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# Optimization of the Enhanced Index Model

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## Abstract

With the development of the domestic economy and the increase in household income, the demand for investment has been growing, and funds are widely favored for their safety and flexibility. Enhanced index funds combine the advantages of both passive and active management, with the potential to outperform the market and reduce tracking errors, attracting the attention of many investors. To address the risk that tracking portfolios may incur significant losses due to market index declines, this paper proposes the introduction of a non-parametric Mean Absolute Deviation (MAD) as a downside risk constraint in the enhanced index model, aiming to effectively control the downside risk of the tracking portfolio. Firstly, the study uses a non-parametric method to estimate the MAD and proves that this estimator is a convex function of the portfolio position. Secondly, an enhanced index model is constructed under the MAD constraint, where the objective function consists of a weighted sum of tracking error and excess return. Specifically, we use downside risk to measure tracking error. Finally, it is proven that the model is a convex optimization problem. Empirical research shows that the enhanced index model proposed in this paper, which considers the non-parametric MAD constraint, effectively controls downside risk.

## Keywords

Enhanced Index Model, Mean Absolute Deviation, Downside Risk

## 1. Introduction

Traditionally, index-based fund management strategies are broadly divided into passive management and active management. Fund managers implementing a passive management strategy aim to replicate the performance of a specific financial market index (the so-called benchmark) as closely as possible, such as the CSI 300 or the CSI 500. This strategy is known as index tracking, and it seeks to mimic the market index by selecting a subset of stocks from the benchmark, thereby minimizing a function that measures how closely the portfolio tracks its benchmark index (tracking error). Fund managers implementing an active management strategy aim to outperform the benchmark. This strategy involves analyzing a company's financial condition, industry prospects, market trends, and other factors to predict the future performance of a stock, selecting stocks with better future prospects to construct a portfolio with the goal of outperforming the benchmark index. Additionally, [1] have shown that a significant number of actively managed funds fail to outperform their benchmark over the long term. Therefore, fund managers typically prefer to adopt a hybrid strategy, often using a passive strategy to manage the majority of the fund's investments, while employing an active strategy to manage a limited portion of the investments (see [2]).

Enhanced index tracking is an investment strategy aimed at achieving higher returns than the benchmark index (excess returns) while minimizing tracking errors. Therefore, the Enhanced Index Tracking Problem (EITP) seeks to minimize tracking error while maximizing excess returns above the benchmark. This investment strategy is an effective combination of passive and active management, providing relatively stable returns, which has attracted the attention of many scholars.

In recent years, many scholars have developed models and solved the Enhanced Index Tracking Portfolio Problem. [3] proposed a related mixed-integer linear programming formulation for the enhanced index tracking problem, which includes transaction costs, constraints on the number of stocks that can be purchased, and limits on the total transaction costs incurred. They provided numerical results using a standard solver (Cplex). [4] proposed a large-scale linear optimization model for enhanced index tracking, which selects the optimal portfolio based on a new stochastic dominance criterion and designed an effective constraint generation technique to solve the model. [5] proposed a partial replication strategy to construct a risk-averse enhanced index fund. By defining asset returns and return covariance terms as random variables to account for parameter estimation risk, they developed a stochastic mixed-integer nonlinear model. [6] presented an empirical study analyzing the effectiveness of a portfolio selection model based on second-order stochastic dominance (SSD) in the context of enhanced indexing.

The goal of EITP is to minimize tracking error while maximizing excess returns relative to the benchmark. Therefore, this problem is essentially a multi-objective optimization problem. [7] proposed a multi-objective optimization approach for EITP, providing a framework where the objectives are defined as maximizing the degree of outperformance relative to the benchmark and minimizing the cumulative error of underperformance, with transaction costs restricted in the constraints. The paper introduced a disturbance-resistant multi-objective optimization algorithm to solve the enhanced index tracking problem. [8] proposed a linear bi-objective optimization method, which maximizes the average excess return of the portfolio relative to the benchmark during the learning phase and minimizes the maximum downside deviation of portfolio returns from the market index, solving it efficiently to optimality using standard linear programming techniques. [9] proposed a bi-objective mixed-integer linear programming formulation, provided computational results for a set of benchmark instances, and then designed a heuristic process to approximate the Pareto optimal solution set.

The multi-objective optimization model results in a set of near-optimal solutions, and the specific solution still requires subjective selection by the decision-maker. Therefore, many scholars have considered converting the bi-objective problem into a single-objective problem. [10] made an appropriate trade-off between the objective functions of tracking error and excess return in the enhanced index tracking problem, and then solved the problem in two steps. First, they selected stocks that statistically represent the index and limited the number of stocks in the tracking portfolio by considering a subset of stocks. Second, the allocation of the tracking portfolio sets the weights for each stock. Other scholars have also established single-objective models based on the ratio of tracking error to excess return. [11] applied the Omega ratio for the first time and proposed two optimization models, showing that both models can be converted into linear programming models.

In addition, some models and methods proposed in the literature have considered risk control to some extent. [12] were the first to attempt using the return-to-risk ratio in the context of enhanced indexing. The authors introduced a nonlinear optimization model based on maximizing the modified Sortino ratio and solved it using a genetic algorithm. [13] were the first to apply the theoretical framework of the risk-return ratio model to the enhanced index tracking problem. They proposed a novel bi-criteria optimization model based on Conditional Value-at-Risk (CVaR), using the risk-return ratio as the objective. [14] used the two-tail mixed Conditional Value-at-Risk (TMCVaR) measure for index tracking. [15] pointed out that when using index-based investment strategies for portfolio management, the tracking portfolio also suffers losses when the target index declines. Therefore, it is necessary to incorporate downside risk constraints into the enhanced index model. CVaR was introduced as a constraint in the general index tracking model to control the downside risk of the portfolio composed of the benchmark index component stocks.

This paper considers that in enhanced index tracking investments, investors seek to have portfolio returns exceed the benchmark index returns while avoiding portfolio returns falling below the benchmark index returns. Based on [10], we have constructed an enhanced index model with the weighted sum of tracking error and excess returns as the objective function. In particular, to better meet the needs of investors, we use downside risk to measure tracking error. According to [15], we incorporate the Mean Absolute Deviation (MAD) as a lower bound constraint to effectively control the downside risk of the tracking portfolio. Compared to more complex measurement methods, MAD is simple, robust, and easy to implement, offering significant advantages in controlling downside risk and preventing large losses.

The remainder of this paper is organized as follows. In Section 2, we use a non-parametric method to derive an estimator for the MAD. In Section 3, we prove that the non-parametric MAD estimator is a convex function of portfolio positions. In Section 4, We have developed an enhanced index model with the MAD constraint and proved that the model is a convex optimization problem. In Section 5, we conducted an empirical study that specifically analyzes the model's ability to control downside risk. Section 6 provides a conclusion.

## 2. Non-Parametric Estimation of the MAD

Let the asset return be a random variable $X$, and the target return $\alpha$ is a value set in advance based on the investor's risk preference or wealth status, typically taken as 0, the risk-free rate, or the expected return. The Mean Absolute Deviation (MAD) can be defined as

$$\begin{aligned}\mathrm{MAD}_{\alpha}(X) &= \mathbb{E}\left[|\alpha - X|\right] \\ &= \int_{-\infty}^{\infty} |\alpha - x| f(x)\mathrm{d}x \\ &= \int_{-\infty}^{\alpha} (\alpha - x) f(x)\mathrm{d}x + \int_{\alpha}^{\infty} (x-\alpha) f(x)\mathrm{d}x.\end{aligned} \tag{2.1}$$

To obtain the analytical expression of the MAD in Equation (2.1), the density function of asset returns must be defined. However, in practice, the density function is usually unknown and must be estimated from historical return data. Common estimation methods include parametric, semi-parametric, and non-parametric approaches. Parametric and semi-parametric methods assume a specific distribution and estimate its parameters, but they depend on model assumptions, which may introduce biases. In contrast, non-parametric methods avoid assumptions and estimate the distribution directly from historical data. This typically provides more accurate and reliable risk assessments. Let $x_t, t = 1, 2, \cdots, T$ be the sample of $X$, then the non-parametric kernel estimate of $f(x)$ is

$$\hat{f}(x) = \frac{1}{Th}\sum_{t=1}^{T} k\left(\frac{x - x_t}{h}\right). \tag{2.2}$$

$k(y)$ is the kernel function, and $h$ is the bandwidth, where the Gaussian kernel function is $k(y) = (2\pi)^{-1/2}\mathrm{e}^{-y^2/2}$, and the bandwidth can be selected according to the algorithm rules.

$$h = c_0\hat{\sigma}(X) = c_0\sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(x_t - \bar{x})^2}. \tag{2.3}$$

where $c_0 = 1.06\times T^{-1/5}$ is a constant, and $\bar{x} = \frac{1}{T}\sum_{t=1}^{T} x_t$. The non-parametric estimator of the MAD is then given by

$$\begin{aligned}\widehat{\mathrm{MAD}}_{\alpha}(X) &= \int_{-\infty}^{\infty} |\alpha - x| f(x)\mathrm{d}x \\ &= \int_{-\infty}^{\alpha} (\alpha - x) f(x)\mathrm{d}x + \int_{\alpha}^{\infty} (x-\alpha) f(x)\mathrm{d}x\end{aligned}$$

$$= \int_{-\infty}^{\alpha} (\alpha - x) \frac{1}{Th} \sum_{t=1}^{T} k\left(\frac{x - x_t}{h}\right) \mathrm{d}x + \int_{\alpha}^{\infty} (x - \alpha) \frac{1}{Th} \sum_{t=1}^{T} k\left(\frac{x - x_t}{h}\right) \mathrm{d}x$$
$$= \frac{1}{T} \sum_{t=1}^{T} \int_{-\infty}^{\frac{\alpha - x_t}{h}} (\alpha - x_t - hy) k(y) \mathrm{d}y + \frac{1}{T} \sum_{t=1}^{T} \int_{\frac{\alpha - x_t}{h}}^{\infty} (x_t + hy - \alpha) k(y) \mathrm{d}y. \tag{2.4}$$

## 3. Convexity of the Non-Parametric MAD Estimator in Portfolio Positions

Let the return of a stock index in the market be a random variable $r_t$. This index consists of $N$ constituent stocks, and a tracking portfolio is constructed using $n(n \le N)$ of these constituent stocks. Let $\boldsymbol{r} = (r_1, r_2, \cdots, r_n)^\top$ be the return vector of the $n$ constituent stocks, and $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)^\top$ be the portfolio weights invested in the $n$ constituent stocks. Then, the return of the tracking portfolio is $\boldsymbol{a}^\top \boldsymbol{r}$. Let $\{r_t\}_{t=1}^{\top}$ and $\{r_{I,t}\}_{t=1}^{\top}$ represent the return samples of the $n$ constituent stocks and the index, respectively, where $\boldsymbol{r}_t = (r_{1t}, r_{2t}, \cdots, r_{nt})^\top$. Then, the return sample of the tracking portfolio is $\boldsymbol{a}^\top \boldsymbol{r}_t$, where $t = 1, 2, \cdots, T$.

Let $X = \boldsymbol{a}^\top \boldsymbol{r}$ and $x_t = \boldsymbol{a}^\top \boldsymbol{r}_t$. Then, according to Equation (2.4), the non-parametric estimator of the MAD for the tracking portfolio is given by

$$\widehat{\mathrm{MAD}}_{\alpha}\left(\boldsymbol{a}^\top \boldsymbol{r}\right) = \frac{1}{T} \sum_{t=1}^{T} \int_{-\infty}^{\frac{\alpha - \boldsymbol{a}^\top \boldsymbol{r}_t}{h}} \left(\alpha - \boldsymbol{a}^\top \boldsymbol{r}_t - hy\right) k(y) \mathrm{d}y + \frac{1}{T} \sum_{t=1}^{T} \int_{\frac{\alpha - \boldsymbol{a}^\top \boldsymbol{r}_t}{h}}^{\infty} \left(\boldsymbol{a}^\top \boldsymbol{r}_t + hy - \alpha\right) k(y) \mathrm{d}y.$$

Let $\xi_t = \frac{\alpha - x_t}{h}$, $\Phi_0(\xi_t) = \int_{-\infty}^{\xi_t} k(y) \mathrm{d}y$, $\Phi_1(\xi_t) = \int_{-\infty}^{\xi_t} y k(y) \mathrm{d}y$, $\Phi_0'(\xi_t) = \int_{\xi_t}^{\infty} k(y) \mathrm{d}y$, $\Phi_1'(\xi_t) = \int_{\xi_t}^{\infty} y k(y) \mathrm{d}y$. The above expression can be simplified as

$$\widehat{\mathrm{MAD}}_{\alpha}\left(\boldsymbol{a}^\top \boldsymbol{r}\right) = -\frac{1}{T} \sum_{t=1}^{T} \left[\left(\boldsymbol{a}^\top \boldsymbol{r}_t - \alpha\right) \Phi_0(\xi_t) + h \Phi_1(\xi_t)\right] + \frac{1}{T} \sum_{t=1}^{T} \left[\left(\boldsymbol{a}^\top \boldsymbol{r}_t - \alpha\right) \Phi_0'(\xi_t) + h \Phi_1'(\xi_t)\right]. \tag{3.1}$$

The bandwidth is determined according to Equation (2.3).

$$\begin{aligned} h &= c_0 \hat{\sigma}\left(\boldsymbol{a}^\top \boldsymbol{r}\right) = c_0 \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} \left(\boldsymbol{a}^\top \boldsymbol{r}_t - \boldsymbol{a}^\top \bar{\boldsymbol{r}}\right)^2} \\ &= c_0 \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} \boldsymbol{a}^\top (\boldsymbol{r}_t - \bar{\boldsymbol{r}})(\boldsymbol{r}_t - \bar{\boldsymbol{r}})^\top \boldsymbol{a}} \\ &= c_0 \sqrt{\boldsymbol{a}^\top \hat{\boldsymbol{\Sigma}} \boldsymbol{a}}. \end{aligned} \tag{3.2}$$

where $\hat{\boldsymbol{\Sigma}} = \frac{1}{T-1} \sum_{t=1}^{T} (\boldsymbol{r}_t - \bar{\boldsymbol{r}})(\boldsymbol{r}_t - \bar{\boldsymbol{r}})^\top$, $\bar{\boldsymbol{r}} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{r}_t$.

**Lemma 3.1.** *The bandwidth* $h = c_0 \sqrt{\boldsymbol{a}^\top \hat{\boldsymbol{\Sigma}} \boldsymbol{a}}$ *is a convex function of the portfolio position* $\boldsymbol{a}$.

**Proof.** The derivative of $h$ with respect to $\boldsymbol{a}$ is

$$\frac{\partial h}{\partial \boldsymbol{a}} = C_0 \frac{\hat{\boldsymbol{\Sigma}} \boldsymbol{a}}{\sqrt{\boldsymbol{a}^\top \hat{\boldsymbol{\Sigma}} \boldsymbol{a}}}$$

Further, taking the derivative with respect to $\boldsymbol{a}^\top$

$$\frac{\partial h}{\partial \boldsymbol{a}\partial \boldsymbol{a}^\top}=C_0\frac{\hat{\Sigma}(\boldsymbol{a}^\top\hat{\Sigma}\boldsymbol{a})^{\frac{1}{2}}-\hat{\Sigma}\boldsymbol{a}(\boldsymbol{a}^\top\hat{\Sigma}\boldsymbol{a})^{-\frac{1}{2}}\boldsymbol{a}^\top\hat{\Sigma}}{\boldsymbol{a}^\top\hat{\Sigma}\boldsymbol{a}}=C_0\frac{\hat{\Sigma}\boldsymbol{a}\hat{\Sigma}\boldsymbol{a}-\hat{\Sigma}\boldsymbol{a}\boldsymbol{a}^\top\hat{\Sigma}}{\left(\boldsymbol{a}^\top\hat{\Sigma}\boldsymbol{a}\right)^{\frac{3}{2}}}.$$

It follows that $\boldsymbol{a}^\top\boldsymbol{\Sigma}\boldsymbol{a}\ge 0$. Let $\hat{\Sigma}=\boldsymbol{p}\boldsymbol{p}^\top$, and since

$$\begin{aligned}&\boldsymbol{X}^\top\left(\hat{\Sigma}\boldsymbol{a}\hat{\Sigma}\boldsymbol{a}-\hat{\Sigma}\boldsymbol{a}\boldsymbol{a}^\top\hat{\Sigma}\right)\boldsymbol{X}\\&=\boldsymbol{X}^\top\hat{\Sigma}\boldsymbol{a}\hat{\Sigma}\boldsymbol{a}\boldsymbol{X}-\boldsymbol{X}^\top\hat{\Sigma}\boldsymbol{a}\boldsymbol{a}^\top\hat{\Sigma}\boldsymbol{X}\\&=\boldsymbol{X}^\top\boldsymbol{p}\boldsymbol{p}^\top\boldsymbol{a}\boldsymbol{p}\boldsymbol{p}^\top\boldsymbol{a}\boldsymbol{X}-\boldsymbol{X}^\top\boldsymbol{p}\boldsymbol{p}^\top\boldsymbol{a}\boldsymbol{a}^\top\boldsymbol{p}\boldsymbol{p}^\top\boldsymbol{X}\\&=\left(\boldsymbol{p}^\top\boldsymbol{a}\right)^\top\left(\boldsymbol{p}^\top\boldsymbol{a}\right)\left(\boldsymbol{p}^\top\boldsymbol{X}\right)^\top\left(\boldsymbol{p}\boldsymbol{X}\right)-\left(\left(\boldsymbol{p}^\top\boldsymbol{a}\right)^\top\boldsymbol{p}^\top\boldsymbol{X}\right)^2\\&=\left\|\boldsymbol{p}^\top\boldsymbol{a}\right\|^2\left\|\boldsymbol{p}^\top\boldsymbol{X}\right\|^2-\left(\boldsymbol{p}^\top\boldsymbol{a}\cdot\boldsymbol{p}^\top\boldsymbol{X}\right)^2\ge 0\end{aligned}$$

Therefore, it follows that $\frac{\partial h}{\partial \boldsymbol{a}\partial \boldsymbol{a}^\top}$ is a positive semi-definite matrix, *i.e.*, the bandwidth $h=c_0\sqrt{\boldsymbol{a}^\top\hat{\Sigma}\boldsymbol{a}}$ is a convex function of the portfolio position $\boldsymbol{a}$.

**Proposition 3.1.** *The non-parametric estimator of the MAD,* $\widehat{MAD}_\alpha\left(\boldsymbol{a}^\top\boldsymbol{r}\right)$, *is a convex function of the portfolio position* $\boldsymbol{a}$.

**Proof.** According to Equation (3.1), let

$$\begin{aligned}F\left(\boldsymbol{a}\right)=\widehat{\mathrm{MAD}}_\alpha\left(\boldsymbol{a}^\top\boldsymbol{r}\right)=&-\frac{1}{T}\sum_{t=1}^{T}\left[\left(\boldsymbol{a}^\top\boldsymbol{r}_t-\alpha\right)\Phi_0\left(\xi_t\right)+h\Phi_1\left(\xi_t\right)\right]\\&+\frac{1}{T}\sum_{t=1}^{T}\left[\left(\boldsymbol{a}^\top\boldsymbol{r}_t-\alpha\right)\Phi_0'\left(\xi_t\right)+h\Phi_1'\left(\xi_t\right)\right].\end{aligned}\tag{3.3}$$

Take the derivative of both sides of Equation (3.3) with respect to $\xi_i$, and based on the definition of $\xi_i$, we get

$$\begin{aligned}\frac{\partial F\left(\boldsymbol{a}\right)}{\partial \xi_t}&=-\frac{2}{T}\sum_{t=1}^{T}\left(\left(\boldsymbol{a}^\top\boldsymbol{r}_t-\alpha\right)k\left(\xi_t\right)+h\xi_t k\left(\xi_t\right)\right)\\&=-\frac{2}{T}\sum_{t=1}^{T}k\left(\xi_t\right)\left(\boldsymbol{a}^\top\boldsymbol{r}_t-\alpha+h\xi_t\right)=0\end{aligned}\tag{3.4}$$

Take the derivative of $\xi_t$ with respect to $\boldsymbol{a}$, and based on the definition of $\xi_t$, we get

$$\frac{\partial \xi_t}{\partial \boldsymbol{a}}=-\frac{1}{h}\boldsymbol{r}_t-\frac{\alpha-\boldsymbol{a}^\top\boldsymbol{r}_t}{h^2}\frac{\partial h}{\partial \boldsymbol{a}}=-\frac{1}{h}\boldsymbol{r}_t-\frac{\xi_t}{h}\frac{\partial h}{\partial \boldsymbol{a}}\tag{3.5}$$

Take the derivative of both sides of Equation (3.3) with respect to $\boldsymbol{a}$, and using Equation (3.4), we get

$$\begin{aligned}\frac{\partial F\left(\boldsymbol{a}\right)}{\partial \boldsymbol{a}}=&-\frac{1}{T}\sum_{t=1}^{T}\left(\Phi_0\left(\xi_t\right)\boldsymbol{r}_t+\Phi_1\left(\xi_t\right)\frac{\partial h}{\partial \boldsymbol{a}}\right)\\&+\frac{1}{T}\sum_{t=1}^{T}\left(\Phi_0'\left(\xi_t\right)\boldsymbol{r}_t+\Phi_1'\left(\xi_t\right)\frac{\partial h}{\partial \boldsymbol{a}}\right)\end{aligned}\tag{3.6}$$

Furthermore, take the derivative of both sides of Equation (3.6) with respect to

$\boldsymbol{a}$, and using Equation (3.5), we get

$$\begin{aligned}
\frac{\partial^2 F(\boldsymbol{a})}{\partial \boldsymbol{a} \partial \boldsymbol{a}^{\top}} &= -\frac{1}{T}\sum_{t=1}^{T}\left(k(\xi_t)\boldsymbol{r}_t \frac{\partial \xi_t}{\partial \boldsymbol{a}^{\top}} + \xi_t k(\xi_t)\frac{\partial h}{\partial \boldsymbol{a}}\frac{\partial \xi_t}{\partial \boldsymbol{a}^{\top}} + \Phi_1(\xi_t)\frac{\partial^2 h}{\partial \boldsymbol{a}\partial \boldsymbol{a}^{\top}}\right) \\
&\quad + \frac{1}{T}\sum_{t=1}^{T}\left(-k(\xi_t)\boldsymbol{r}_t \frac{\partial \xi_t}{\partial \boldsymbol{a}^{\top}} - \xi_t k(\xi_t)\frac{\partial h}{\partial \boldsymbol{a}}\frac{\partial \xi_t}{\partial \boldsymbol{a}^{\top}} + \Phi_1'(\xi_t)\frac{\partial^2 h}{\partial \boldsymbol{a}\partial \boldsymbol{a}^{\top}}\right) \\
&= -\frac{2}{T}\sum_{t=1}^{T} k(\xi_t)\left(\boldsymbol{r}_t + \xi_t \frac{\partial h}{\partial \boldsymbol{a}}\right)\frac{\partial \xi_t}{\partial \boldsymbol{a}}^{\top} \\
&\quad - \frac{1}{T}\frac{\partial^2 h}{\partial \boldsymbol{a}\partial \boldsymbol{a}^{\top}}\sum_{t=1}^{T}\Phi_1(\xi_t) + \frac{1}{T}\frac{\partial^2 h}{\partial \boldsymbol{a}\partial \boldsymbol{a}^{\top}}\sum_{t=1}^{T}\Phi_1'(\xi_t) \\
&= \frac{2}{T}\sum_{t=1}^{T} k(\xi_t)\frac{\partial \xi_t}{\partial \boldsymbol{a}}\frac{\partial \xi_t}{\partial \boldsymbol{a}}^{\top} - \frac{1}{T}\frac{\partial^2 h}{\partial \boldsymbol{a}\partial \boldsymbol{a}^{\top}}\sum_{t=1}^{T}\Phi_1(\xi_t) + \frac{1}{T}\frac{\partial^2 h}{\partial \boldsymbol{a}\partial \boldsymbol{a}^{\top}}\sum_{t=1}^{T}\Phi_1'(\xi_t).
\end{aligned} \tag{3.7}$$

Since the kernel function $k(\xi_t) \geq 0$, the sample size $T > 0$, and the bandwidth $h > 0$, it follows that $\frac{2h}{T}\sum_{t=1}^{T} k(\xi_t)\frac{\partial \xi_t}{\partial \boldsymbol{a}}\frac{\partial \xi_t}{\partial \boldsymbol{a}}^{\top}$ is a positive semi-definite matrix. The function $k(y)$ is the density function of the standard normal distribution, and after simple derivation, we obtain

$$\Phi_1(\xi_t) = \int_{-\infty}^{\xi_t} y k(y)\,\mathrm{d}y = -k(\xi_t) \leq 0$$

$$\Phi_1'(\xi_t) = \int_{-\infty}^{\xi_t} y k(y)\,\mathrm{d}y = k(\xi_t) \geq 0$$

Since $-\frac{1}{T} < 0$, $\Phi_1(\xi_t) \leq 0$, $\Phi_1'(\xi_t) \geq 0$, and according to Lemma 3.1, $\frac{\partial^2 h}{\partial \boldsymbol{a}\partial \boldsymbol{a}^{\top}}$ is a positive semi-definite matrix, it follows that the last two terms are also positive semi-definite matrices. Therefore, combining everything, we conclude that $\frac{\partial^2 F(\boldsymbol{a})}{\partial \boldsymbol{a}\partial \boldsymbol{a}^{\top}}$ is a positive semi-definite matrix, meaning that $F(\boldsymbol{a})$ is a convex function of $\boldsymbol{a}$.

## 4. The Enhanced Index Model under MAD Constraint

In this section, we construct the enhanced index model and discuss the objective function and constraints of the model. This paper emphasizes that traditional enhanced index models do not include a downside risk constraint, which may lead to the risk of the tracking portfolio deviating negatively from the benchmark index, a major concern in the current Chinese market. Therefore, we incorporate a constraint based on the MAD into the model. We also prove that the enhanced index model with the non-parametric MAD constraint is a convex optimization problem.

For the enhanced index tracking problem, our goal is to generate a portfolio that seeks to achieve relatively high excess returns while minimizing tracking error. Tracking error refers to the difference between the actual returns of the portfolio and the returns of the benchmark index. This difference can be adjusted according to specific circumstances and preferences, for example, by using metrics such as mean squared error, root mean squared error, downside risk, or other risk measures. In enhanced index investing, investors expect the portfolio to outperform the benchmark index, rather than merely tracking it. Therefore, we consider using downside risk to measure tracking error, and define excess return as the average difference between the portfolio's actual return and the benchmark index's return, which better aligns with the risk perception of enhanced index investors. The objective of the enhanced index model is to minimize the linear combination of tracking error $TE$ and excess return $ER$

$$\lambda TE-(1-\lambda)ER=\lambda\left(\sum_{t=1}^{T}\omega_t\left(\max\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}^{\top}\boldsymbol{r}_t,0\right)\right)^{\gamma}\right)^{1/\gamma}-(1-\lambda)\sum_{t=1}^{T}\omega_t\left(\boldsymbol{a}^{\top}\boldsymbol{r}_t-\boldsymbol{r}_{I,t}\right). \tag{4.1}$$

where $\omega_t$ represents the probability of the *t*-th outcome, typically taken as equal probability, *i.e.*, $\omega_t=\frac{1}{T}$. $\gamma$ is any positive integer greater than zero, and different values of $\gamma$ can be set. When $\gamma=2$, the tracking error is the Lower Partial Deviation. The $\gamma$-th power of the tracking error is used to eliminate the influence of dimensionality, ensuring that the units of the tracking error and excess return are consistent.

We introduce the MAD of the tracking portfolio to control the downside risk. By embedding the non-parametric estimator of MAD from Equation (3.1) into model (4.1), and assuming that the maximum downside risk the investor can bear is $v$, we obtain the enhanced index model based on the non-parametric MAD constraint. We require that the portfolio weights $\boldsymbol{a}=(a_1,a_2,\cdots,a_n)^{\mathrm{T}}$ invested in $n$ constituent stocks do not involve short positions, *i.e.*, $\boldsymbol{a}_i\geq 0,\forall i=1,\cdots,n$, and that the investment weights in each stock are normalized, *i.e.*, $\sum_{i=1}^{n}\boldsymbol{a}_i=1$.

$$P(\gamma,\lambda)=\begin{cases}\min\limits_{\boldsymbol{a}\in\mathbb{R}^n}\lambda\left(\sum_{t=1}^{T}\omega_t\left(\max\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}^{\top}\boldsymbol{r}_t,0\right)\right)^{\gamma}\right)^{1/\gamma}-(1-\lambda)\sum_{t=1}^{T}\omega_t\left(\boldsymbol{a}^{\top}\boldsymbol{r}_t-\boldsymbol{r}_{I,t}\right).\\ \text{s.t.}\\ \widehat{\mathrm{MAD}}_{\alpha}\left(\boldsymbol{a}^{\top}\boldsymbol{r}\right)=-\frac{1}{T}\sum_{t=1}^{T}\left[\left(\boldsymbol{a}^{\top}\boldsymbol{r}_t-\alpha\right)\Phi_0\left(\xi_t\right)+h\Phi_1\left(\xi_t\right)\right]\\ \qquad\qquad+\frac{1}{T}\sum_{t=1}^{T}\left[\left(\boldsymbol{a}^{\top}\boldsymbol{r}_t-\alpha\right)\Phi_0'\left(\xi_t\right)+h\Phi_1'\left(\xi_t\right)\right]\leq v.\end{cases}$$

**Theorem 4.1.** *For any positive integer* $\gamma\geq 1$, *if the feasible set* $\Omega$ *is nonempty, the enhanced index model* $P(\gamma,\lambda)$ *based on the non-parametric MAD is a convex optimization problem.*

**Proof.** In the model $P(\gamma,\lambda)$, besides the non-parametric Mean Absolute Deviation (MAD) constraint, all other constraints are linear, and the set of linear constraints is necessarily a convex set. According to Theorem 3.1, $\widehat{\mathrm{MAD}}_{\alpha}\left(\boldsymbol{a}^{\top}\boldsymbol{r}\right)$ is a convex function of the portfolio position $\boldsymbol{a}$. According to optimization theory, the lower level set of a convex function is a convex set. Therefore, the constraint set of the non-parametric MAD, $\mathrm{MAD}_{\alpha}\left(\boldsymbol{a}^{\top}\boldsymbol{r}\right)\leq v$, is a convex set, and thus the feasible set $\Omega$ of model $P(\gamma,\lambda)$ is a convex set. The objective function consists of two parts, with the second part being a linear function of the decision variable $\boldsymbol{a}$, and thus also a convex function of $\boldsymbol{a}$. Therefore, the following key result is to prove that $f(\boldsymbol{a})=\left(\sum_{t=1}^{T}\omega_t\left(\max\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}^{\top}\boldsymbol{r}_t,0\right)\right)^{\gamma}\right)^{1/\gamma}$ is a convex function of $\boldsymbol{a}$. To this end, we first present Lemma 4.1 and Lemma 4.2.

**Lemma 4.1 (Minkowski Inequality).** *If* $x_t, y_t > 0, t=1,2,\cdots,T$ *and* $\gamma \ge 1$, *then the following holds*

$$\left(\sum_{t=1}^{T}(x_t+y_t)^{\gamma}\right)^{1/\gamma} \le \left(\sum_{t=1}^{T}x_t^{\gamma}\right)^{1/\gamma}+\left(\sum_{t=1}^{T}y_t^{\gamma}\right)^{1/\gamma}.$$

**Lemma 4.2 (Triangle Inequality).** *If* $x, y \in \mathbb{R}$, *then the following holds*

$$|x+y| \le |x|+|y|.$$

**Proof.** For any two decision vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, and any real number $\kappa \in [0,1]$, based on Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned}
& f\left(\kappa\boldsymbol{a}_1+(1-\kappa)\boldsymbol{a}_2\right) \\
&=\left(\sum_{t=1}^{T}\omega_t\left(\max\left(\boldsymbol{r}_{I,t}-\left(\kappa\boldsymbol{a}_1+(1-\kappa)\boldsymbol{a}_2\right)^{\top}\boldsymbol{r}_t,0\right)\right)^{\gamma}\right)^{1/\gamma} \\
&=\left(\sum_{t=1}^{T}\omega_t\left(\frac{\boldsymbol{r}_{I,t}-\left(\kappa\boldsymbol{a}_1+(1-\kappa)\boldsymbol{a}_2\right)^{\top}\boldsymbol{r}_t}{2}+\left|\frac{\boldsymbol{r}_{I,t}-\left(\kappa\boldsymbol{a}_1+(1-\kappa)\boldsymbol{a}_2\right)^{\top}\boldsymbol{r}_t}{2}\right|\right)^{\gamma}\right)^{1/\gamma} \\
&=\left(\sum_{t=1}^{T}\omega_t\left(\frac{\kappa\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_1^{\top}\boldsymbol{r}_t\right)}{2}+\frac{\left|\kappa\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_1^{\top}\boldsymbol{r}_t\right)\right|}{2}\right.\right. \\
&\qquad\left.\left.+\frac{(1-\kappa)\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_2^{\top}\boldsymbol{r}_t\right)}{2}+\frac{\left|(1-\kappa)\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_2^{\top}\boldsymbol{r}_t\right)\right|}{2}\right)^{\gamma}\right)^{1/\gamma} \\
&\le\left(\sum_{t=1}^{T}\omega_t\left(\kappa\max\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_1^{\top}\boldsymbol{r}_t,0\right)+(1-\kappa)\max\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_2^{\top}\boldsymbol{r}_t,0\right)\right)^{\gamma}\right)^{1/\gamma} \\
&=\left(\sum_{t=1}^{T}\left(\sqrt[\gamma]{\omega_t}\,\kappa\max\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_1^{\top}\boldsymbol{r}_t,0\right)+\sqrt[\gamma]{\omega_t}\,(1-\kappa)\max\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_2^{\top}\boldsymbol{r}_t,0\right)\right)^{\gamma}\right)^{1/\gamma} \\
&\le\left(\sum_{t=1}^{T}\left(\sqrt[\gamma]{\omega_t}\,\kappa\max\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_1^{\top}\boldsymbol{r}_t,0\right)\right)^{\gamma}\right)^{1/\gamma}+\left(\sum_{t=1}^{T}\left(\sqrt[\gamma]{\omega_t}\,(1-\kappa)\max\left(\boldsymbol{r}_{I,t}-\boldsymbol{a}_2^{\top}\boldsymbol{r}_t,0\right)\right)^{\gamma}\right)^{1/\gamma} \\
&=\kappa f\left(\boldsymbol{a}_1\right)+(1-\kappa)f\left(\boldsymbol{a}_2\right).
\end{aligned}$$

Therefore, $f(\boldsymbol{a})$ and $\lambda f(\boldsymbol{a})$ are convex functions of the portfolio position $\boldsymbol{a}$.

## 5. Empirical Analysis

To further evaluate the performance of the enhanced index model proposed in this paper in real financial markets, we conducted an empirical analysis using the CSI 300 Index and its constituent stocks. The data is sourced from the baostock economic and financial database, specifically collecting daily closing price data of the CSI 300 Index and its constituent stocks from January 1, 2015, to December 31, 2024. The time trend of the CSI 300 Index is shown in Figure 1. To test the model's risk control ability during the downtrend of the index, this paper focuses on three downtrend periods of the index: February 20, 2017, to April 3, 2019 (Period 1), June 3, 2021, to May 5, 2023 (Period 2), and July 23, 2020, to February 4, 2021 (Period 3).


**Figure 1.** CSI 300 index trend and long-term downturn periods.

After performing the logarithmic difference, the CSI 300 index returns data (in %) are obtained. Table 1 presents the descriptive statistics of the index returns data for the three sample periods. From the mean and median values, the overall performance in all three periods is poor, indicating a downtrend. From the maximum, minimum, and standard deviation values, Period 1 and Period 3 show high volatility and risk, indicating significant market weakness. From the skewness, kurtosis, and JB statistic, the return distributions in all three periods deviate from normal distribution, indicating frequent extreme fluctuations in these periods. From the lower partial moment, Period 1 has the highest downside risk, while Periods 2 and 3 have relatively lower downside risks.

**Table 1.** Descriptive statistics table.

| Indicator | Period Length | Mean | Median | Standard Deviation | Minimum | Maximum | Skewness | Kurtosis | JB Statistic | Lower Partial Moment |
|---|---|---|---|---|---|---|---|---|---|---|
| **Period 1** | 519 | −0.0744 | −0.0775 | 1.4614 | −17.9352 | 7.9413 | −3.2446 | 4.4193 | 430.6320 | 1.2700 |
| **Period 2** | 466 | −0.0594 | 0.0000 | 0.9606 | −6.0865 | 2.7895 | −0.8387 | 4.3937 | 42.8552 | 0.1350 |
| **Period 3** | 134 | −0.0398 | −0.1930 | 1.1212 | −3.9515 | 4.7067 | 0.6954 | 2.6010 | 48.2127 | −0.0940 |

Next, we use the subgradient descent (SGD) to assess the model's ability to control risk during the long-term downtrend of the index. Specifically, the data is divided into a training set (Period 1), a validation set (Period 3), and a test set (Period 2). This division ensures the temporal order of the data and allows the training, validation, and testing processes to more reasonably reflect the model's effectiveness and stability.

In addition, the core of the enhanced index model is to track the index and generate excess returns using a small number of constituent stocks. Therefore, before optimizing the model, it is necessary to determine the set of constituent stocks that will be included in the tracking portfolio. To reduce the complexity of model solving, this paper focuses on the fundamental issues of the model itself, using the Beta values of the constituent stocks for stock selection, and tracking the index and generating excess returns using the selected stocks. Specifically, the Beta values of the constituent stocks in the CSI 300 index were calculated, and the 30 constituent stocks with Beta values closest to 1 were selected to track the index.

Since the enhanced index model can better achieve the goal of generating excess returns while tracking the index, the enhanced index model is used in the empirical analysis with $\lambda = 0.5$. Additionally, to evaluate performance under different MAD constraints, three different risk constraint values are selected: $v = 1.1$, $v = 1.2$, and $v = 1.3$. Based on the earlier definition of symbols, the return of the tracking portfolio is $a_r r_t$, and the return of the index is $r_{I_t}$, where $t = 1, 2, \cdots, T$. The following indicators are defined to compare the performance of the tracking portfolio: Excess Return ($ER$), Information Ratio $IR = \frac{ER}{TE}$, Standard Deviation $\hat{\sigma} = \sqrt{\boldsymbol{a}^\top \hat{\Sigma} \boldsymbol{a}}$, Sharpe Ratio $SR = \frac{ER}{\hat{\sigma}}$, and Downside Risk based on the risk-free rate.

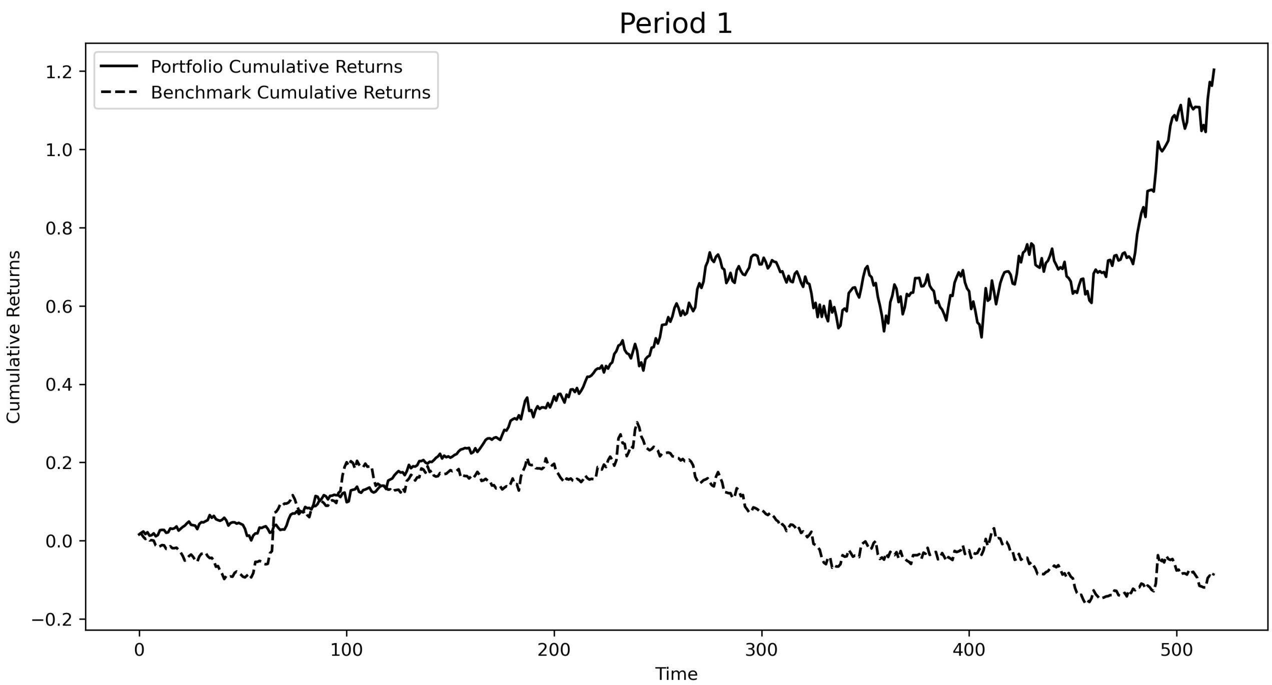
**Table 2.** Sample period indicators.

| Sample | Indicator | Excess Return | Information Ratio | Standard Deviation | Sharpe Ratio | Downside Risk |
|---|---|---|---|---|---|---|
| **Period 1** | $v = 0.05$ | 0.0067 | 0.0967 | 0.0117 | 0.1371 | 0.0122 |
| | $v = 0.10$ | 0.0088 | 0.1001 | 0.0083 | 0.1446 | 0.0075 |
| | $v = 0.15$ | 0.0068 | 0.0997 | 0.0120 | 0.1343 | 0.0173 |
| | **Index** | 0.0000 | 0.0000 | 0.0123 | 0.0000 | 0.0324 |
| **Period 3** | $v = 0.05$ | 0.0054 | 0.0209 | 0.0327 | 0.0290 | 0.0133 |
| | $v = 0.10$ | 0.0083 | 0.0490 | 0.0130 | 0.0461 | 0.0127 |
| | $v = 0.15$ | 0.0054 | 0.0202 | 0.0428 | 0.0278 | 0.0135 |
| | **Index** | 0.0000 | 0.0000 | 0.0246 | 0.0000 | 0.0168 |
| **Period 2** | $v = 0.05$ | 0.0061 | 0.0209 | 0.0115 | 0.0760 | 0.0079 |
| | $v = 0.10$ | 0.0090 | 0.0709 | 0.0112 | 0.0920 | 0.0076 |
| | $v = 0.15$ | 0.0070 | 0.0675 | 0.0113 | 0.0870 | 0.0077 |
| | **Index** | 0.0000 | 0.0000 | 0.0189 | 0.0000 | 0.0084 |

Table 2 presents a comparison of the returns and risks of the tracking portfolio under different parameters $v$ for each sample period. From Table can be observed that in all sample periods, the investment strategy based on the enhanced index model consistently generates positive excess returns, information ratio, and positive Sharpe ratio. This indicates that, from the perspective of returns and risk-adjusted returns, the investment strategy developed in this paper outperforms the traditional index strategy. Further analysis of the standard deviation and downside risk shows that the standard deviation and downside risk of the investment strategy are lower than those of the benchmark index (except in Period 3). Specifically, in Period 3, when $v = 0.10$, both the standard deviation and downside risk of the investment strategy are smaller than those of the benchmark index, suggesting that the investment strategy developed in this paper performs better in terms of risk control compared to the benchmark index.

Additionally, comparing the results for different $v$ values reveals that when $v = 0.10$, the excess return is maximized, the standard deviation is minimized, and consequently, the information ratio and Sharpe ratio also reach their maximum values. Under this condition, the downside risk is minimized. Meanwhile, $v = 0.05$ and $v = 0.15$ yield similar results, both outperforming the benchmark index and showing an ability to control risk. This indicates that by introducing an appropriate risk constraint level, risk can be effectively controlled while generating excess returns. Therefore, the enhanced index model developed in this paper proves to be effective in investment management within real financial markets.

Figure 2 visually demonstrates the cumulative returns of the investment strategy when $v = 0.10$ across different sample periods. As can be seen from the figure, the cumulative returns of the tracking portfolio constructed in this paper



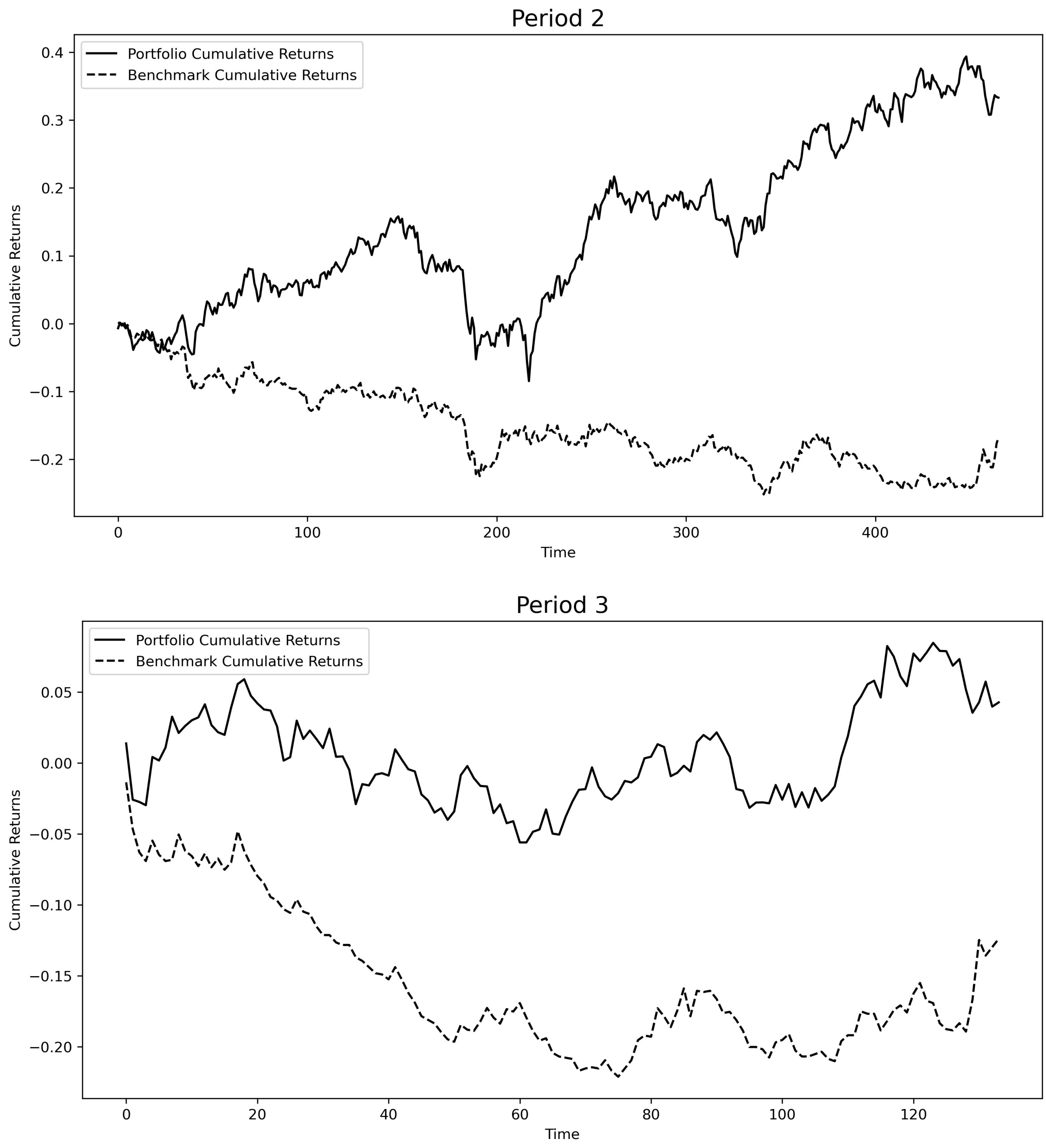



**Figure 2.** Comparison of portfolio cumulative returns and benchmark index cumulative returns.

closely follow the cumulative returns of the index, while exceeding the index returns. This indicates that the investment strategy developed in this paper achieves the investor's objective by both closely tracking the index's trend and generating excess returns. Specifically, in Periods 1 and 3, the downside risk is well controlled, as the cumulative return of the portfolio does not decline along with the index's cumulative return. This suggests that our model is better at controlling downside risk.

## 6. Conclusion

In this paper, we consider the risk of significant losses in the tracking portfolio when the market index experiences a downward jump, as the tracking portfolio tends to follow the index trend. Therefore, we introduce a downside risk constraint into the traditional enhanced index model and construct an enhanced index model under the constraint of MAD, aiming to effectively control the downside risk of the tracking portfolio and achieve excess returns. We prove that the model is a convex optimization problem and optimize it using SGD. Empirical research shows that the enhanced index model proposed in this paper, which considers the non-parametric MAD constraint, can effectively control downside risk.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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