

Control Values of Connected Components in Graph Games

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Abstract

This paper studies an allocation rule for graph games with a control structure. In such graph games, a virtual player C is introduced as the control structure; only sub-coalitions that combine with C can obtain the coalition payoff. Based on this model, we propose three novel axioms-null player property, control fairness, and linearity of loss-and introduce the control value as the allocation rule. The control value first assigns a Shapley payoff to every player, and the Shapley payoff of the virtual player C is then evenly redistributed among all players. We prove that the control value is the unique allocation rule satisfying efficiency, representative null player property, control fairness, and linearity of loss.

Keywords

Cooperative Games, Graph Structure, Single-Layer Restriction, Allocation Rule

1. Introduction

The central question in cooperative game theory is how to allocate the worth of coalitions in a fair and reasonable manner. When the payoff of any coalition can be transferred among its members without loss, one speaks of a cooperative game with transferable utility, briefly a TU game [1].

In a TU game every subset of players may form a coalition and earn the associated worth; real-world interaction, however, is often subject to frictions that render some coalitions infeasible. This observation motivated the study of TU games with restricted cooperation. Myerson (1977) initiated this line of research by introducing graph games [2]. In his model, only players who are connected in an undirected communication graph may cooperate; disconnected coalitions earn the sum of the worths of their connected components. Myerson used the Shapley value [3] to

define the Myerson value and characterised it by component efficiency and fairness [2]. Subsequent work extended Myerson's framework to hypergraphs [4] [5], networks [6] [7], union stable systems [8] [9], and directed graphs [10] [11], yielding a rich family of graph-restricted cooperative games.

Parallel to these structural extensions, a variety of allocation rules have been proposed. Besides the Myerson value, the two-step Shapley value [12], Owen value [13], and position value [14] adapt the Shapley idea to different informational settings. More egalitarian approaches, such as the r-Shapley value (Yokote *et al.*, 2018) [15] and the Shley-egalitarian hybrid solution (Li & Shan, 2024) [16], have also been axiomatically characterised. Yet, the literature has paid little attention to control structures inside the graph environment. Dietzenbacher *et al.* (2017) took a first step by embedding a network-control mapping into communication games [17]; the worth of a coalition is then the aggregate worth of the components it controls. Their work, however, treats the control mapping as an exogenously given partition, leaving open the question of how to allocate the surplus when the controller itself is a strategic virtual player.

In most existing graph games, cooperation is assumed to emerge spontaneously among connected agents. In practice, however, the architecture often matters economically: trading platforms, supply-chain orchestrators, or system operators must be present for transactions to occur. We therefore treat the graph structure as a virtual player C and posit that a coalition generates its worth only if it includes C . This single-layer control idea gives rise to a new graph game with control structure: We introduce a single-layer control structure, where a virtual player C must be included in any coalition for it to generate worth, modeling real-world controllers like platforms or regulators, and to a natural allocation rule—the control value—whose axiomatic foundation is the subject of the present paper.

2. Theoretical Foundations

A pair (N, v) denotes a cooperative game with transferable utility (TU game), where $N = 1, \dots, n$ ($n \geq 2$) is the set of players, $v: 2^N \rightarrow \mathcal{R}$ is the characteristic function satisfying $v(\emptyset) = 0$. Any subset $S \subseteq N$ is called a coalition, $v(S)$ is the worth of coalition S . Let \mathcal{G}^N be the set of all TU games on player set N . Whenever no confusion arises, we identify a game by its characteristic function v , *i.e.*, $v \in \mathcal{G}^N$. For every non-empty coalition $\forall S \subseteq N$, the unanimity game (N, u_S) is defined by $u_S(T) = 1$, $S \subseteq T$, otherwise $u_S(T) = 0$. Moreover, if $\forall i \in N$, $v(i) = 0$, then the characteristic function $v \in \mathcal{G}_0^N$ is a zero-normalised game on player set N . For any $S \subseteq N$, denote $|S|$ as the cardinality of set S , *i.e.*, the number of elements in set S .

It is well known that the Shapley value is the unique solution on TU games satisfying efficiency, additivity, symmetry and the null player property. For a given $v \in \mathcal{G}^N$, its Shapley value is defined as:

$$Sh_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} (v(S \cup \{i\}) - v(S)), \forall i \in N. \quad (1)$$

A pair (N, L) represents an undirected graph, where N denotes the node set (player set), the edge set $L \subseteq \{\{i, j\} : i, j \in N, i \neq j\}$, and $\{i, j\} \in L$ is an edge indicating that players i and j have bilateral communication. For simplicity, we usually refer to the graph (N, L) and the edge $\{i, j\}$ by the edge set L and the symbol ij , respectively. Likewise, we use the symbol \mathcal{L}^N to denote the set of all graphs on node set N . A graph $L \in \mathcal{L}^N$ is called complete if there exists an edge between every pair of distinct nodes.

For any non-empty subset $S \subseteq N$, $(S, L(S))$ denotes the subgraph of L induced by S , where $L(S) = \{ij \in L : i, j \in S\}$, $L_i = \{ij \in L : j \in N\}$ is the set of edges incident to player i , and $L_{-i} = L \setminus L_i$ is the set of edges not incident to player i .

A sequence of distinct nodes (i_1, \dots, i_k) , $k \geq 2$, is called a path from node i_1 to node i_k in L if $i_h i_{h+1} \in L$ for every $h = 1, 2, \dots, k-1$. If there exists a path between every pair of nodes in L , the graph is said to be connected; if, in addition, every pair of nodes is connected by exactly one path, the connected graph is a tree.

If a non-empty subset $S \subseteq N$ is connected, then its induced subgraph $L(S)$ is connected. In a graph game $L \in \mathcal{L}^N$, a largest connected subset is called a (connected) component: a non-empty subset $T \subseteq N$ is a component of L if T is connected and $T \cup \{i\}$ is disconnected for any $i \in N \setminus T$. The set of all connected components of L is denoted by N/L ; for any $S \subseteq N$, we use S/L to denote the set of all connected components of the subgraph $L(S)$, and if $L = \{\emptyset\}$, then $S/L = \{\{i\} \mid i \in S\}$. Moreover, for any $i \in N$, the connected component containing i is denoted by T_i , where $T_i \in C$ and $i \in T_i$.

A triple (N, v, L) defines an undirected graph game on player set N . We use the symbol \mathcal{G}_L^N to denote the set of all undirected graph games defined on player set N , and write $(v, L) \in \mathcal{G}_L^N$ for short. If any undirected graph game $(v, L) \in \mathcal{G}_L^N$ is assigned a unique payoff vector $f(v, L) \in \mathbb{R}^n$, we call f a single-valued solution for undirected graph games. The earliest solution for undirected graph games was proposed by Myerson (1977) and is therefore referred to as the Myerson value; it is defined by:

$$\mu(v, L) = Sh(v^L), \quad \forall (v, L) \in \mathcal{G}_L^N, \quad (2)$$

where v^L is the graph-restricted game (briefly, the graph game); for any $S \subseteq N$, one has $v^L(S) = \sum_{T \in S/L} v(T)$. Moreover, the Myerson value is the unique solution satisfying component efficiency and fairness.

Component Efficiency. For every graph game $(v, L) \in \mathcal{G}_L^N$ and every component $T \in N/L$, the following holds:

$$\sum_{i \in T} f_i(v, L) = v(T). \quad (3)$$

Fairness. For every graph game $(v, L) \in \mathcal{G}_L^N$ and every edge $ij \in L$, the following holds:

$$f_i(v, L) - f_i(v, L \setminus \{ij\}) = f_j(v, L) - f_j(v, L \setminus \{ij\}). \quad (4)$$

3. Single-Layer Restricted Game and Control Value

In the graph game $(N, L) \in \mathcal{G}_L^N$, we postulate a single-layer control structure (hereafter, control structure) outside the player set N whose role is to regulate the connectivity among players. Specifically, this control structure enters the game as a virtual player C , and only coalitions that contain C can generate the worth prescribed by the characteristic function; any coalition $S \subseteq N$ that does not include C earns merely the sum of stand-alone payoffs of its members. Such control structures are ubiquitous in practice. Consider a large manufacturer: the firm acts as the control structure (virtual player C) overseeing its suppliers (players in N). A group of suppliers ($S \subseteq N$) obtains stable orders and profits (coalition worth) only if it cooperates with the firm (forming $S \cup C$) and satisfies the firm's purchasing standards, quality requirements and delivery schedules. Suppliers who bypass the firm (thus failing to form $S \cup C$) face uncertain demand and fierce competition, so their earnings are reduced to stand-alone levels.

Definition 3.1 In the graph game $(N, L) \in \mathcal{G}_L^N$, the single-layer restricted game is defined as $w \in \mathcal{G}^{N \cup C}$, where for all $S \subseteq N \cup C$ we have:

$$w(S) = \begin{cases} \sum_{R \in (S \setminus C)/L} v(R), & S \cap C = C, \\ \sum_{i \in S} v(i), & S \cap C = \{\emptyset\}. \end{cases} \quad (5)$$

In the single-layer restricted game, the grand coalition is $N \cup C$, which contains $|N|+1$ players. Any sub-coalition $S \subseteq N \cup C$ consists of members from the original player set N and the virtual player C . Equation (5) shows that the control structure is determined by $S \cap C$. When $S \cap C = C$, players $i \in S \setminus C$ can form connected components; otherwise, if $S \cap C = \{\emptyset\}$, every player $i \in S \setminus C$ remains isolated. Consequently, in the single-layer restricted game $w \in \mathcal{G}^{N \cup C}$, only sub-coalitions that include the virtual player C earn the coalition worth, while any coalition without C receives merely the sum of stand-alone payoffs.

We first present the definition of the control value φ (hereafter referred to as the φ -value):

Definition 3.2 For any graph game $(v, L) \in \mathcal{G}_L^N$, the φ -value assigned to player $i \in N$ is

$$\varphi_i = Sh_i(N \cup C, w) + \frac{1}{|N|} Sh_c(N \cup C, w). \quad (6)$$

We interpret the φ -value in two steps. First, the single-layer restricted game $w \in \mathcal{G}^{N \cup C}$ yields a Shapley value for every real player $i \in N$ and for the virtual player C , denoted by $Sh_i(N \cup C, w)$ and $Sh_c(N \cup C, w)$, respectively. Second, the φ -value is the solution assigned to each real player $i \in N$ in the graph game $(v, L) \in \mathcal{G}_L^N$ and is given by the sum of $Sh_i(N \cup C, w)$ and $\frac{1}{|N|} Sh_c(N \cup C, w)$.

Hence, the Shapley payoff of the virtual player C is equally divided among all real players $i \in N$. Consequently, the φ -value is built on the Shapley value and simultaneously embodies the idea of equal division.

We now provide an example to demonstrate that the φ -value is a novel allocation rule distinct from existing solutions for graph games.

Example 3.1 We consider the graph game $(v, L) \in \mathcal{G}_L^N$ with player set $N = \{1, 2, 3, 4, 5\}$ and characteristic function $v = u_{\{1,2\}} + u_{\{4,5\}}$. As shown in **Figure 1**, the graph is $L = \{\{1, 2\}, \{1, 3\}, \{4, 5\}\}$. Applying the definition of the φ -value, the Shapley payoffs allocated to players $i \in N$ in the single-layer restricted game $w \in \mathcal{G}^{N \cup C}$ are $\left\{ \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3} \right\}$, while the Shapley payoff assigned to the virtual player C is $\left\{ \frac{2}{3} \right\}$. Consequently, the φ -value for the players is

$$\varphi(v, L) = \left\{ \frac{7}{15}, \frac{7}{15}, \frac{2}{15}, \frac{7}{15}, \frac{7}{15} \right\},$$

whereas the Myerson value of this game is

$$\mu(v, L) = \left\{ \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \right\}.$$

Comparing the two solutions shows that player 3, whose marginal contribution is zero, receives zero payoff under the Myerson value. In contrast, the φ -value equally distributes the virtual player's payoff $f_C(v, L)$ among all players, yielding a more even allocation across the coalition.

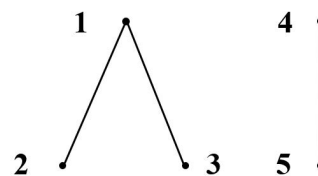


Figure 1. The graph structure (N, L) in Example 3.1.

4. Axiomatic Characterization of the φ -Value

In the single-layer restricted game $w \in \mathcal{G}^{N \cup C}$, we assume that the payoff vector assigned to the virtual player C is given by a function $f_C(v, L)$ that depends on the characteristic function v and the edge set L . Note that $f_C(v, L)$ is not necessarily unique; it is an unknown quantity we introduce to facilitate the axiomatic characterization of solutions for the graph game $(v, L) \in \mathcal{G}_L^N$.

Before presenting the formal definition, we note that subtracting a fraction of the control player's payoff is meaningful because it isolates the contribution of each real player, independent of the control surplus redistribution.

Definition 4.1 The *representative payoff* $f^*(v, L)$ assigned to each player $i \in N$ in the graph game $(v, L) \in \mathcal{G}_L^N$ is defined as

$$f_i^*(v, L) = f_i(v, L) - \frac{1}{|N|} f_C(v, L). \tag{7}$$

It is easy to see that the representative payoff $f^*(v, L)$ implies that the payoff vector $f_C(v, L)$ is equally distributed among every player in T . When the payoff vector $f_C(v, L)$ is known, the player payoff vector $f(v, L)$ depends only on the representative payoff $f^*(v, L)$. Therefore, we propose two new axioms based on the representative payoff $f^*(v, L)$:

Definition 4.2 For any graph game $(v, L) \in \mathcal{G}_L^N$ and any edge $ij \in L$, if the following equation holds, then the graph game $(v, L) \in \mathcal{G}_L^N$ is said to satisfy control fairness:

$$f_i^*(v, L) - f_i^*(v, L \setminus \{ij\}) = f_j^*(v, L) - f_j^*(v, L \setminus \{ij\}) = f_c(v, L) - f_c(v, L \setminus \{ij\}). \quad (8)$$

Control fairness requires not only that the two players incident to a removed edge experience identical gains or losses in their representative payoffs, but also that these gains or losses match the change in the payoff vector $f_c(v, L)$. Thus, control fairness is an edge-deletion axiom analogous to fairness, yet strictly more demanding. Unlike standard fairness, which only balances payoffs between two players, control fairness further links their payoff changes to the control player's gain or loss, making it strictly more demanding.

Notice that if deleting edge ij does not create any new connected component, control fairness reduces to the standard fairness property.

This axiom captures the idea that when an edge is removed, the total loss of unaffected components should scale proportionally to the control player's loss, reflecting a systemic risk-sharing principle.

Definition 4.3 For any graph game $(v, L) \in \mathcal{G}_L^N$, any edge $ij \in L(T_{ij})$ with $T_{ij} \in N/L$, and any connected component $T \in (N/L) \setminus \{T_{ij}\}$, if the following equality holds, then the graph game $(v, L) \in \mathcal{G}_L^N$ is said to satisfy linearity of loss:

$$\sum_{i \in T: T \in (N/L) \setminus \{T_{ij}\}} f_i(v, L) - f_i(v, L \setminus \{ij\}) = \frac{|T|}{|N|} (f_c(v, L) - f_c(v, L \setminus \{ij\})). \quad (9)$$

Linearity of loss states that after edge ij is removed, the aggregate loss of any connected component not containing ij is proportional to the loss incurred by the representative payoff $f_c(v, L)$, with the proportion equal to the ratio of the cardinality of that component to the cardinality of the grand coalition. This axiom complements control fairness by specifying the quantitative relationship between the losses of components unaffected by the edge deletion.

Definition 4.4 For any player $i \in N$ in a graph game $(v, L) \in \mathcal{G}_L^N$ such that $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$, if the following equality holds, then the graph game $(v, L) \in \mathcal{G}_L^N$ is said to satisfy the representative null-player property:

$$f_i^*(v, L) = 0. \quad (10)$$

Representative null-player property is inspired by the standard null-player property of the Shapley value. It requires that a player who contributes nothing to any coalition should receive zero representative payoff; however, such a player may still obtain a share of $f_c(v, L)$. This is the key distinction between representative null-player property and the usual null-player property.

Definition 4.5 For any single-layer restricted game $w \in \mathcal{G}^{N \cup C}$, the game $w^{L \setminus e} \in \mathcal{G}^{N \cup C}$ after edge $e \in L$ is removed is defined for all $S \subseteq N \cup C$ by

$$w^{L \setminus e}(S) = \begin{cases} \sum_{R \in (S \setminus C) / (L \setminus e)} v(R), & S \cap C = C, \\ \sum_{i \in S} v(i), & S \cap C = \{\emptyset\}. \end{cases} \quad (11)$$

Lemma 4.1 In the graph game $(N, v, L) \in \mathcal{G}_L^N$, the φ -value satisfies efficiency, representative null-player property, control fairness and linearity of loss.

Proof We first prove that the φ -value satisfies efficiency. For any graph game $(v, L) \in \mathcal{G}_L^N$ and the virtual player C , we have

$$\sum_{i \in N} \varphi_i = \sum_{i \in N} Sh_i(N \cup C, w) + Sh_c(N \cup C, w).$$

Since the Shapley value satisfies efficiency, we obtain $\sum_{i \in N} Sh_i(N \cup C, w) + Sh_c(N \cup C, w) = w(N \cup C) = v^L(N)$. Hence the φ -value satisfies efficiency.

To prove that the φ -value satisfies control fairness, set $f_c(v, L) = Sh_c(N \cup C, w)$; then $f_i^*(v, L) = Sh_i(N \cup C, w)$. It suffices to show that for every graph game $(v, L) \in \mathcal{G}_L^N$ and every edge $e = ij \in L$,

$$\begin{aligned} Sh_i(N \cup C, w) - Sh_i(N \cup C, w^{L^e}) &= Sh_j(N \cup C, w) - Sh_j(N \cup C, w^{L^e}) \\ &= Sh_c(N \cup C, w) - Sh_c(N \cup C, w^{L^e}). \end{aligned} \tag{12}$$

By additivity and symmetry of the Shapley value, Equation (11) holds provided that for all $S \subseteq N \cup C$ with $i, j \notin S$,

$$w(S \cup i) - w(S) - (w^{L^e}(S \cup i) - w^{L^e}(S)) = w(S \cup j) - w(S) - (w^{L^e}(S \cup j) - w^{L^e}(S)).$$

Simplifying gives

$$w(S \cup i) - w^{L^e}(S \cup i) = w(S \cup j) - w^{L^e}(S \cup j).$$

Because $e = ij \notin L(S \cup i)$, we have $L(S \cup i) = (L \setminus e)(S \cup i)$; thus by the definition of w , $w(S \cup i) - w^{L^e}(S \cup i) = 0$. Similarly, $w(S \cup j) - w^{L^e}(S \cup j) = 0$, so (11) holds.

Equation (12) is verified by distinguishing two cases: $S \cap C = \{\emptyset\}$ or $S \cap C = C$. When $S \cap C = \{\emptyset\}$, the definition of w yields $w(S \cup C) - w^{L^e}(S \cup C) = 0$; when $S \cap C = C$, $w(S \cup C) - w^{L^e}(S \cup C) = 0$ always holds. Hence

$$\begin{aligned} w(S \cup i) - w^{L^e}(S \cup i) &= w(S \cup j) - w^{L^e}(S \cup j) \\ &= w(S \cup C) - w^{L^e}(S \cup C) = 0. \end{aligned}$$

Additivity and symmetry of the Shapley value then imply that the φ -value satisfies control fairness. Linearity of loss and the representative null-player property follow directly from additivity and the null-player property of the Shapley value.

Lemma 4.3 In the graph game $(N, v, L) \in \mathcal{G}_L^N$, there exists a unique payoff vector $f(v, L) \in \mathbb{R}^n$ and a unique payment vector $f_c(v, L) \in \mathbb{R}^n$ that satisfy efficiency, representative null-player property, control fairness and linearity of loss.

Proof To prove Lemma 4.3, it suffices to show that for any graph game (N, v, L) there is at most one allocation rule and one payment vector $f_c(v, L)$ satisfying the four properties. We proceed by contradiction. Assume that two distinct allocation rules with associated payment vectors $f(v, L), f_c(v, L) \in \mathbb{R}^n$ and

$g(v, L), d(v, L) \in \mathbb{R}^n$ all satisfy efficiency, representative null-player property, control fairness and linearity of loss. For an edge $e = ij \in L(T_{ij})$ with $T_{ij} \in N/L$, applying efficiency and linearity of loss to f and g gives

$$\begin{aligned} & \sum_{i \in T_{ij}} (f_i(v, L) - f_i(v, L \setminus e)) \\ &= w(T_{ij}) - w^{L \setminus e}(T_{ij}) - \frac{|N| - |T_{ij}|}{|N|} (f_C(v, L) - f_C(v, L \setminus e)). \\ & \sum_{i \in T_{ij}} (g_i(v, L) - g_i(v, L \setminus e)) \\ &= w(T_{ij}) - w^{L \setminus e}(T_{ij}) - \frac{|N| - |T_{ij}|}{|N|} (g_C(v, L) - g_C(v, L \setminus e)). \end{aligned}$$

Subtracting the two equations and substituting

$$f_i^*(v, L) = f_i(v, L) - \frac{1}{|N|} f_C(v, L) \text{ yields}$$

$$\begin{aligned} & \sum_{i \in T_{ij}} (f_i^*(v, L) - f_i^*(v, L \setminus e) - (g_i^*(v, L) - g_i^*(v, L \setminus e))) \\ &= g_C(v, L) - g_C(v, L \setminus e) - (f_C(v, L) - f_C(v, L \setminus e)). \end{aligned} \tag{13}$$

We now use mathematical induction to prove

$$f(v, L) - g(v, L) = f_C(v, L) - g_C(v, L) = 0.$$

Base case: For any connected component $T \in N/L$ in the graph game $(N, v, L) \in \mathcal{G}_L^N$, if $|L(T)| = 0$, then T is an isolated node and representative null-player property gives $f_T^*(v, L) = g_T^*(v, L) = 0$. If $|L(T)| = 1$ with $e = ij \in L(T)$, then $f_T^*(v, L) = g_T^*(v, L) = 0$ implies $f_i^*(v, L \setminus e) = g_i^*(v, L \setminus e) = f_j^*(v, L \setminus e) = g_j^*(v, L \setminus e) = 0$; plugging these into control fairness and differencing gives

$$\begin{aligned} f_i^*(v, L) - g_i^*(v, L) &= f_j^*(v, L) - g_j^*(v, L) \\ &= f_C(v, L) - g_C(v, L) - (f_C(v, L \setminus e) - g_C(v, L \setminus e)). \end{aligned}$$

Substituting into (13) yields $f_i^*(v, L) - g_i^*(v, L) = f_j^*(v, L) - g_j^*(v, L) = 0$, and control fairness then implies $f_C(v, L) - g_C(v, L) = f_C(v, L \setminus e) - g_C(v, L \setminus e) = 0$. Hence for $|L(T)| = 0$ or $|L(T)| = 1$ we have

$$f_i(v, L) - g_i(v, L) = f_C(v, L) - g_C(v, L) = 0 \text{ for all } i \in T.$$

Inductive hypothesis: For any connected component $T \in N/L$ in the graph game $(N, v, L) \in \mathcal{G}_L^N$, if $|L(T)| = k$, then

$$f_i(v, L) - g_i(v, L) = f_C(v, L) - g_C(v, L) = 0 \text{ for all } i \in T.$$

Inductive step: For any connected component $T \in N/L$ with $|L(T)| = k + 1$, Equation (13), control fairness and the induction hypothesis imply

$$f_i(v, L) - g_i(v, L) = f_C(v, L) - g_C(v, L) = 0 \text{ for all } i \in T.$$

Therefore, for every connected component $T \in N/L$ in any graph game $(N, v, L) \in \mathcal{G}_L^N$ we have $f_i(v, L) - g_i(v, L) = f_C(v, L) - g_C(v, L) = 0$ for all $i \in T$, i.e., $f_i(v, L) - g_i(v, L) = f_C(v, L) - g_C(v, L) = 0$ for all $i \in N$. Thus $f(v, L) - g(v, L) = f_C(v, L) - g_C(v, L) = 0$, contradicting the assumption. Con-

sequently, for any graph game (N, v, L) there exists at most one allocation rule and one payment vector $f_c(v, L)$ satisfying efficiency, representative null-player property, control fairness and linearity of loss. \square

Theorem 4.1 In the graph game $(N, v, L) \in \mathcal{G}_L^N$, the φ -value is the unique solution that satisfies efficiency, representative null-player property, control fairness and linearity of loss.

Proof This follows immediately from Lemma 4.3 and Lemma 4.3. \square

The four allocation rules below demonstrate the logical independence of the axioms: omitting any one of them prevents the remaining three from uniquely determining an allocation rule.

Definition 4.6 Let f^1 be an allocation rule on \mathcal{G}_L^N defined by

$$f_i^1(v, L) = 0, \quad \forall i \in N. \quad (14)$$

Evidently, this rule meets representative null-player property, control fairness and linearity of loss in Theorem 4.1, but fails efficiency.

Definition 4.7 Let f^2 be an allocation rule on \mathcal{G}_L^N defined by

$$f_i^2(v, L) = \mu_i(v, L), \quad \forall i \in N. \quad (15)$$

This rule satisfies efficiency, representative null-player property and linearity of loss in Theorem 4.1, but violates control fairness.

Definition 4.8 Let f^3 be an allocation rule on \mathcal{G}_L^N with, for every sub-coalition $\forall S \subseteq N \setminus \{i\}$, $f_c(v, L) = Sh_c(N \cup C, w)$. The expression of f^3 is

$$f_i^3(v, L) = \begin{cases} 1 + \frac{1}{|N|} f_c(v, L), & i \in N, v(S \cup i) = v(S), \\ \varphi_i - \frac{\sum_{i \in N: v(S \cup i) = v(S)} 1}{\sum_{i \in N: v(S \cup i) \neq v(S)} 1}, & i \in N, v(S \cup i) \neq v(S). \end{cases} \quad (16)$$

This rule satisfies efficiency, control fairness and linearity of loss in Theorem 4.1, but fails representative null-player property.

Definition 4.9 Let f^4 be an allocation rule on \mathcal{G}_L^N defined component-wise: for each connected component $T \in N/L$,

$$f_i^4(v, L) = \frac{v(T)}{\sum_{i \in T} \varphi_i} \varphi_i, \quad \forall i \in T. \quad (17)$$

This rule satisfies efficiency, control fairness and representative null-player property in Theorem 4.1, but violates linearity of loss.

5. Conclusions and Suggestions

This paper introduces an allocation rule and the control value for graph games with a control structure, and proposes three new axioms-representative null-player property, control fairness and linearity of loss. We prove that the control value is the unique solution satisfying efficiency, representative null-player property, control fairness and linearity of loss. These results not only provide a solid

theoretical foundation for analysing graph games, but also offer practical insights for applications in airport runway cost sharing. For instance, the airport authority acts as the virtual player C , and airlines (real players) can only form effective coalitions for cost sharing if they coordinate through the authority. Future research will focus on multi-layer control structures and the axiomatic characterization of the control value in that richer framework, so as to broaden both the applicability and the theoretical depth of graph games.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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