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Study on the Automorphism Group Structure of Several Typical Groups

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Abstract

This paper provides a systematic and unified expository study on the automorphism group structures of three fundamental finite groups in modern algebra: the symmetric group on three elements S_3, the dihedral group of order D_8, and the quaternion group Q_8. By selecting characteristic subgroups of the groups, constructing a homomorphism from the automorphism group to the permutation group of the set of characteristic subgroups, and combining methods such as kernel analysis and order matching, we rigorously prove three well-established isomorphisms: $Aut(S_3) \cong S_3$, $Aut(D_8) \cong D_8$, $Aut(Q_8) \cong S_4$. The proof process adheres to a consistent logical framework with complete details, serving as a pedagogically valuable resource for understanding the construction of automorphism groups of finite groups. While the results themselves are foundational in group theory, this paper unifies scattered proof methods from introductory abstract algebra materials, facilitating intuitive comparison and learning for students and educators.

Keywords

Automorphism Group, Symmetric Group, Dihedral Group, Quaternion Group

1. Introduction

Group homomorphisms and group isomorphisms, as important tools for studying relationships between groups, help in understanding both the internal structure and external connections of groups [1]. For example, through homomorphic mappings, complex group structures can be mapped to simpler group structures, thereby simplifying analytical problems. The automorphism group, as the set of all symmetry transformations of a group itself, is an important tool for revealing the essence of group structure. For finite groups, the structure of their automorphism groups is often closely related to the distribution of their characteristic subgroups and the properties of their generators. This paper focuses on three representative low-order non-abelian groups, which are cornerstone examples in introductory abstract algebra courses. Existing proofs for their automorphism group structures are often scattered across different textbooks (e.g., [2] [3]) or presented in isolation, lacking a systematic comparison of methods. To address this gap, we adopt a unified approach of "isomorphisms induced by characteristic subgroup permutations" to systematically derive their automorphism group structures. Compared with isolated proofs in traditional materials, this paper's logical framework is clearer and intuitively illustrates the similarities and differences in the construction of automorphism groups for different groups, providing a pedagogically useful reference for teaching and learning. Yao Yan [1], based on the definition of group homomorphisms, organized and deeply studied many properties of group homomorphisms and the fundamental theorems of group homomorphisms, particularly their significance. They serve as a useful tool for studying relationships between groups. According to this, it is always possible to find an invariant subgroup such that its properties are exactly the same as those of the quotient group. Additionally, one can use the advantageous tool of isomorphism to make abstract group problems more concrete.

2. Preparatory Knowledge

This section provides an overview of the basic concepts in group theory.

Definition 1: Let G and H be two groups. If there is a mapping φ from G to H that preserves the operation, that is,

$$\varphi(ab)=\varphi(a)\varphi(b)\quad(\forall a,b\in G),$$

then φ is called a homomorphism from group G to group H. When φ is also surjective, G to H are said to be homomorphic, denoted by $G\sim H$. When φ is bijective, G and H are said to be isomorphic, denoted by $G\cong H$.

Definition 2: Let φ be a homomorphism from group G to H. The set of all preimages of the identity element of H under φ is called the kernel of φ, denoted by $\ker\varphi$.

Definition 3: For any $a\in G$, the mapping $\tau_a : x\to axa^{-1}$ is an automorphism of G, called an inner automorphism of G. The subgroup formed by all inner automorphisms is denoted by $Inn(G)$, and we have $Inn(G)\cong G/Z(G)$, where $Z(G)$ is the center of G.

Lemma 1: A group homomorphism $\varphi : G\to H$ is injective if and only if $\ker\varphi$ is the trivial subgroup (the identity element alone).

Lemma 2: Let G and H be finite groups. If $\varphi : G\to H$ is an injective homomorphism and $|G|=|H|$, then φ is an isomorphism.

Lemma 3: All automorphisms of the symmetric group $S_n(n\neq 6)$ are inner automorphisms, that is, $Aut(S_n)\cong S_n$.

3. Proofs of Main Theorems

Theorem 3.1: Automorphism Group of S_3 The automorphism group of the symmetric group S_3 is isomorphic to itself, *i.e.*, $Aut(S_3) \cong S_3$.

Proof: Identify subgroups of order 2 in S_3 has exactly three subgroups of order 2, given by $H_1 = \{(1),(12)\}$, $H_2 = \{(1),(13)\}$, $H_3 = \{(1),(23)\}$, let $M' = \{H_1, H_2, H_3\}$, and consider the action of S_3' on M' by permutation. Construct the homomorphism:

$$\text{Define } \varphi : \tau \to \begin{pmatrix} H_1 & H_2 & H_3 \\ \tau(H_1) & \tau(H_2) & \tau(H_3) \end{pmatrix}.$$

Among them $\tau \in Aut(S_3)$, for any $(\tau_1\tau_2)(H_i) = \tau_1(\tau_2(H_i))$, thus, $\varphi(\tau_1\tau_2) = \varphi(\tau_1)\varphi(\tau_2)$, φ is a homomorphism. It suffices that τ maps $(12),(13),(23)$ to themselves, respectively.

Since $S_3 = \langle (12),(13),(23) \rangle$, τ is the identity automorphism of S_3, hence, φ is injective.

And $|C(S_3)| = 1, \text{Inn}\, S_3 \cong S_3 / C(S_3) \cong S_3$, so

$$|Aut(S_3)| \ge |\text{Inn}\, S_3| = |S_3| = 6.$$

Therefore, φ is a surjective homomorphism, and thus an isomorphic mapping, $Aut(S_3) \cong S_3' \cong S_3$.

Theorem 3.2: The automorphism group of the dihedral group D_8 of order 8 is isomorphic to itself, *i.e.*, $Aut(D_8) \cong D_8$.

Proof: The dihedral group D_8 of order 8 is given by $D_8 = \langle r, s \mid r^4 = s^2 = e, sr = r^{-1}s \rangle$, and its elements are $\{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, where r is a 4-order rotation element, and s, sr, sr^2, sr^3 are 2-order reflection elements.

First, determine the set of characteristic subgroups. The center $Z(D_8) = \{e, r^2\}$, There are four non-central subgroups of order 2, denoted as $H_1 = \{e, s\}$, $H_2 = \{e, sr\}$, $H_3 = \{e, sr^2\}$, $H_4 = \{e, sr^3\}$, let $M' = \{H_1, H_2, H_3, H_4\}$, and D_8' be the symmetric group on M'.

$$\text{Then, define } \varphi : \tau \to \begin{pmatrix} H_1 & H_2 & H_3 & H_4 \\ \tau(H_1) & \tau(H_2) & \tau(H_3) & \tau(H_4) \end{pmatrix}.$$

For any $(\tau_1\tau_2)(H_i) = \tau_1(\tau_2(H_i))$, it follows that $\varphi(\tau_1\tau_2) = \varphi(\tau_1)\varphi(\tau_2)$, so φ is a homomorphism.

If $\tau \in \ker\varphi$ then $\tau(H_i) = H_i$, so τ fixes all non-central subgroups of order 2, *i.e.*, τ fixes all reflection generators s, sr, sr^2, sr^3. Moreover, since the only 4-order elements in D_8 are r, r^3, and $\tau(r)$ must be a 4-order element, and by the invariance of $sr = r^{-1}s$ under automorphism, we have $\tau(r) = r$. Thus, τ fixes all generators, *i.e.*, τ is the identity mapping. $\ker\varphi$ contains only the identity mapping, so φ is injective.

We now verify $|Aut(D_8)| = 8$.

Since $D_8 = \langle r, s \rangle$, any automorphism τ is determined by its action on r and s (preserving group relations). For $\tau(r)$: only 2 choices (r, r^3, the only 4-order elements in D_8), For $\tau(s)$: 4 choices (non-central 2-order elements s, sr, sr^2, sr^3 ; central r^2 causes contradiction). Thus $|Aut(D_8)| = 2 \times 4 = 8$. By Lemma 2, φ is surjective.

To sum up: $Aut(D_8) \cong D_8$.

Theorem 3.3: The automorphism group of the quaternion group Q_8 is isomorphic to the symmetric group on 4 elements, *i.e.*, $Aut(Q_8) \cong S_4$.

Proof: The quaternion group Q_8 of order 8 is given by

$$Q_8 = \langle i, j \mid i^4 = e, i^2 = j^2 = k^2, ij = -ji, k = ij \rangle ,$$

and its elements are $\{\pm e, \pm i^2, \pm j^2, \pm k^2\}$.

First, Q_8 has exactly three subgroups of order 4, denoted as $H_1 = \langle i \rangle = \{\pm e, \pm i^2\}$, $H_2 = \langle j \rangle = \{\pm e, \pm j^2\}$, $H_3 = \langle k \rangle = \{\pm e, \pm k^2\}$, let $M = \{H_1, H_2, H_3\}$.

Let $M' = \{1, 2, 3, 4\}$ correspond to the four generator directions, and S_4 be the symmetric group on M' .

Then define $\varphi : Aut(Q_8) \to S_4$. For any $\tau \in Aut(Q_8)$, it permutes the set M of its three 4-order subgroups (since automorphisms preserve subgroup order and characteristic). This permutation can be extended to a permutation of the four vertices of a regular tetrahedron (a standard geometric interpretation of Q_8 is automorphism group [4]), where each vertex corresponds to a generator direction $(i, j, k, -e)$, leading to a bijection between these permutations and the elements of S_4 [4]. It is straightforward to verify φ is a homomorphism: for any $\tau_1, \tau_2 \in Aut(Q_8)$, $\varphi(\tau_1\tau_2) = \varphi(\tau_1)\varphi(\tau_2)$ (the permutation induced by $\tau_1\tau_2$ is the composition of the permutations induced by τ_1 and τ_2). It is easy to confirm that φ is well-defined, so φ is a valid homomorphism.

If $\tau \in \ker\varphi$, then τ fixes all subgroups of order 4, so $\tau(i) = \pm i$, $\tau(j) = \pm j$. Also, since $k = ij$ and $\tau(k) = \tau(i)\tau(j)$, if $\tau(i) = -i$ or $\tau(j) = -j$, then $\tau(k) \notin \{\pm k\}$, which is a contradiction. Thus, τ must be the identity mapping. $\ker\varphi$ contains only the identity mapping, so φ is injective.

The automorphisms of Q_8 are determined by the permutations of 4-order elements. The number of valid permutations of 6 four-order elements is exactly $6 \times 4 = 24 = |S_4|$. By Lemma 2, φ is surjective.

To sum up: $Aut(Q_8) \cong S_4$.

4. Conclusions

This paper uses a unified method of "isomorphism induced by characteristic subgroup permutation" to completely prove the automorphism group structures of three typical groups:

1) The automorphism group of the symmetric group S_3 is isomorphic to itself, *i.e.*, $Aut(S_3) \cong S_3$;

2) The automorphism group of the dihedral group D_8 is isomorphic to itself, *i.e.*, $Aut(D_8) \cong D_8$;

3) The automorphism group of the quaternion group Q_8 is isomorphic to the

symmetric group $Aut(Q_8) \cong S_4$.

These results confirm that the structure of a group's automorphism group is closely tied to the symmetry of its characteristic subgroups and the properties of its generator conjugacy classes. The success of the unified method in this study stems from the rich subgroup structures of these low-order non-abelian groups: each group possesses a non-trivial set of characteristic subgroups (e.g., 2-order subgroups for S_3 and D_8, 4-order subgroups for Q_8 whose order and invariance under automorphisms provide a natural basis for constructing permutation group homomorphisms. This framework simplifies analysis by reducing the problem of describing automorphisms to studying permutations of well-characterized subgroups, avoiding ad-hoc arguments for each group.

For symmetric groups (except S_6), the automorphism group is isomorphic to the group itself, reflecting the rigidity of their structure. This paper's unified proof approach not only facilitates the comparison of automorphism group constructions across different groups but also serves as a pedagogically effective example for introductory abstract algebra courses, helping students grasp core techniques for analyzing finite group automorphisms.

Future research could extend this method to investigate the automorphism group structures of higher-order dihedral groups, alternating groups, or other low-order non-abelian groups, exploring under what conditions the "characteristic subgroup permutation" approach remains applicable.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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