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Biconservative Submanifolds in 5-Dimensional Pseudo-Euclidean Space

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Abstract

This paper focuses on the geometric rigidity of 3-dimensional proper biconservative submanifolds M_r^3 with a parallel normalized mean curvature vector field in the 5-dimensional pseudo-Euclidean space E_s^5. Grounded in the theories of harmonic maps, biharmonic maps, and stress-energy tensors, we employ fundamental tools from pseudo-Riemannian geometry including the Gauss equation, Weingarten formula, and Codazzi equation to derive the equivalent conditions characterizing biconservative submanifolds. We further establish that if the shape operator of such a submanifold is diagonalizable and admits at most two distinct principal curvatures in the direction of the mean curvature vector field H, its scalar curvature takes the form of a fractional-power polynomial in the mean curvature f with non-vanishing coefficients. In particular, any such submanifold with constant scalar curvature necessarily possesses constant mean curvature, revealing a striking rigidity property.

Keywords

Pseudo-Euclidean Space Form, Principal Curvatures, Proper Biconservative, Scalar Curvature

1. Introduction

Let M_r^m and N_q^{m+p} be pseudo-Riemannian manifolds with indexs r and q $(q \ge r \ge 0)$, respectively, and we consider a smooth map $\phi : M_r^m \to N_q^{m+p}$. The energy functional is defined by

$$E(\phi) = \int_{M_r^m} e(\phi) v_g,$$

where $e(\phi) = \frac{1}{2}|d(\phi)|^2$ is called the energy density of ϕ. Critical points of

$E(\phi)$ are called harmonic maps. The theory of harmonic maps has been applied to various fields in differential geometry, we refer to [1] [2] for a review.

The study of biharmonic submanifolds began in the mid-1980s, proposed by Chen in his research on finite type submanifolds in Euclidean and pseudo-Euclidean spaces [3]. Meanwhile, In [4] and [5], Jiang introduced the concept of k-harmonic mapping proposed by Eells and Sampson in [6] to study the double harmonic isometric immersion between Riemannian manifolds. Harmonic mappings are characterized by the vanishing of the tension field as the criterion, corresponding to the critical points of the energy functional, and describe the smoothest mapping relationship between manifolds [6], while biharmonic mappings are defined by the vanishing of the biharmonic field and are the critical points of the bienergy functional [7].

In order to understand the geometric characteristics of biharmonic systems, some geometers have begun to focus on studying doubly conservative submanifolds [8]-[11]. For example, the general notion of biconservative submanifolds was introduced in [8]. Aslo, the complete classification of biconservative hypersurfaces in Euclidean spaces with three distinct principal curvatures is obtained by the second named author in [10].

The stress-energy tensor was intitated by G. Y. Jiang in [12] and afterwards developed by E. Loubeau, S. Montaldo and C. Oniciuc in [13], defining the stress-energy tensor S_2 with

$$div\, S_2 = -\langle \tau_2(\varphi), d\varphi \rangle$$

Moreover, Jiang [12] demonstrated that φ is biharmonic if and only if it satisfies the Euler-Lagrange equation related to the dual energy functional, *i.e.*, $\tau_2(\varphi) = 0$. where $\tau_2(\varphi)$ is the bitension field of φ defined by

$$\tau_2(\varphi) = \Delta \tau(\varphi) - tr\tilde{R}(d\varphi, \tau(\varphi))d\varphi,$$

where Δ is the Rough-Laplacian. If the condition $\langle \tau_2(\varphi), d\varphi \rangle = 0$ is satisfied, then φ is called a biconservative mapping. Pseudo-Riemannian manifolds are generalizations of Riemannian manifolds. The condition for a Riemannian submanifold to be biconservative can be used to obtain the condition for a pseudo-Riemannian submanifold to be bi-conservative as follows (cf [8] [14])

$$traceA_{D(.)H(.)} + \frac{n}{4} grad \langle H, H \rangle = 0,$$

where D is the normal connection, A_ξ is the shape operator with respect to the normal vector field ξ and H is the mean curvature vector filed of M_r^m. In [11], authors studied geometrical properties of PNMCV surfaces of E^4 and proved that a biharmonic PNMCV surface in E^4 is minimal. Motivated by above paragraphs, in this paper, we will study 3-dimensional biconservative of E_s^5 with parallel normal mean curvture vector (PNMCV). we prove

Theorem 1.1. *If a proper biconservative submanifold* M_r^3 $(\|grad\, f \neq 0\|)$ *in* E_s^5 *has a diagonalizable shape operator and at most two distinct principal* *curvatures in the direction of H, then its scalar curvature is a power-law polynomial in the mean curvature f with non-zero coefficients. Furthermore, the expression for the scalar curvature is derived from the Gauss equation and the results concerning the shape operator.*

Theorem 1.2. *Let M_r^3 be a biconservative submanifold with constant scalar curvature in the pseudo-Euclidean space E_s^5, possessing a parallel normaliazed mean curvature vector field. If, in the direction of f, it has at most two distinct principal curvatures and its shape operator is diagonalizable, then M_r^3 necessarily has constant mean curvature.*

Remark 1.3. It is easy to see that a PNMC submanifold is CMC if and only if it is PMC. As a corollary of Theorem 1.3, let M_r^3 be a biconservative pnmc submanifold with constant scalar curvature in E_s^5. Then M_r^3 is PMC provided that it has diagonalizable shape operator with two distinct principal curvatures in the direction of f.

Remark 1.4. Let M_r^3 be a biconservative pnmc submanifold with constant scalar curvature in the pseudo-Euclidean space from E_s^5. Assume that M_r^3 has two distinct principal curvatures in the direction of f. As an immediate consequence of Theorem 1.3, we know that M_r^3 has constant mean curvature.

2. Preliminaries

Let E_s^n denote the pseudo-Euclidean n-space with the metric tensor $\tilde{g}$ given by

$$\tilde{g} = \langle \ldots \rangle = -\sum_{i=1}^{s} dx_i \otimes dx_i + \sum_{j=s+1}^{n} dx_j \otimes dx_j, \tag{1}$$

Let $\varphi : M_r^3 \to E_s^5$ be an isometric immersion of an 3-dimensional pseudo-Riemannian manifold M_r^3 into a pseudo-Euclidean 5-space. Denote the Levi-Civita connections of M_r^3 and E_s^3 by ∇ and $\tilde{\nabla}$, respectively. Then the Gauss and Weingarten formulae are given by (cf [15] [16])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2}$$

and

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi, \tag{3}$$

respectively, for any vectors X, Y tangent to M_r^3 and ξ normal to M_r^3, where h and A_ξ are the second fundamental form and the shape of M_r^3 along the normal direction ξ, respectively and $\nabla^\perp$ is the normal connection, It is well known that h and A_ξ are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle. \tag{4}$$

If R and $\tilde{R}$ stand for the curvature tensor of M_r^3 and E_s^3 respectively, then, the Codazzi equation $\left(\tilde{R}(X,Y)Z\right)^\perp = 0$ and the Gauss equation $\left(\tilde{R}(X,Y)Z\right)^\top = 0$ become

$$\left(\tilde{\nabla}_X h\right)(Y,Z) = \left(\tilde{\nabla}_X h\right)(Y,Z). \tag{5}$$

where $\tilde{\nabla} h$ is defined by

$$\left(\tilde{\nabla}_X h\right)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (6)$$

$$R(X,Y)Z,W) = R(X,Y,Z,W) = \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle. \quad (7)$$

where

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \quad (8)$$

Let e_1, e_2, e_3, e_4, e_5 be a pseudo-Euclidean orthonormal field on E_s^3 such that e_1, e_2, e_3 are tangent to M_r^3 and e_4, e_5 are normal to M_r^3, and we denote the connection forms corresponding to this fram field by ω_{ij}. Then, we have

$$\nabla_{e_i} e_j = \sum_{k=1}^{5} \varepsilon_k \omega_{ij}(e_k), 1 \le i, j \le 5, \omega_{ii} = 0, \omega_{ij} + \omega_{ji} = 0, (i \ne j). \quad (9)$$

The mean curvature vector f of M_r^3 is defined by

$$H = \frac{1}{3}\mathrm{tr}h = \frac{1}{3}\sum_{i=1}^{3} \varepsilon_i h(e_i, e_i) = \frac{1}{3}\sum_{4}^{5} \varepsilon_4 \mathrm{tr} A_4 e_4, \quad (10)$$

where $A_4 = A_{e_4}$. The mean curvature f of M_r^3 in E_s^5 is expressed as $f = \langle H, H\rangle^{1/2}$.

At a point $p \in M$, a 2-dimensional linear subspae π of the tangent space $T_p M$ is called a plane section. For a given basis ν, ω of the palne section π, we define a real number by

$$Q(\nu,\omega) = \langle \nu,\nu\rangle\langle \omega,\omega\rangle - \langle \nu,\omega\rangle^2. \quad (11)$$

The plane section π is called nondegenerate if and only if $Q(\nu,\omega) \ne 0$. $Q(\nu,\omega)$ is positive when $g|_\pi$ is definite, and is negative when $g|_\pi$ is indefinte. The absolute value $\|Q(\mu,\nu)\|$ is the square of the area of the parallelogram with sides μ and ν.

For a nondegenerate plane setion π at p, the number

$$K(\mu,\nu) = \frac{\langle R(\mu,\nu)\nu, \mu\rangle}{Q(\mu,\nu)}. \quad (12)$$

is independent of the choice of basis ν, ω for π, which is called the setional curvature $K(\pi)$ of π.

3. Some Key Lemmas

According to [17]-[19]:

Lemma 2.1 Let $\varphi: M_r^3 \to E_s^5$ be an isometric immersion of an 3-dimensional pseudo-Riemannian manifold M_r^3 into a pseudo-Euclidean space. φ is biconservative if and only if the equation

$$m\nabla\langle H, H\rangle + 4\sum_{i=1}^{m} \varepsilon_i A_{\nabla_{e_i}^{\perp} H}(e_i) = 0. \quad (13)$$

is satisfied, where m is the imension of M. By Lemma 2.1, we can obtain

Lemma 2.2 (cf [18]) Let M_r^3 be a submanifolds with parallel normalized mean curvature vector field in E_s^5. Then M_r^3 is biconservative if and only if the equation holds:

$$A_{e_4}\left(\nabla f\right)=\frac{-3\varepsilon_4 f}{2}\left(\nabla f\right), \tag{14}$$

where

$$\nabla f=\sum_{i}^{3}\varepsilon_i e_i\left(f\right)e_i. \tag{15}$$

Proof. Essentially this lemma is a special case of Lemma 2.2 in [18] We can choose a pesudo-Riemannian orthonormal frame field e_1,e_2,e_3,e_4,e_5, such that e_1 is parallel to ∇f, e_2,e_3 are tangent to M_r^3, $e_4=\frac{H}{f}$, e_5 are normal to M_r^3, then

$$H=fe_4. \tag{16}$$

Note that $\nabla_{e_i}^{\perp}e_4=\nabla_{e_i}^{\perp}\left(\frac{H}{f}\right)=0$, $i=1,2,3$, then it follows form (16) we have

$$\nabla_{e_i}^{\perp}H=\nabla_{e_i}^{\perp}\left(fe_4\right)=e_i\left(f\right)e_4, \tag{17}$$

which together with (15) and (16) we have

$$\sum_{i=1}^{3}\varepsilon_i A_{\nabla_{e_i}^{\perp}H}\left(e_i\right)=A_4\left(\nabla f\right). \tag{18}$$

using (16), we can obtain

$$\nabla\left\langle H,H\right\rangle=\nabla\left\langle fe_4,fe_4\right\rangle=\varepsilon_4\nabla f^2=2\varepsilon_4 f\nabla f \tag{19}$$

Combining with (13), (18) and (19) we can obtain $A_{e_4}\left(\nabla f\right)=\frac{-3\varepsilon_4 f}{2}\left(\nabla f\right)$.

According to [17]-[20]:

Lemma 2.3 Assum that φ has parallel normalized mean curvature vector e_4. In this case, the Ricci equation $\left(\tilde{R}\left(X,Y\right)\xi\right)^{\top}=0$ yields that all the shape operators of φ can be diagonalized simulataneously (see [21]). Therefore, by abusung the terminology, we are going to call X as principal direction of φ, if $A_{e_4}X=\lambda X$, where the smooth function λ is going to be called as the corrrs-ponding principal curvature. Note that there exist an orthonormal fram field e_1,e_2,e_3,e_4,e_5 such that

$$A_{e_4}=diag\left(\lambda_1,\lambda_2,\lambda_3\right),A_{e_5}=diag\left(\mu_1,\mu_2,\mu_3\right). \tag{20}$$

for some smooth fuctions λ_i, μ_i satisfying $\lambda_1+\lambda_2+\lambda_3=3\varepsilon_4 f$ and $\mu_1+\mu_2+\mu_3=0$. We are going to call a biconservative PNMCV immersion as proper if $\left\|\nabla f\right\|$ does not vanish. Assum that φ is proper biconservative PNMCV immersion. By calculating $\nabla_{e_i}^{\perp}\left\langle e_4,e_4\right\rangle=0$ and $\nabla_{e_i}^{\perp}\left\langle e_4,e_5\right\rangle=0$, where $i=1,2,3$. we obtain

$$\nabla_{e_i}^{\perp}e_4=\nabla_{e_i}^{\perp}e_5=0 \tag{21}$$

where e_5 is a unit normal vector field orthogonal to e_4. If e_1 is chosen to be proportional to $\|\nabla f\|$, then (14) implies

$$e_1(f) \neq 0, e_2(f) = e_3(f) = 0. \tag{22}$$

and $\lambda_1 = \frac{-3\varepsilon_4 f}{2}$.

Lemma 2.4 Let $\varphi: M_r^3 \to E_s^5$ be an isometric immersion with two distinct principal curvatures, if φ is proper biconservative PNMCV then there exists an orthonormal frame field e_1, e_2, e_3, e_4, e_5 such that the shape operators A_{e_4} has the form

$$A_{e_4} = \begin{pmatrix} \frac{-3\varepsilon_4 f}{2} & & \\ & \frac{9\varepsilon_4 f}{4} & \\ & & \frac{9\varepsilon_4 f}{4} \end{pmatrix} \tag{23}$$

and A_{e_5} has the form

$$\text{(I)}\quad A_{e_5} = 0, \text{ or} \tag{24}$$

$$\text{(II)}\quad A_{e_5} = \begin{pmatrix} 0 & & \\ & g_2 f^{\frac{3}{5}} & \\ & & -g_3 f^{\frac{3}{5}} \end{pmatrix} \tag{25}$$

where g_2 and g_3 are nonzero constants with $g_2 + g_3 = 0$, or.

$$\text{(III)}\quad A_{e_5} = \begin{pmatrix} c_1 f^{\frac{9}{5}} & & \\ & -\frac{c_1}{2} f^{\frac{9}{5}} + g_2 f^{\frac{3}{5}} & \\ & & -\frac{c_1}{2} f^{\frac{9}{5}} - g_3 f^{\frac{3}{5}} \end{pmatrix} \tag{26}$$

for c_1 and g_2, g_3 are nonzero constants with $g_2 + g_3 = 0$.

Proof. We have from (10) and (16) that

$$f = \frac{\varepsilon_4}{3} traceA_{e_4}, traeA_{e_5} = 0. \tag{27}$$

When M_r^3 has the same principal curvatures in the direction of H, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{-3\varepsilon_4 f}{2}$, from (27), it follows that

$$(2+3) f = 0, \tag{28}$$

which shows $f = 0$, it is contradition. There for we are going to consider the case $\lambda_1 \neq \lambda_2 = \lambda_3$. from (20) and (22) we have λ_2, λ_3 satisfy

$$\lambda_2 = \lambda_3 = \frac{9\varepsilon_4 f}{4}. \tag{29}$$

Now, we start to derive the explicit expressions of the shape operator A_{e_5} of

M_r^3 Combing with (4) and (20) yields, for any $i,j=1,2,3$

$$h(e_i,e_j)=\sum_{\beta=4}^{5}\varepsilon_\beta\left\langle h(e_i,e_j),e_\beta\right\rangle e_\beta=\varepsilon_4\left\langle e_i,e_j\right\rangle\lambda_i e_4+\varepsilon_5\left\langle e_i,e_j\right\rangle\mu_i e_5 \quad (30)$$

which means that

$$\begin{aligned}&h(e_1,e_2)=0,h(e_1,e_3)=0,h(e_1,e_1)=\varepsilon_4\varepsilon_1\lambda_1 e_4+\varepsilon_5\varepsilon_1\mu_1 e_5,\\&h(e_2,e_2)=\varepsilon_4\varepsilon_2\lambda_2 e_4+\varepsilon_5\varepsilon_1\mu_2 e_5,\\&h(e_3,e_3)=\varepsilon_4\varepsilon_1\lambda_3 e_4+\varepsilon_5\varepsilon_1\mu_3 e_5.\end{aligned}$$

Calculating $(\tilde{\nabla}e_1 h)(e_A,e_1)=(\tilde{\nabla}e_A h)(e_1,e_1)$, for $A=2,3$. Uising (6), (9), (22), (29), (30), we get

$$\varepsilon_5\varepsilon_1 e_A(\mu_1)e_5=\omega_{1A}(e_1)\varepsilon_4(\lambda_1-\lambda_A)e_4+\varepsilon_5(\mu_1-\mu_A)e_5. \quad (31)$$

from (22) and (29) we have

$$\varepsilon_5\varepsilon_1 e_A(\mu_1)e_5=\omega_{1A}(e_1)\varepsilon_4\left(-\frac{15\varepsilon_4 f}{4}\right)e_4+\varepsilon_5(\mu_1-\mu_A)e_5. \quad (32)$$

Furthermore, we have

$$\varepsilon_5\left(\varepsilon_1 e_A(\mu_1)-\omega_{1A}(e_1)(\mu_1-\mu_A)\right)e_5=-\frac{15\varepsilon_4}{4}\omega_{1A}(e_1)fe_4. \quad (33)$$

whih implies that

$$\omega_{1A}(e_1)=0,e_A(\mu_1)=0,A=2,3 \quad (34)$$

Similary, calculating $(\tilde{\nabla}e_A h)(e_1,e_A)=(\tilde{\nabla}e_A h)(e_A,e_A)$, from (6), (9), (22), (29), (30), we find

$$e_1(\mu_A)=\varepsilon_A\omega_{1A}(e_A)(\mu_1-\mu_A) \quad (35)$$

and

$$e_1(\lambda_A)=\varepsilon_A\omega_{1A}(e_A)(\lambda_1-\lambda_A)=\varepsilon_A\omega_{1A}(e_A)\left(-\frac{15\varepsilon_4}{4}\right). \quad (36)$$

This together with (29) and (36) deduces to

$$\omega_{1A}(e_A)=-\frac{3\varepsilon_A}{5}\frac{e_1(f)}{f}. \quad (37)$$

Substituting (37) into (35), we have

$$e_1(\mu_A)=\frac{3}{5}\frac{e_1(f)}{f}(\mu_A-\mu_1). \quad (38)$$

Furthermore we have

$$e_1\left(\sum_{A=2}^{3}\mu_A\right)=\frac{3}{5}\frac{e_1(f)}{f}\left(\sum_{A=2}^{3}\mu_A-2\mu_1\right) \quad (39)$$

using (21), (16) and (39), we get

$$e_1(\mu_1)=\frac{9}{5}\frac{e_1(f)}{f}\mu_1. \quad (40)$$

case (1) when $\mu_1 = 0$, then (38) becomes

$$e_1(\mu_A) = \frac{3}{5}\frac{e_1(f)}{f}\mu_A, A = 2,3. \tag{41}$$

if $\mu_2 = \mu_3 = 0$, then A_{e_5} has the form (I).

If μ_2, μ_3 are not equal to 0, Then it follows from (36) that

$$\frac{e_1(\mu_A)}{\mu_A} = \frac{3}{5}\frac{e_1(f)}{f}, A = 2,3. \tag{42}$$

Integrating (42), we have

$$\mu_A = g_A f^{\frac{3}{5}}, A = 2,3. \tag{43}$$

where g_A are nonzero constents. Substituting (43) into (20) we get

$$g_2 + g_3 = 0, A = 2,3. \tag{44}$$

Thus A_{e_5} takes the form

$$A_{e_5} = \begin{pmatrix} 0 & & \\ & g_2 f^{\frac{3}{5}} & \\ & & -g_3 f^{\frac{3}{5}} \end{pmatrix}$$

Case (2) If $\mu_1 \neq 0$ for $A = 2,3$, we have form (36) that

$$\mu_1 = c_1 f^{\frac{3}{5}}. \tag{45}$$

where c_1 is a nonzero constant, substituting (45) into (38) we have

$$e_1(\mu_A) = \frac{3}{5}\frac{e_1(f)}{f}\left(\mu_A - c_1 f^{\frac{3}{5}}\right). \tag{46}$$

By integrating (46). It fllows that

$$\mu_A = -\frac{c_1}{2} f^{\frac{9}{5}} + g_A f^{\frac{3}{5}}. \tag{47}$$

where $e_1(g_A) = 0$ and g_A are smooth function. Also $g_2 + g_3 = 0$. Thus

$$A_{e_5} = \begin{pmatrix} c_1 f^{\frac{9}{5}} & & \\ & -\frac{c_1}{2} f^{\frac{9}{5}} + g_2 f^{\frac{3}{5}} & \\ & & -\frac{c_1}{2} f^{\frac{9}{5}} - g_3 f^{\frac{3}{5}} \end{pmatrix} \tag{48}$$

In the following, we will prove that g_A ($A = 2,3$) are constants.

If g_2, g_3 are not constants, then we know from (47) that $e_A(g_2)$ and $e_A(g_3)$ are not equal to 0 for $A = 2,3$.

Calculating the equation $(\tilde{\nabla} e_1 h)(e_2, e_3) = (\tilde{\nabla} e_2 h)(e_1, e_3)$ and $(\tilde{\nabla} e_1 h)(e_3, e_2) = (\tilde{\nabla} e_2 h)(e_1, e_2)$, combing (6), (9), we have

$$\omega_{23}(e_1)\left(\varepsilon_2 h(e_2,e_2)-\varepsilon_3 h(e_3,e_3)\right)=\omega_{31}(e_2)\left(\varepsilon_3 h(e_3,e_3)-\varepsilon_1 h(e_1,e_1)\right).$$

$$\omega_{32}(e_1)\left(\varepsilon_3 h(e_3,e_3)-\varepsilon_2 h(e_2,e_2)\right)=\omega_{21}(e_3)\left(\varepsilon_2 h(e_2,e_2)-\varepsilon_1 h(e_1,e_1)\right).$$

Combing (30) we obtain

$$\begin{aligned}&\varepsilon_4\omega_{23}(e_1)(\lambda_2-\lambda_3)e_4+\varepsilon_5\omega_{23}(e_1)(\mu_2-\mu_3)e_5\\&=\varepsilon_4\omega_{31}(e_2)(\lambda_3-\lambda_1)e_4+\varepsilon_5\omega_{31}(e_1)(\mu_3-\mu_1)e_5.\\&\varepsilon_4\omega_{32}(e_1)(\lambda_3-\lambda_2)e_4+\varepsilon_5\omega_{32}(e_1)(\mu_3-\mu_2)e_5\\&=\varepsilon_4\omega_{21}(e_3)(\lambda_2-\lambda_1)e_4+\varepsilon_5\omega_{21}(e_1)(\mu_2-\mu_1)e_5.\end{aligned}\tag{49}$$

Note that $\lambda_2=\lambda_3$, then it follows from (46) that

$$\omega_{13}(e_2)=0,\omega_{12}(e_3)=0,\omega_{23}(e_1)=0.\tag{50}$$

A short calulation from (9) shows that

$$\begin{aligned}\nabla_{e_1}\nabla_{e_2}e_1&=e_1\left(\sum_{k=1}^{3}\varepsilon_k\omega_{1k}(e_2)\right)e_k+\sum_{k=1}^{3}\varepsilon_k\omega_{1k}(e_2)\sum_{t=1}^{3}\varepsilon_t\omega_{kt}(e_1)e_t\\\nabla_{e_1}\nabla_{e_3}e_1&=e_1\left(\sum_{k=1}^{3}\varepsilon_k\omega_{1k}(e_3)\right)e_k+\sum_{k=1}^{3}\varepsilon_k\omega_{1k}(e_3)\sum_{t=1}^{3}\varepsilon_t\omega_{kt}(e_1)e_t\\\nabla_{e_2}\nabla_{e_1}e_1&=e_2\left(\sum_{k=1}^{3}\varepsilon_k\omega_{1k}(e_1)\right)e_k+\sum_{k=1}^{3}\varepsilon_k\omega_{1k}(e_1)\sum_{t=1}^{3}\varepsilon_t\omega_{kt}(e_2)e_t\\\nabla_{e_3}\nabla_{e_1}e_1&=e_3\left(\sum_{k=1}^{3}\varepsilon_k\omega_{1k}(e_1)\right)e_k+\sum_{k=1}^{3}\varepsilon_k\omega_{1k}(e_1)\sum_{t=1}^{3}\varepsilon_t\omega_{kt}(e_3)e_t\\\nabla_{[e_1,e_2]}e_1&=\sum_{k=1}^{3}\varepsilon_k\left(\omega_{2k}(e_1)-\omega_{1k}(e_2)\right)\sum_{t=1}^{3}\varepsilon_t\omega_{1t}(e_k)e_t\\\nabla_{[e_1,e_3]}e_1&=\sum_{k=1}^{3}\varepsilon_k\left(\omega_{3k}(e_1)-\omega_{1k}(e_3)\right)\sum_{t=1}^{3}\varepsilon_t\omega_{1t}(e_k)e_t\end{aligned}\tag{51}$$

Then it follows from (8), (9), (34), (51)

$$\begin{aligned}R(e_1,e_2)e_1,e_2)&=e_1\left(\omega_{12}(e_2)\right)-e_2\left(\omega_{12}(e_1)\right)-\sum_{k=1}^{3}\varepsilon_k\left(\omega_{2k}(e_1)-\omega_{1k}(e_2)\right)\omega_{12}(e_k)\\&\quad+\sum_{k=1}^{3}\varepsilon_k\left(\omega_{1k}(e_2)\omega_{k2}(e_1)-\omega_{1k}(e_1)\omega_{k2}(e_2)\right)\\&=e_1\left(\omega_{12}(e_2)\right)+\varepsilon_2\left(\omega_{12}(e_2)\right)^2\\R(e_1,e_3)e_1,e_3)&=e_1\left(\omega_{12}(e_3)\right)-e_3\left(\omega_{12}(e_1)\right)-\sum_{k=1}^{3}\varepsilon_k\left(\omega_{3k}(e_1)-\omega_{1k}(e_3)\right)\omega_{13}(e_k)\\&\quad+\sum_{k=1}^{3}\varepsilon_k\left(\omega_{1k}(e_3)\omega_{k2}(e_1)-\omega_{1k}(e_1)\omega_{k3}(e_3)\right)\\&=e_1\left(\omega_{13}(e_3)\right)+\varepsilon_3\left(\omega_{13}(e_3)\right)^2\end{aligned}\tag{52}$$

Also using Guauss equation we from (30) that

$$\begin{aligned}R(e_1,e_2)e_1,e_2)&=-\varepsilon_1\varepsilon_2\left(\varepsilon_4\lambda_1\lambda_2+\varepsilon_5\mu_1\mu_2\right)\\R(e_1,e_3)e_1,e_3)&=-\varepsilon_1\varepsilon_3\left(\varepsilon_4\lambda_1\lambda_3+\varepsilon_5\mu_1\mu_3\right)\end{aligned}\tag{53}$$

Those two facts shows that

$$\begin{aligned} e_1\left(\omega_{12}\left(e_2\right)\right)+\varepsilon_2\left(\omega_{12}\left(e_2\right)\right)^2 &= -\varepsilon_1\varepsilon_2\left(\varepsilon_4\lambda_1\lambda_2+\varepsilon_5\mu_1\mu_2\right) \\ &= -\varepsilon_1\varepsilon_2\left(-\frac{27\varepsilon_4 f^2}{8}+\varepsilon_5\left(-\frac{c_1^2}{2}f^{\frac{12}{5}}+c_1g_2f^{\frac{6}{5}}\right)\right) \\ e_1\left(\omega_{13}\left(e_3\right)\right)+\varepsilon_3\left(\omega_{13}\left(e_3\right)\right)^2 &= -\varepsilon_1\varepsilon_3\left(\varepsilon_4\lambda_1\lambda_3+\varepsilon_5\mu_1\mu_3\right) \\ &= -\varepsilon_1\varepsilon_2\left(-\frac{27\varepsilon_4 f^2}{8}+\varepsilon_5\left(-\frac{c_1^2}{2}f^{\frac{12}{5}}+c_1g_3f^{\frac{6}{5}}\right)\right) \end{aligned} \tag{54}$$

Calculating $\left(\nabla_{e_1}e_2-\nabla_{e_2}e_1\right)f=\left[e_1,e_2\right](f)=e_1e_2(f)-e_2e_1(f)$ and $\left(\nabla_{e_1}e_3-\nabla_{e_3}e_1\right)f=\left[e_1,e_3\right](f)=e_1e_3(f)-e_3e_1(f)$, yeid

$$e_2e_1(f)=0, e_3e_1(f)=0. \tag{55}$$

Calculating $\left(\nabla_{e_1}e_2-\nabla_{e_2}e_1\right)\left(e_1f\right)=\left[e_1,e_2\right]\left(e_1f\right)=e_1e_2e_1(f)-e_2e_1e_1(f)$ and $\left(\nabla_{e_1}e_3-\nabla_{e_3}e_1\right)\left(e_1f\right)=\left[e_1,e_3\right]\left(e_1f\right)=e_1e_3e_1(f)-e_3e_1e_1(f)$, yeid

$$e_2e_1e_1(f)=0, e_3e_1e_1(f)=0. \tag{56}$$

Differentiating (54) along e_A, for $A=2,3$ we have

$$\mu_1e_A\mu_A=0 \tag{57}$$

Combing (45), (47), (57), we obtain

$$\frac{-c_1^2}{2}e_A\left(f^{\frac{9}{5}}\right)f^{\frac{9}{5}}+c_1e_A\left(g_A\right)f^{\frac{12}{5}}+c_1g_Ae_A\left(f^{\frac{3}{5}}\right)f^{\frac{9}{5}}=0. \tag{58}$$

Noting that $e_A\left(g_A\right)\neq 0$ and $f\neq 0$, then it follows from (22) and (56) that $c_1=0$, which together with (45) proves $\mu_1=0$, a contradiction. There for g_2 and g_3 are constant function.

4. Proofs of Main Theorems

Proof of Theorem 1.1. we choose a pseudo-Riemannian orthonormal frame field $e_1,e_2,\cdots,e_n$. For $i<j$, we have from (11) that

$$\left\|e_i\wedge e_j\right\|^2=\left\langle e_i,e_i\right\rangle\left\langle e_j,e_j\right\rangle-\left\langle e_i,e_j\right\rangle^2=\varepsilon_i\varepsilon_j. \tag{59}$$

Using the Gauss equation, it follows from (30) that

$$\begin{aligned} R\left(e_i,e_j,e_i,e_j\right) &= \left\langle h\left(e_j,e_i\right),h\left(e_i,e_j\right)\right\rangle-\left\langle h\left(e_i,e_i\right),h\left(e_j,e_j\right)\right\rangle \\ &= -\varepsilon_i\varepsilon_j\left(\varepsilon_5\lambda_i\lambda_j+\varepsilon_5\mu_i\mu_j\right). \end{aligned} \tag{60}$$

Combing (11), (12), (20) we have

$$\begin{aligned} R &= \sum_{i,j=1}^{3}K\left(e_i,e_j\right) \\ &= \sum_{i,j=1}^{3}\frac{\varepsilon_i\varepsilon_j(\varepsilon_4\lambda_i\lambda_j+\varepsilon_5\mu_i\mu_j}{\varepsilon_i\varepsilon_j} \\ &= \sum_{i,j=1}^{3}\left(\varepsilon_4\lambda_i\lambda_j+\varepsilon_5\mu_i\mu_j\right) \\ &= \varepsilon_4\left(\lambda_1+\lambda_2+\lambda_3\right)^2-\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)+\varepsilon_5\left(\mu_1+\mu_2+\mu_3\right)^2-\left(\mu_1^2+\mu_2^2+\mu_3^2\right) \\ &= \varepsilon_4\left(\lambda_1+\lambda_2+\lambda_3\right)^2-\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)-\varepsilon_5\left(\mu_1^2+\mu_2^2+\mu_3^2\right). \end{aligned} \tag{61}$$

When A_{e_5} has the form (I), Combining with (24) and (61) we obtain

$$\begin{aligned} R &= \varepsilon_4\left(\lambda_1+\lambda_2+\lambda_3\right)^2-\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right) \\ &= \varepsilon_4\left(\frac{-3\varepsilon_4 f}{2}+\frac{9\varepsilon_4 f}{4}+\frac{9\varepsilon_4 f}{4}\right)^2-\varepsilon_4\left(\left(\frac{-3\varepsilon_4 f}{2}\right)^2+\left(\frac{9\varepsilon_4 f}{4}\right)^2+\left(\frac{9\varepsilon_4 f}{4}\right)^2\right) \\ &= -\frac{27\varepsilon_4 f^2}{8}. \end{aligned} \tag{62}$$

When A_{e_5} has the form (II), Combining with (25) and (61) we obtain

$$R=-\frac{27\varepsilon_4 f^2}{8}-\varepsilon_5 g_2^2 f^{\frac{6}{5}}-\varepsilon_5 g_3^2 f^{\frac{6}{5}}.$$

where g_2 and g_3 are constants with $g_2+g_3=0$.

When A_{e_5} has the form (III), Combining with (26) and (61) we obtain

$$R=-\frac{27\varepsilon_4 f^2}{8}-\varepsilon_5 g_2^2 f^{\frac{6}{5}}-\varepsilon_5 g_3^2 f^{\frac{6}{5}}-\frac{3\varepsilon_5 c_1{}^2}{2} f^{\frac{18}{5}}.$$

where g_2 and g_3 are constants satisfying $g_2+g_3=0$. From the above, it follows that R is expressed as a fractional power polynomial in the mean curvature with non-zero coefficients, which completes the proof of Theorem 1.1.

Assuming that f is non-constant, a contradiction can be derived by applying Theorem 1.2, thereby concluding the proof of Theorem 1.2.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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