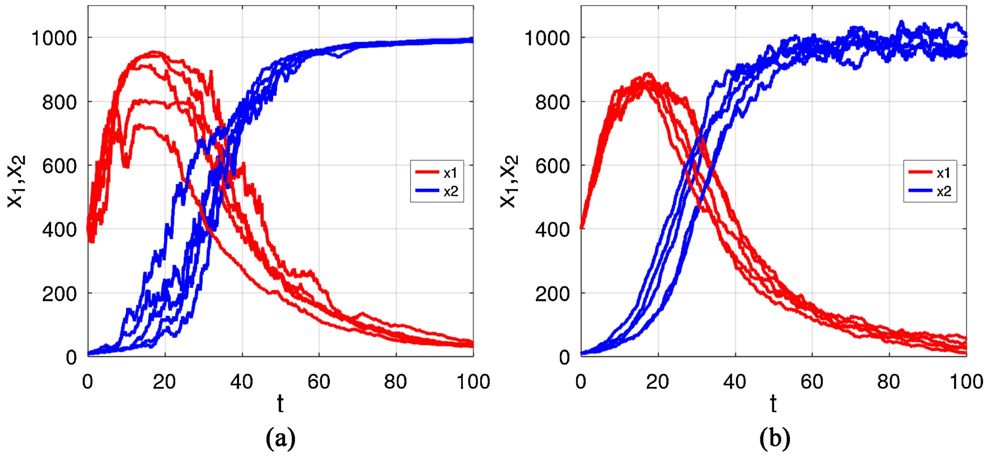

Figure 22. Models 3 and 4 with $N = 5$ simulations. (a) Model 3; (b) Model 4.

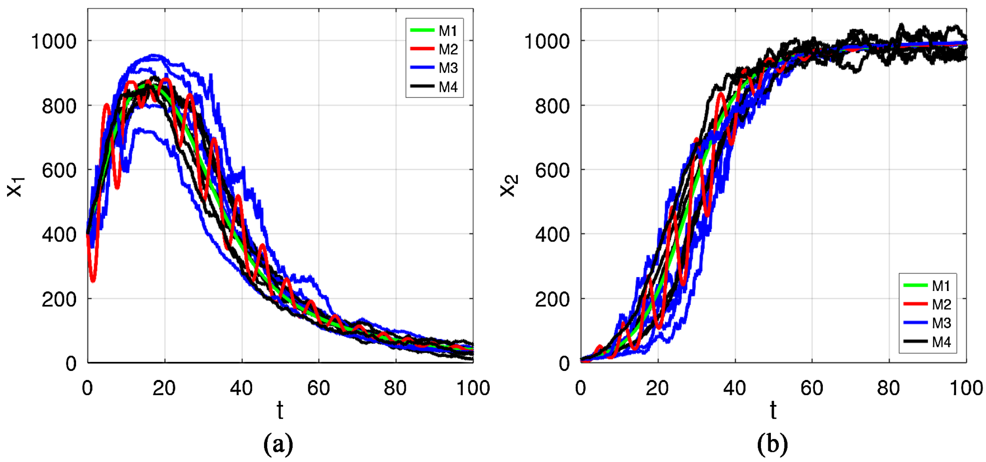

Figure 23. Comparison between numerical solutions with $N = 5$ simulations. (a) Company A_1; (b) Company A_2.

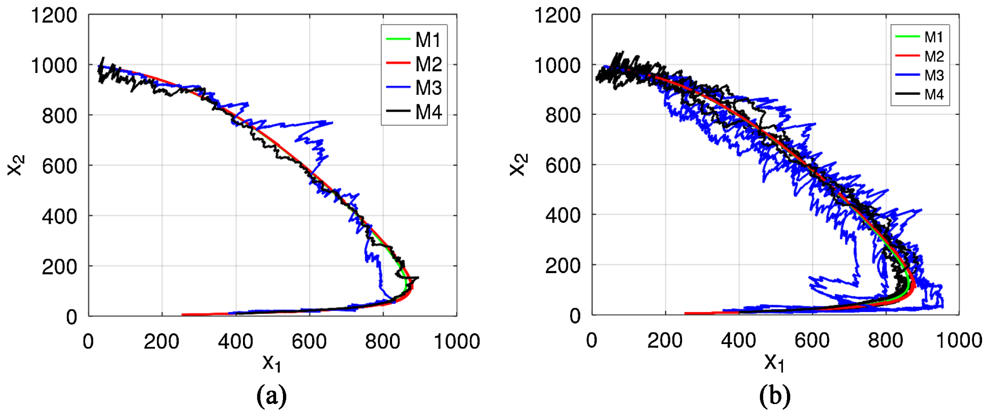

Figure 24. Trajectories in the phase plane. (a) Phase Plane 1; (b) Phase Plane 2.

A case similar to this inspired the beginning of this research. Two Brazilian banks competed in the niche of *Gamers*, providing credit cards with exclusive features for this environment. The A_1 bank, which dominated the niche of customers who are athletes or fans of electronic games, had a product that was a classic credit card, with just a few exclusive benefits. The A_2 bank, newly arrived in the *Gamers* sector, created a new card inserted in the universe of electronic game fans. The success of this new credit card was such that it profoundly affected the bank's product A_1, to the point of dominating the niche of *Gamers* and extinguish the product of A_1 bank. The A_1 bank tried to renew his credit card, for the world of electronic game players, but was unable to reinsert himself and ended his activities in this segment.

It is interesting to note that, even if market fluctuations are added, both periodic and random, these are not enough to change the asymptotic behavior of future dominance of the new company (A_2 bank) and annihilation of the product of A_1 bank. For A_1 bank to avoid the decline of its credit card (or other product) to extinction, business measures need to be taken and these must be sufficient to reduce the value of δ_1, so that it is between zero and one (as in the case of Subsection 4.2).

5. Conclusion

The competitive environment in each business niche has important characteristics that determine the decision-making process within a company. Knowing and trying to predict the evolutionary process can help in taking measures that can decide the direction of the business.

Thus, in this work, mathematical models via differential equations were developed, inspired by models of competition between species, to develop scenarios of competition between two companies in the same business niche.

A qualitative study was carried out to establish the possible resulting scenarios, after a sufficiently large amount of time, namely, coexistence of the companies or the survival of only one of them. The models were then solved computationally, via Euler and Euler-Maruyama methods, written in their own codes in the language *R*.

With the simulations carried out, even the models that involve periodic oscillations, via trigonometric functions, as well as the models that involve: 1) white noise in business growth and 2) modeling via stochastic differential equations, had tendencies to follow the modeling of Subsection 2.5.

The only scenario that allows the coexistence of two companies competing with each other was obtained when $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$. And yet, even with small market disruptions, represented by the models (2), (3) and (4), the asymptotic behavior of coexistence was maintained.

In future works, the aims are to expand and add other characteristics inherent to the complexity of business dynamics, so that mathematical models can produce more detailed and realistic scenarios.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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