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The Report of Homological Algebra

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Abstract

The current article intends to introduce the reader to the concept of injective and projective modules and to describe the CFT. We present a clear view to show the homological algebra and injective and projective modules.

Keywords

Homological Algebra, Category, Projective and Injective Module

1. Introduction

Before we get into the main text, let's talk about homological algebra. This work originated from the generalization of some simple topological relations.

This framework is sketched in Chapter 2, we'll start with simplicial sets, while discussing simple categories. And in this chapter we would like to talk more about simplicial complexes.

We'll start with module in Chapter 3, and from an algebraic point of view, discuss the projective and injective module with the free module, whose theme is the same as that of [1].

In Chapter 4, we'll show more detail about Projective and Injective, and knowledge about Yoneda Lemma. Here we can refer to [1] [2] more details can be referred to [3]-[5].

We will abstract the concept of simple sets from algebraic topology, and give a more fundamental understanding and thinking of homology theory. This will provide a transcendental reflection on many of our subsequent study directions which is to obtain global information from local information (Although the method may still be categorical). For example, we can think of sheaf theory as a kind of singular homology which I'll talk a lot about that in Chapter 1. The article can be thought as a simple one of Sur quelques points d'algèbre homologique [TOHOKU] [1957] without some derived category and with some abstract algebraic topology. More detail can be referred to [6]-[8].

2. Simplicial Sets and Cohomology

2.1. Simplicial Sets

In this section, we will present some brief observations of simplicial sets.

Definition 1.1. *Before defining other definitions, we need a bit of the analytic geometry of Euclidean space. Given a set* $\{a_0,\cdots,a_n\}$ *of points of* $\mathbb{R}^{\mathbb{N}}$, *this set is said to be geometrically independent if for any scalars* t_i, *the equations*

$$\sum_{i=0}^{n} t_i = 0 \text{ and } \sum_{i=0}^{n} t_i a_i = 0 \tag{1}$$

imply that $t_0 = t_1 = \cdots = t_n = 0$.

Definition 1.2. *Let* $\{a_0,\cdots,a_n\}$ *be a geometrically independent set in* $\mathbb{R}^{\mathbb{N}}$. *We define the n-simplex* Δ_n *spanned by* $a_0,\cdots,a_n$ *to be the set of all points x of* $\mathbb{R}^{\mathbb{N}}$ *such that*

$$x = \sum_{i=0}^{n} t_i a_i \text{, where } \sum_{i=0}^{n} t_i = 1 \tag{2}$$

and $t_i \ge 0$ for all *i*.

Definition 1.3. *The points* $a_0,\cdots,a_n$ *that span* Δ_n *are called the vertices of* Δ_n; *the number n is called the dimension of* Δ_n. *Any simplex spanned by a subset of* $\{a_0,\cdots,a_n\}$ *is called a face of* Δ_n. *The face of* Δ_n *spanned by* $a_0,\cdots,a_n$ *is called the face opposite* a_0. *Their union is called the boundary of* Δ_n, *which we denote as* $Bd\Delta_n$. *The interior of* Δ_n *is defined by the equation* $\text{Int}\Delta_n = \Delta_n - Bd\Delta_n$.

Definition 1.4. *A simplicial set is a family of sets* $X = (X_n), n = 0,1,\cdots$, *elements of* X_n *are n-simplices, and of map* $X(f): X_n \to X_m$, *one for each nondecreasing map* $f:[m]\to[n]$ *such the following conditions are satisfied:*

$$X(id) = id, \quad X(f \circ g) = X(g) \circ X(f) \tag{3}$$

For any nondecreasiong map $f:[m]\to[n]$ we define the "f-th face" Δ_f as the linear map $\Delta_m \to \Delta_n$ that maps any vertex $e_i \in \Delta_m$ into the vertex $e_{f(i)} \in \Delta_n, i = 0,\cdots,m$. The geometric realization of the simplicial set $\Delta[p]$ is the *p*-dimensional simplex Δ_p. We can describe $\Delta[p]$ as the simplicial set of all singular simplices of Δ_p that are compatible with the standard triangulation of Δ_p.

A topological space $|X|$ with the underlying set $\coprod_{n=0}^{\infty}(\Delta_n \times X_{(n)})/R$, where *R* is the weakest equivalence relation that identified $(s,x) \in \Delta_n \times X_{(n)}$ and $(t,y) \in \Delta_m \times X_{(m)}$ with

$$y = X(f)x, \quad s = \Delta_f(t) \tag{4}$$

for some increasing mapping $f:[m]\to[n]$. We denote the situation as in (4) by $(t,y) \mapsto (s,x)$. The canonical topology on $|X|$ is the weakest topology for which the canonical mapping $\coprod_{n=0}^{\infty}(\Delta_n \times X_{(n)})/R \to |X|$ is continuous.

Definition 1.5. *Let* Δ_p *be a simplex. Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. Each of these classes is called an orientation of* Δ_p. *The oriented simplex* $[v_0, v_1, v_2]$ *is indicated in the figure by drawing an arrow in the direction from* v_0 *to* v_1 *to*

v_2. *We can check that* $[v_1, v_2, v_0]$ *and* $[v_2, v_0, v_1]$ *are indicated by the same arrow. In particular,* 0*-simplex has only one element and only one orientation.*

Definition 1.6. *Let X be a simplicial set. An n-simplex* $x \in X_n$ *is said to be degenerate if there exists a surjective nondecreasing map* $f:[n] \to [m]$, $m < n$, *and an element* $y \in X_m$ *such that* $x = X(f)y$. *One can easily check that if x is nondegenerate and* $x = X(f)y$ *for some f and y, then f is an injection.*

Example 1.7. (Triangulation of the Product of Two Simplices) The product of two segments $[0,1] \times [0,1]$ is not a triangle but a square; it can be naturally divided into two triangles by a diagonal, of two diagonals one is singled out by the fact that its vertices are naturally ordered: $[00,11]$.

Generalizing this construction, we define the canonical triangulation $(X_{(n)}, X(f))$ of the product $\Delta_p \times \Delta_q$. One element of $X_{(n)}$ is a set of $n+1$ different pairs of integers $(i_0, j_0), \cdots, (i_n, j_n)$, where they are a directed set. Any increasing mapping $f:[m] \to [n]$ define $X(f)$ as follows:

$$X(f)\{(i_0, j_0), \cdots, (i_n, j_n)\} = \{(i'_0, j'_0), \cdots, (i'_n, j'_n)\} \tag{5}$$

where $i'_k = i_{f(k)}$, $j'_k = j_{f(k)}$.

Define a mapping

$$\theta_n : \Delta_n \times X_{(n)} \to \Delta_p \times \Delta_q \tag{6}$$

as follows: with the *x*-th simplex,

$$x = \{(i_0, j_0), \cdots, (i_n, j_n)\} \in X_n \tag{7}$$

θ_n associates the simplex Δ'_n in $\Delta_p \times \Delta_q \in \mathbb{R}^{p+q+2}$ spanned by the points (e_{i_a}, e'_{j_a}), $0 \le a \le n$, where e_i (resp. e'_i) is the *i*-th vertex of Δ_p (resp. Δ_q). Or, more formally, $\theta_n(\cdot, x): \Delta_n \to \Delta_p \times \Delta_q$ is a linear order-preserving mapping with the image Δ'_n.

We claim that there exists a commutative diagram.

$$\begin{tikzcd}
& \amalg_n(\triangle_n \times X_{(n)}) \arrow[dl] \arrow[dr, "\theta=\amalg_n\theta_n"] & \\
{|X|} \arrow[rr, "\varphi"'] & & \triangle_p \times \triangle_q
\end{tikzcd}$$

where φ is a bijection.

Definition 1.8. *Let G be a group. Let*

$$(BG)_n = G^n \tag{8}$$

and for $f:[m] \to [n]$ let

$$BG(f)(g_1, \cdots, g_n) = (h_1, \cdots, h_m) \tag{9}$$

where

$$h_i = \prod_{j=f(i-1)+1}^{f(i)} g_j, \quad h_i = e \quad \text{if} \quad f(i-1) = f(i) \tag{10}$$

Taking a example: the mapping $f:[3] \to [4]$ with $f(0) = 0$, $f(1) = f(2) = 2$,

$f(3)=4$, we can see $h_1 = g_1 g_2$; $h_2 = e$; $h_3 = g_3 g_4$ The geometric realization $|BG|$ of BG is called the classifying space of *G*.

The structure of the geometric realization of a simplicial set *X* can be clarified like that:

$$\tau' : \coprod_{n=0}^{\infty} \left(Ind\Delta_n \times X_{(n)} \right) \to |X| \tag{11}$$

where $X_{(n)}$ = the set of nondegenerate *n*-simplices of *X*.

2.2. Category I

In this section we will briefly introduce some category knowledge, the main of which is the addition category and abelian categories. More details can be referred to [7] [8]

Definition 1.9. *The aim is to fix some notations and to recall the axioms of universes. We do not intend neither to enter Set Theory, nor to say more about universes than what we need.*

For a set *u*, we denote by $P(u)$ the set of subsets of $u : P(u) = \{x; x \subset u\}$. For $x_1, \cdots, x_n$, we denote by $x_1, \cdots, x_n$ the set whose elements are $x_1, \cdots, x_n$.

A universe *U* is a set satisfying the following properties:

1) $\phi \in U$,

2) $u \in U$ implies $u \subset U$, (equivalently, $x \in U$ and $y \in x$ implies $y \subset U$, or else $U \subset P(U)$),

3) $u \in U$ implies $u \in U$,

4) $u \in U$ implies $P(u) \in U$,

5) if $I \in U$ and $u_i \in U$ for all $i \in I$, then $i \in I$, $u_i \in U$,

6) $\mathbb{N} \in U$

Following Grothendieck, we shall add an axiom to the Zermelo-Fraenkel theory, asking that for any set *X* there exists a universe *U* such that $X \in U$.

Let *U* be a universe.

1) A set is called a *U*-set if it belongs to *U*.

2) A set is called *U*-small if it is isomorphic to a set belonging to *U*.

We can also define **order**. But that's enough for what we'll talk about below.

Definition 1.10. *A category* $\mathcal{C}$ *consists of the following data.*

1) A family $Ob(\mathcal{C})$, whose members are called the objects of $\mathcal{C}$.

2) for all pairs (X,Y) of $Ob(\mathcal{C})$, a set $Hom_{\mathcal{C}}(X,Y)$, whose elements are called morphisms from *X* to *Y*.

3) for any triple (X,Y,Z) of $Ob(\mathcal{C})$, a map from $Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z)$ to $Hom_{\mathcal{C}}(X,Z)$, called the composition map, and denoted $(f,g) \to g \circ f$.

These data satisfying: the composition of morphisms is associative, for any $X \in Ob(\mathcal{C})$ there exists $id_X \in Hom_{\mathcal{C}}(X,X)$ such that $f \circ id_X = f$ and $id_X \circ g = g$ for an $f \in Hom_{\mathcal{C}}(X,Y)$ and any $g \in Hom_{\mathcal{C}}(Y,X)$.

Definition 1.11 *A subcategory* $\mathcal{C}'$ *of* $\mathcal{C}$ *is a category* $\mathcal{C}'$ *such that* $Ob(\mathcal{C}') \subset Ob(\wp)$ *and for any pair* (X,Y) *of* $Ob(\mathcal{C}')$, $Hom_{\mathcal{C}'}(X,Y) \subset Hom_{\mathcal{C}}(X,Y)$, *with the induced composition law, and*

$id_X \in Hom_{\mathcal{C}'}(X,X)$.

If moreover $Hom_{\mathcal{C}'}(X,Y)=Hom_{\mathcal{C}}(X,Y)$, then $\mathcal{C}'$ is called a full subcategory of $\mathcal{C}$.

Definition 1.12. *Let* $\mathcal{C}$ *be a category. The opposite category, denoted* $\mathcal{C}^\circ$, *is defined by:*

$$Ob(\mathcal{C}^\circ)=Ob(\mathcal{C})$$

$$Hom_{\mathcal{C}^\circ}(X,Y)=Hom_{\mathcal{C}}(Y,X)$$

Definition 1.13. *Let* $f:X\to Y$ *be a morphism in* $\mathcal{C}$. *One says that f is a monomorphism if for any* $W\in Ob(\mathcal{C})$ *and any pair* (g,g') *of* $Hom_{\mathcal{C}}(W,X)$ *such that* $f\circ g=f\circ g'$, *one has* $g=g'$. *In the dual way, we can define epimorphism.*

In a category $\mathcal{C}$ an object P is called initial if $Hom_{\mathcal{C}}(P,Y)$ has exactly one element for any $Y\in Ob(\mathcal{C})$. Similarly an object Q is called final if $Hom_{\mathcal{C}}(X,Q)$ has only one element, that is, if Q is initial in $\mathcal{C}^\circ$. Note that two initial (resp.final) object are naturally isomorphic.

Definition 1.14. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be two categories. A functor* $F:\mathcal{C}\to\mathcal{C}'$ *consists of a map* $F:Ob(\mathcal{C})\to Ob(\mathcal{C}')$ *and of maps* $F:Hom_{\mathcal{C}}(X,Y)\to Hom_{\mathcal{C}}(F(X),F(Y))$ *for all* $X,Y\in\mathcal{C}$, *such that*

$$F(id_X)=id_{F(X)}, X\in C,$$

$$F(g\circ f)=F(g)\circ F(f), f:X\to Y, g:Y\to Z$$

A contravariant functor from $\mathcal{C}$ to $\mathcal{C}'$ is a functor from $\mathcal{C}^\circ$ to $\mathcal{C}'$. In other words, it satisfies $F(g\circ f)=F(f)\circ F(g)$.

Definition 1.15. *A diagram in a category* $\mathcal{C}$ *is a family of symbols representing objects of* $\mathcal{C}$ *and a family of arrows between these symbols representing morphisms of these objects. One defines in an obvious way the notion of a commutative diagram.*

Definition 1.16. *Let* $F:\mathcal{C}\to\mathcal{C}'$ *be a functor. We say that F is faithful (resp. full, fully faithful) if* $Hom_{\mathcal{C}}(X,Y)\to Hom_{\mathcal{C}'}(F(X),F(Y))$, *is injective (resp. surjective, bijective) for any* X,Y *in* $\mathcal{C}$.

Let F_1 and F_2 be two functors from $\mathcal{C}$ to $\mathcal{C}'$. A morphism θ from F_1 to F_2 consists of the follow data.

$$X\in Ob(\mathcal{C}),\quad \theta(X)\in Hom_{\mathcal{C}'}(F_1(X),F_2(X)) \tag{12}$$

These data satisfying: for any $f\in Hom_{\mathcal{C}}(X,Y)$ the following diagram commutes:

$$\begin{tikzcd}
F_1(X) \arrow[r, "\theta(X)"] \arrow[d, "F_1(X)"] & F_2(X) \arrow[d, "F_2(f)"] \\
F_1(Y) \arrow[r, "\theta(Y)"] & F_2(Y)
\end{tikzcd}$$

We gets a new category whose objects are functors from $\mathcal{C}$ to $\mathcal{C}'$ and morphisms are morphism of such functors.

Definition 1.17. *An additive category* $\mathcal{C}$ *is a category* $\mathcal{C}$ *such that*:

1) for any pair (X,Y) of $Ob(\mathcal{C})$, $Hom_{\mathcal{C}}(X,Y)$ has a structure of additive (*i.e.* abelian) group, and the composition law is bilinear.

2) there exists an object 0 such that $Hom_{\mathcal{C}}(0,0)=0$.

An additive functor $F:\mathcal{B}\to\mathcal{A}$ between *Ab*-category $\mathcal{B}$ and $\mathcal{A}$ is a functor such that each $Hom_{\mathcal{B}}(B',B)\to Hom_{\mathcal{A}}(FB',FB)$ is a group homomorphism.

In any additive category $\mathcal{A}$, a kernel of a morphism $f:B\to C$ is defined to be a map $i:A\to B$ such that $fi=0$ and that is universal with respect to this property. Dually, a cokernel of *f* is a map $e:C\to D$, which is universal with respect to having $ef=0$. In $\mathcal{A}$, am map $i:A\to B$ is monic if $ig=0$ implies $g=0$ for every map $g:A'\to A$, and a map $e:C\to D$ is an epi if $he=0$ implies $h=0$ for every map $h:D\to D'$ (The definition of monic and epi in a non-abelian category is slightly different). It's easy to see that every kernel is monic and that every cokernel is an epi.

2.3. Homology and Cohomology I

The boundary of the geometric simplex Δ_1 is the difference of its vertices (1)-(0). In such a form the boundary appears in the Leibniz formula

$\int_0^1 f(x)\mathrm{d}x=f(1)-f(0)$. Similarly, the boundary of Δ_n is the alternating sum of its faces.

To make these definitions precise we need the following notions.

An n-dimensional chain (or simply n-chain) of a simplicial set *X* is an element of the free abelian group $C_n(X)$ generated by all n-simplices of *X*. So an n-dimensional chain is a formal linear combination $\sum_{x\in X_n}a(x)x$, where $a(x)\in Z$ and $a(x)=0$ for a finite number of simplices *x*. Let $\partial_n^i:[n-1]\to[n]$ be the unique strictly increasing mapping whose image does not contain $i\in[n]$.

Definition 1.18. *The boundary of an n-chain* $c\in C_n(x)$ *is the* $(n\text{-}1)$*-chain* $d_n c$ *defined by the following formula*:

$$d_n\left(\sum_{x\in X_n}(a(x)x)\right)=\sum_{x\in X_n}a(x)\sum_{i=1}^{n}(-1)^i X\left(\partial_n^i\right)(x) \tag{13}$$

The so defined boundary operator $d_n:C_n(X)\to C_{n-1}(X)$ is clearly a group homomorphism. For $n=0$, we set $d_0=0$. There exists an obvious generalization of this construction, namely, chains with coefficients in an abelian group A. Such a chain is a formal linear combination $\sum_{x\in X_n}a(x)x$, $a(x)\in A$, we set $C_n(X)=C_n(X,\mathcal{Z})$. The boundary operator $d_n:C_n(X,A)\to C_{n+1}(X,A)$ is again defined by before. Dually, one can define cochains with coefficients in $A:C_n(X,A)$ is the group of functions on X_n with values in *A*. The coboundary $d^n:C^n(X,A)\to C^{n+1}(X,A)$ is given by the formula:

$$\left(d^n f\right)(x)=\sum_{i=0}^{n+1}(-1)^i f\left(X\left(\partial_{n+1}^i\right)(x)\right) \tag{14}$$

Formally, chains can be considered as special cases of cochains: there exists an inclusion $C_n(X,A) \subset C^n(X,A)$ that maps a chain $\sum_{x \in X_n} a(x)x$ into the function $a : X_n \to A$. However, this inclusion is incompatible with the action of d_n and d^n (they act in opposite directions) and, even more important, it is incompatible with the behaviour of C_n and C^n under simplicial maps $X \to Y$.

Lemma 1.19. *We have* $d_{n-1} \circ d_n = 0$ *for* $n \geq 1$, $d^{n+1} \circ d^n = 0$ *for* $n \geq 0$.

Proof: Note first that for any $0 \leq j < i \leq n-1$ we have

$$\partial_n^i \circ \partial_{n-1}^j = \partial_n^j \circ \partial_{n-1}^{j-1}$$

indeed, both sides of the equality give unique increasing mapping of $[n-2]$ into $[n]$ not taking values *i* and *j*.

To prove the lemma it suffices to check that $d_{n-1} \circ d_n(x) = 0$ for any $x \in X_n$. Compositions $\partial_n^i \circ \partial_{n-1}^j$ for different i, j all yield increasing maps of $[n-2]$ into $[n]$, and the map whose image does not contain *i* and *j* appears exactly twice: the first time as $\partial_n^i \circ \partial_{n-1}^j$ with the sign $(-1)^{i+j}$ and the second time as $\partial_n^j \circ \partial_{n-1}^{i-1}$ with the opposite sign $(-1)^{i+j-1}$. Hence $d_{n-1} \circ d_n = 0$.

Similarly one proves another. ■

Definition 1.20. *Let us define several algebraic notions. A chain complex is a sequence of abelian groups and homomorphisms.*

$$C_\bullet : \cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \tag{15}$$

with the property $d_{n-1} \circ d_n = 0$ for all *n*. Homomorphisms d_n are called boundary maps or boundary operators.

A cochain complex is a similar sequence

$$C^\bullet : \cdots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots \tag{16}$$

$d^{n+1} \circ d^n = 0$ Any chain complex can be transformed into a cochain complex by setting $D^n = C_{-n}$, $d^n = d_{-n-1}$. So we will usually consider only cochain complexes.

Definition 1.21. *Homology groups of a chain complex* $C_\bullet$ *are*

$$H_n(C_\bullet) = Kerd_n / Imd_{d+1}$$

Cohomology groups of a cochain complex $C^\bullet$ are

$$H^n(C^\bullet) = Kerd^n / Imd^{d+1}$$

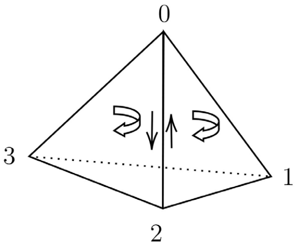
A substantial part of homological algebra can be considered as a collection of methods for computing (co)homology of various complexes, which we'll talk about that later. For a simplicial set *X* we will use the following notations: $H^n(X,A) = H^n(C^\bullet(X,A))$, $H_n(X,A) = H_n(C_\bullet(X,A))$.

Elements of the group $H_n(X,A)$ are called homology classes, and those of $H^n(X,A)$ are called cohomology classes (of the simplicial set *X* with coefficients)

Definition 1.22. *Each homology* (*resp. cohomology*) *class is represented by an n-chain c* (*resp. cochain f*) *such that* $d_n c = 0$ (*resp.* $d^n f = 0$). *Such chains* (*resp. cochains*) *are called cycles* (*resp. cocycles*). *A cycle c in a given homology class is defined up to a summand of the form* $b = d_{n+1}c'$; *such chains are called boundaries. Similarly, cochains of the form* $d^{n-1}c$ *are called coboundaries. Two chains* *whose difference is a boundary are said to be homological.*

2.4. Geometry of Chains and Cochains with Coefficient System

Example 1.23. *Look at the picture of a tetrahedron* Δ_3.



On any edge two adjacent faces induce opposite orientation. Therefore the corresponding terms in $d_2 d_3(\Delta_3)$ have opposite signs.

Example 1.24. *Zero-dimensional homology yields somewhat different information. Since,* $d_0 = 0$, *any* 0*-chain is a cycle. Let us show that there exists a natural isomorphism.* $H_0(X,\mathbb{Z})$ *= free abelian group generated by piecewise connected components of* $|X|$. *Denote for a moment the group on the right-hand side by* $\pi_0(X)$. *Define a map* $\pi_0(X)\to H_0(X,\mathbb{Z})$ *by associating with a component of* $|X|$ *the class of a chain consisting of one (arbitrary) point in this component.*

Example 1.25. *An important role in geometry is played by various modifications of topological spaces that eliminate some (co)homology classes or generate new ones.*

Let X be a triangulated space. The cone CX over X is a triangulated space obtained from X in the following manner:

$$\{\text{Vertices of } CX\} = \{\text{vertices of } X\} \cup \{*\}$$

and for $n \geq 1$

$$\begin{aligned}&\{n\text{-simplices of } CX\}\\&= \{n\text{-simplices of } X\} \cup \{\text{cones of } (n-1)\text{-simplices of } X \text{ with the vertex } *\}\end{aligned}$$

so that $*$ is the vertex of the cone.

We claim that any hole in X is filled in CX (by the cone over the boundary of the hole) and no new holes appear.

Indeed, let us define the complex of chains $\hat{C}_\bullet(X)$ of a triangulated space X as follows:

$$\hat{C}_\bullet(X) = \text{the free abelian group generated by } X_{(n)}$$

the boundary operator on $\hat{C}$ is same as before, we claim that $H_n\left(\hat{C}_\bullet(CX)\right) = 0$, $n > 0$ and $= \mathbb{Z}$, $n = 0$. We have $\hat{C}_n(CX) \cong C_n(X) \oplus C_{n-1}(X)$. Later we define the cone of any complex.

Definition 1.26. *We can construct chains and cochains of a simplicial set using as coefficients something more involved than just abelian groups. There are two* *types of coefficient systems: for homology and for cohomology. A homological coefficient system A on a simplicial set X is a family of abelian groups* $\{Ax\}$, *one for each simplex* $x \in X_n$, *and a family of homomorphisms* $A(f,x): A_x \to A_{X(f)x}$, *one for each pair* $x \in X_n$, $f:[m] \to [n]$, *such that the following conditions are satisfied:*

$$A(id,x) = id$$

$$A(f \circ g, x) = A(g, X(f)x)A(f,x)$$

The second equality means that the following diagram is commutative:

$$\begin{tikzcd}
& A_{X(f)x} \arrow[dl] \arrow[dr] & \\
A_x \arrow[rr] & & A_{X(f\circ g)x}
\end{tikzcd}$$

A cohomological coefficient system *B* on a simplicial set *X* is a family of abelian groups $\{B_x\}$, one for each simplex $x \in X_n$, and a family of ho momorphisms $B(f,x): B_{X(f)x}\beta B_x$, one for each pair (x, f), such that the following conditions are satisfied:

$$B(id,x) = id$$

$$B(f \circ g, x) = B(f,x)B(g, X(f)x)$$

The second equality is equivalent to the commutativity of a diagram similar to the one above.

Definition 1.27. *Let A be a homological coefficient system on a simplicial set X. An n-dimensional chain of X with coefficients in A is a formal linear combination* $\sum_{x\in X_n} a(x)x$, $a(x) \in A_x$. *Such chains form an abelian group (under addition) which is denoted by* $C_n(X,A)$. *The boundary of an n-dimensional chain* $c = a(x)x \in C_n(X,A)$ *is an* $(n-1)$ *-dimensional chain* $d_n c \in C_{n-1}(X,A)$ *defined by:*

$$d_n c = \sum_{x\in X_n}\sum_{i=0}^{n} A(\partial_n^i, x)(a(x))(-1)^i X(\partial_n^i)(x)$$

$$d_0 = 0$$

As before we can easily see that

$$C_\bullet(X,A): \cdots \xrightarrow{d_{n+1}} C_n(X,A) \xrightarrow{d_n} C_{n-1}(X,A) \xrightarrow{d_{n-1}} \cdots$$

is a chain complex, *i.e.* $d_{n-1} \circ d_n = 0$.

Homology groups of the complex $C_\bullet(X,A)$ are called the homology groups of the simplicial set *X* with coefficients in *A*; they are denoted by $H_n(X,A)$.

Similarly and dually, we can define cohomology groups.

2.5. The Exact Sequence I

We defined groups $H_n(X,A)$ and $H^n(X,B)$, where *X* is a simplicial set, and

A and B are coefficient systems. In some simple cases these groups can be computed directly. But the main technique consists in the study of the behaviour of these groups under the change of X or the change of A.

In this subsection we fix X and study the dependence of homology and cohomology on coefficients. The main tool here is the theorem about the exact sequence.

Definition 1.28. *An exact sequence of abelian groups is a complex* $C^{\bullet}$ *with all cohomology groups vanishing* (*for chain complexes the definition is the same*). *This means that* $Kerd^{n} = Imd^{n-1}$ *for all n. Usually such a sequence is written as*

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0 \tag{17}$$

To give such a triple is the same as to give an abelian group B and its subgroup A. The homomorphism theorem says that

$$Imi = Ker\pi \quad implies \quad C = B/A \tag{18}$$

Theorem 1.29. *Let X be a simplicial set. Any exact triple of abelian groups canonically determines a cohomology exact sequence*:

$$\begin{aligned} 0 &\to H^{0}(X,A) \to H^{0}(X,B) \to H^{0}(X,C) \to H^{1}(X,A) \\ &\to H^{1}(X,B) \to \cdots \to H^{n}(X,A) \to \cdots \end{aligned} \tag{19}$$

and a similar homology sequence.

To proof the theorem we must define the morphisms of complexes and construct the boundary homomorphism.

Theorem 1.30. *Let* $B^{\bullet}, C^{\bullet}$ *be two complexes. A morphism* $f^{\bullet}: B^{\bullet} \to C^{\bullet}$ *is a family of homomorphisms* $f^{n}: B^{n} \to C^{n}$ *commuting with differentials*:

$$d^{n} \circ f^{n} = f^{n+1} \circ d^{n} \tag{20}$$

Given $f^{n}: B^{n} \to C^{n}$, let us construct a family of homomorphisms

$$H^{n}(f): H^{n}(B^{\bullet}) \to H^{n}(C^{\bullet}) \tag{21}$$

as follows. Let $b \in H^{n}(B^{\bullet})$ be represented by a cocycle $\hat{b} \in Kerd^{n} \subset B^{n}$, Then $f^{n}(\hat{b}) \in Kerd^{n} \subset C^{n}$, and we define $H^{n}(f)(b)$ to be the class of $f^{n}(\hat{b})$ in $H^{n}(C^{\bullet})$. this class does not depend on the choice of a representative of b modulo Imd^{n-1}.

It is clear also that if $g^{\bullet}: B^{\bullet} \to C^{\bullet}$ is another morphism of complexes then $H^{n}(f \circ g) = H^{n}(f) \circ H^{n}(g)$. Let $Kerf^{\bullet} = (Kerf^{n})$, $Cokerf^{\bullet} = (Cokerf^{n})$, so $Kerf^{\bullet}$ and $Cokerf^{\bullet}$ are complexes.

Lemma 1.31. *Let* $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ *be an exact triple of abelian groups. Then the sequences of groups of chains and of cochains*

$$0 \to C_{\bullet}(X,A) \xrightarrow{i_{\bullet}} C_{\bullet}(X,B) \xrightarrow{\pi_{\bullet}} C_{\bullet}(X,C) \to 0$$

$$0 \to C^{\bullet}(X,A) \xrightarrow{i^{\bullet}} C^{\bullet}(X,B) \xrightarrow{\pi^{\bullet}} C^{\bullet}(X,C) \to 0$$

are exact.

Proof: An element of $C_{n}(X,A)$ is a formal linear combination $\sum_{x \in X_{n}} a(x)x$,

$a(x)\in A_x$. The image of this element under the mapping $i_n:C_n(X,A)\to C_n(X,B)$ is $\sum_{x\in X_n} i(a(x))x$ and since *i* is an injection, i_n is also an injection. Similarly one proves that π_n is a surjection. Further, $\pi_n i_n\left(\sum_{x\in X_n} a(x)x\right)=\sum(\pi\circ i)(a(x))x=0$ for $\sum a(x)x\in C_n(X,A)$. Let now $\beta=\sum b(x)x\in C_n(X,B)$ and $\pi_n(\beta)=0$. Then $\pi(b(x))=0$ for all $x\in X_n$, *i.e.*, $b(x)=i(a(x))$, $a(x)\in A$ and $\beta=i_n(\alpha)$ for $\alpha=\sum a(x)x\in C_n(X,A)$. The second sequence is treated similarly. ■

Theorem 1.32 *Proof of* 1.29: 1) *Exactness at* $H^n(B^\bullet)$. *First of all,* $H^n(\pi^\bullet)\circ H^n(i^\bullet)=H^n(\pi^\bullet\circ i^\bullet)=0$, *because* $\pi^\bullet\circ i^\bullet=0$. *Next, let* $b\in H^n(B^\bullet)$ *and* $H^n(\pi^\bullet)(b)=0$. *We construct* $a\in H^n(A^\bullet)$ *with* $b=H^n(i^\bullet)(a)$ *as follows. Let* $\hat{b}\in B^n$ *be a representative of b so that* $d\hat{b}=0$. *Since* $H^n(\pi^\bullet)(b)=0$, *we have* $\pi^n(\hat{b})=d\hat{c}$ *for some* $\hat{c}\in C^{n-1}$ *and,* $p^{n-1}:B^{n-1}\to C^{n-1}$ *being a surjection,* $\hat{c}=p^{n-1}(\hat{b}_1)$ *for some* $\hat{b}_1\in B^{n-1}$. *It is clear that* $p^n(\hat{b}-d\hat{b}_1)=0$ *so that by the exactness of before,* $b-d\hat{b}_1=i^n(\hat{a})$ *for some* $\hat{a}\in A^n$; *moreover,* $i^{n+1}(d\hat{a})=di^n(\hat{a})=d\hat{b}=0$. *As* i^{n+1} *is injective,* $d\hat{a}=0$. *Now one can easily check that* $a=\hat{a} \mod Imd^{n-1}\in H^n(B^\bullet)$ *satisfies the required property.*

2) $\partial^n(i^\bullet,\pi^\bullet)\circ H^n(\pi^\bullet)=0$. Let $c=H^n(\pi^\bullet)(b)$ for some $b\in H^n(B^\bullet)$ and let $\hat{b}\in B^n$, $\hat{c}\in C^n$ be representatives of b,c respectively, so that $d\hat{b}=0$. Then the definition of $\partial^n(i^\bullet,\pi^\bullet)$ shows that $\partial^n(i^\bullet,\pi^\bullet)(c)=0$.

3) $Ker\partial^n(i^\bullet,\pi^\bullet)\subset ImH^n(\pi^\bullet)$. Let $\partial^n(i^\bullet,\pi^\bullet)(c)=0$ and let $\hat{c}\in C^n$ be a cocycle representing the cohomology class *c*. Let $\hat{c}=\pi^n(\hat{b})$ and $d\hat{b}=i^{n+1}(\hat{a})$ for $\hat{b}\in B^n$, $\hat{a}\in A^{n+1}$. Then $\partial^n(i^\bullet,\pi^\bullet)(c)=0$ implies $\hat{a}=d\hat{a}_1$ for some $\hat{a}_1\in A^n$. Let $\hat{b}_1=\hat{b}-i^n(\hat{a}_1)$. Then $d\hat{b}_1=0$ and $\pi^n(\hat{b}_1)=p^n(b)-p^n\circ i^n(\hat{a}_1)=\hat{c}$. Hence $c=H^n(\pi^\bullet)(b_1)$, where $b_1=\hat{b} \mod Imd^n\in H^n(B^\bullet)$.

It's the same on the other side. ■

3. Module: With View of Free Module in Projective and Injective

Modules over a ring are a generalization of abelian groups (which are modules over $\mathbb{Z}$). In the section, we'll cover projective and injective with view of the free module.

3.1. Module and the Exact Sequence II

Definition 2.1. *Let R be a ring. A* (*left*) *R-module is an additive abelian group* A *together with a function* $R\times A\to A$ (*the image of* (r,a) *being denoted by* ra) *such that for all* $r,s\in R$ *and* $a,b\in A$:

1) $r(a+b)=ra+rb$

2) $(r+s)a=ra+sa$

3) $r(sa)=(rs)a$

If *R* has an identity element 1_R and

4) $1_R a=a$ for all $a\in A$

then A is said to be a unitary *R*-module. If *R* is a division ring, then a unitary *R*-module is called a vector space.

Definition 2.2. *Let A and B be modules over a ring R. A function* $f: A \to B$ *is an R-module homomorphism provided that for all* $a, c \in A$ *and* $r \in R$:

$$f(a+c) = f(a) + f(c) \quad \textit{and} \quad f(ra) = rf(a) \tag{22}$$

If R is a division ring, then an R-module homomorphism is called a linear transformation.

Definition 2.3. *Let R be a ring, A an R-module and B a nonempty subset of A. B is a submodule of A provided that B is an additive subgroup of A and* $rb \in B$ *for all* $r \in R$, $b \in B$. *A submodule of a vector space over a division ring is called a subspace.*

If X is a subset of a module A over a ring R, then the intersection of all submodules of A containing X is called the submodule generated by X (or. spanned by X). If X is finite, and X generates the module B, B is said to be finitely generated. Let B be a submodule of a module A over a ring R. Then the quotient group A/B is an R-module with the action of R on A/B given by:

$$r(a+B) = ra + B \quad \textit{for all} \quad r \in R, a \in A$$

Definition 2.4. *A pair of module homomorphisms,* $A \overset{f}{\to} B \overset{g}{\to} C$, *is said to be exact at B provided* $Imf = Kerg$. *A finite sequence of module homomorphisms,* $A_0 \overset{f_1}{\to} A_1 \overset{f_2}{\to} A_2 \overset{f_3}{\to} \cdots \overset{f_{n-1}}{\to} A_{n-1} \overset{f_n}{\to} A_n$, *is exact provided* $Imf_i = Kerf_{i+1}$ *for* $i = 1, 2, \cdots, n-1$. *At the same time we can define the exact of an infinite sequence.*

Lemma 2.5. *Let R be a ring and*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence. Then

1) α, γ monomorphisms $\Rightarrow \beta$ is a monomorphisms;

2) α, γ epimorphisms $\Rightarrow \beta$ is a epimorphisms;

3) α, γ isomorphisms $\Rightarrow \beta$ is a isomorphisms.

Proof: 1) Let $b \in B$ and suppose $\beta(b) = 0$; we must show that $b = 0$. By commutativity we have $\gamma g(b) = g'\beta(b) = g'(0) = 0$. This implies $g(b) = 0$, since γ is a monomorphism. By exactness of the top row at B, we have $b \in Kerg = Imf$, say $a \in A$. By commutativity, $f'\alpha(a) = \beta f(a) = \beta(b) = 0$. By exactness of the bottom row at A', f' is a monomorphism, hence $\alpha(a) = 0$. But α is a monomorphism; therefore $a = 0$ and hence $b = f(a) = f(0) = 0$. Thus β is a monomorphism.

2) Let $b' \in B'$. Then $g'(b') \in C'$; since γ is an epimorphism $g'(b') = \gamma(c)$ for some $c \in C$. By exactness of the top row at C, g is an epimorphism; hence $c = g(b)$ for some $b \in B$. By commutativity, $g'\beta(b) = \gamma g(b) = \gamma(c) = g'(b')$. Thus $g'[\beta(b) - b'] = 0$ and $\beta(b) - b' \in Kerg' = Imf'$ by exactness, say $f'(a') = \beta(b) - b'$, $a' \in A'$. Since α is an epimorphism, $a' = \alpha(a)$ for some $a \in A$. Consider $b - f(a) \in B$: $\beta[b - f(a)] = \beta(b) - \beta f(a)$, by commutativity,

$\beta f(a) = f'\alpha(a) = f'(a') = \beta(b) - b'$, hence $\beta[b - f(a)] = \beta(b) - \beta f(a) = \beta(b) - (\beta(b) - b') = b'$ and β is an epimorphism.

3) is an immediate consequence of 1) and 2). ■

Definition 2.6 *Let R be a ring and* $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$. *a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent.*

1) There is an R-module homomorphism $h : A_2 \to B$ with $gh = 1_{A_2}$;

2) There is an R-module homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$;

3) the given sequence is isomorphic (with identity maps on A_1 and A_2) to the direct sum short exact sequence $0 \to A_1 \xrightarrow{f'} A_1 \oplus A_2 \xrightarrow{g'} A_2 \to 0$; in particular $B \cong A_1 \oplus A_2$.

A short exact sequence that satisfies the equivalent conditions is said to be split or a split exact sequence.

3.2. Free Module

In this subsection we show free modules, the most important examples of which are vector spaces over a division ring. We'll show the objects without proof.

Definition 2.7. *A subset X of an R-module A is said to be linearly independent provided that for distinct* $x_1, \cdots, x_n \in X$ *and* $r_i \in R$. $r_1x_1 + r_2x_2 + \cdots + r_nx_n = 0 \Rightarrow r_i = 0$ *for every i. A set that is not linearly independent is said to be linearly dependent. If A is generated as an R-module by a set Y, then we say that Y spans A. If R has an identity and A is unitary, Y spans A if and only if every element of A may be written as a linear combination*: $r_1y_1 + r_2y_2 + \cdots + r_ny_n = 0$ $(r_i \in R, y_i \in Y)$; *A linearly independent subset of A that spans A is called a basis of A. Observe that the empty set is linearly independent and is a basis of the zero module.*

Definition 2.8. *A unitary module F over a ring R with identity, which satisfies F has a nonempty basis is called a free R-module.*

Definition 2.9. *A maximal linearly independent subset X of a vector space V over a division ring D is a basis of V. Every vector space V over a division ring D has a basis and is therefore a free D-module. More generally every linearly independent subset of V is contained in a basis of V. If V is a vector space over a division ring D and X is a subset that spans V, then X contains a basis of V.*

Definition 2.10. *Let R be a ring with identity and F a free R-module with an infinite basis X. Then every basis of F has the same cardinality as X. If V is a vector space over a division ring D, then any two bases of V have the same cardinality.*

3.3. Projective and Injective Module

Every free module is projective and arbitrary projective modules (which need not be free) have some of the same properties as free modules. Injectivity, which is also studied here, is the dual notion to projectivity. Before we start our section we consider such example when expressed in modern language, the Riemann-Roch theorem give a formula for the difference of the dimensions of two vector spaces attached to algebraic line bundle over a non-singular projective curve. Thus, we can see easily where the projective or injective module come from (we don't expect its historic origin which comes from *Homological Algebra* written by Cartan). At the same time, in order to fit in with the category we mentioned above, we will make sacrifices to use its abstract algebraic language.

Definition 2.11. *A module P over a ring R is said to be projective if given any diagram of R-module homomorphisms*

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

with bottom row exact (that is, g an epimorphism), there exists an R-module homomorphism $h: P \to A$ such that the diagram

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, "h"'] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

is commutative (that is, $gh = f$).

Theorem 2.12 *Every free module F over a ring R with identity is projective.*

Proof: We are given a diagram of homomorphisms of unitary R-modules:

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

with g an epimorphism and F a free R-module on the set X ($\pi: X \to F$ *). For each* $x \in X$, $f(\pi(x)) \in B$. *Since g is an epimorphism, there exists* $a_x \in A$ *with* $g(a_x) = f(\pi(x))$. *Since F is free, the map* $X \to A$ *given by* $x \to a_x$ *induces an R-module homomorphism* $h: F \to A$ *such that* $h(\pi(x)) = a_x$ *for all* $x \in X$. *Consequently,* $gh\pi(x) = g(a_x) = f\pi(x)$ *for all* $x \in X$ *so that* $gh\pi = f\pi: X \to B$, *we have* $gh = f$. *Therefore F is projective. We also can see every module A over a ring R is the homomorphic image of a projective R-module.*

■

Theorem 2.13 *Let R be a ring. The following conditions on an R-module P are equivalent.*

1) *P is projective;*

2) *there is a free module F and an R-module K such that* $F \cong K \oplus P$;

3) *every short exact sequence* $0\to A\xrightarrow{f}B\xrightarrow{g}P\to 0$ *is split exact.*

Proof: 1) $\Rightarrow$ 3) Consider the diagram

$$\begin{tikzcd}
& P \arrow[d, "1_p"] & \\
B \arrow[r, "g"] & P \arrow[r] & 0
\end{tikzcd}$$

with bottom row exact by hypothesis. Since P is projective there is an R-module homomorphism $h:P\to B$ such that $gh=1_p$. Therefore, the short exact sequence $0\to A\xrightarrow{f}B\xrightarrow{g}P\to 0$ is split exact.

3) $\Rightarrow$ 2) There is a free R-module F and an epimorphism $g:F\to P$. If $K=Kerg$, then $0\to K\xrightarrow{\subset}F\xrightarrow{g}P\to 0$. By hypothesis the sequence splits.

2) $\Rightarrow$ 1) Let π be the composition $F\cong K\oplus P\to P$ where the second map is the canonical projection. Similarly let τ be the composition $P\to K\oplus P\cong F$ with the first map the canonical injection. Given a diagram of R-module homomorphisms

$$\begin{tikzcd}
& P \arrow[d, "f"] & \\
A \arrow[r, "g"] & B \arrow[r] & 0
\end{tikzcd}$$

with exact bottom row, consider the diagram

$$\begin{tikzcd}
& F \arrow[d, shift left, "\pi"] & \\
& P \arrow[u, shift left, "\tau"] \arrow[d, "f"] & \\
A \arrow[r, "g"] & B \arrow[r] & 0
\end{tikzcd}$$

Since F is projective, there is an R-module homomorphism $h_1:F\to A$ such that $gh_1=f\pi$. Let $h=h_1\tau:P\to A$. Then $gh=gh_1\tau=(f\pi)\tau=f(\pi\tau)=f1_P=f$. Therefore, P is projective. ■

Definition 2.14 *A module J over a ring R is said to be injective if given any diagram of R-module homomorphisms*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"] & B \\
& J &
\end{tikzcd}$$

with top row exact (that is, g a monomorphism), there exists an R-module homomorphism $h: B \to J$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"] & B \arrow[dl, "h"'] \\
 & J &
\end{tikzcd}$$

is commutative (that is, $hg = f$).

Theorem 2.15. *A right R-module E is injective if and only if for every right ideal J of R, every map* $J \to E$ *can be extended to a map* $R \to E$.

Proof: The "only if" direction is a special case of the definition of injective. Conversely, suppose given an R-module B, a submodule A and a map $\alpha: A \to E$. Let $\mathcal{E}$ be the poset of all extensions $\alpha': A' \to E$ of α to an intermediate submodule $A \subset A' \subset B$; the partial order is that $\alpha' \leq \alpha''$ if α'' extends α'. By Zorn's lemma there is a maximal extension $\alpha': A' \to E$ in $\mathcal{E}$; we have to show that $A' = B$. Suppose there is some $b \in B$ not in A'. The set $J_b = \{r \in R : br \in A'\}$ is a right ideal of R. By assumption, the map $J_b \to A' \xrightarrow{\alpha'} E$ extends to a map $f: R \to E$. Let A'' be the submodule $A' + bR$ of B and define $\alpha'': A'' \to E$ by $a''(a + br) = a'(a) + f(r)$, $a \in A'$ and $r \in R$. This is well defined because $\alpha'(br) = f(r)$ for br in $A' \cap bR$, and α'' extends α', contradicting the existence of b. Hence $A' = B$. ■

Definition 2.16. *An abelian group D is said to be divisible if given any* $y \in D$ *and* $0 \neq n \in Z$, *there exists* $x \in D$ *such that* $nx = y$. *For example, the additive group* $\mathbb{Q}$ *is divisible, but* $\mathbb{Z}$ *is not. An abelian group D is divisible if and only if D is an injective (unitary) Z-module.*

Theorem 2.17. *Let R be a ring. The following conditions on an R-module J are equivalent.*

1) *J is injective;*

2) *J is a direct summand of any module B of which it is a submodule;*

3) *every short exact sequence* $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is split exact.*

Proof like before.

Example 2.18. *The divisible abelian groups* $\mathbb{Q}$ *and* $\mathbb{Z}_{p^\infty} = \mathbb{Z}\left[\frac{1}{p}\right] / \mathbb{Z}$ *are injective. Every injective abelian group is direct sum of these. In particular, the injective abelian group* $\mathbb{Q}/\mathbb{Z}$ *is isomorphic to* $\oplus \mathbb{Z}_{p^\infty}$.

If A is an abelian group, let $I(A)$ be the product of copies of the injective group $\mathbb{Q}/\mathbb{Z}$, indexed by the set $Hom_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})$, then $I(A)$ is injective.

Example 2.19. *Nice rings every projective module is a free module like* $\mathbb{Z}$, *fields, division rings…*

4. Homological Algebra

Homological algebra is a tool used to prove nonconstructive existence theorems in algebra. It also provides obstructions to carrying out various kinds of constructions; when the obstructions are zero, the construction is possible. In the section, we will show many theorems without proof because of space. And we'll skip content like δ-function that is interesting.

Definition 3.1. *Let A be an abelian category. Then* $Hom_A(M,-)$ *is a left exact functor from A to Ab for every M in A. That is, given an exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, *the following sequence of abelian groups is also exact:*

$$Hom(M,A) \to Hom(M,B) \xrightarrow{g_*} Hom(M,C) \to 0 \tag{23}$$

$Hom_A(-,M)$ is a left exact contravariant functor.

Theorem 3.2. ***Yoneda Embedding*** *Every additive category A can be embedded in the abelian category* $Ab^{A^{op}}$ *by the functor h sending A to* $h_A = Hom_A(-,A)$. *Since each* $Hom_A(M,-)$ *is left exact, h is a left exact functor. Since the functors* h_A *are left exact, the Yoneda embedding actually lands in the abelian subcategory* $\mathcal{L}$ *of all left exact contravariant functors from A to Ab whenever A is an abelian category.*

Lemma 3.3. *The Yoneda embedding h reflects exactness. That is, a sequence* $A \xrightarrow{f} B \xrightarrow{g} C$ *in A is exact, provided that for every M in A the following sequence is exact:*

$$Hom(M,A) \xrightarrow{f_*} Hom(M,B) \xrightarrow{g_*} Hom(M,C) \tag{24}$$

Theorem 3.4. *M is projective if and only if* $Hom_A(M,-)$ *is an exact functor. That is, the sequence of groups:*

$$0 \to Hom(M,A) \xrightarrow{f_*} Hom(M,B) \xrightarrow{g_*} Hom(M,C) \to 0 \tag{25}$$

is exact for every exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in *A*.

Definition 3.5. *Let M be an object of A. A left resolution of M is a complex* $P_\bullet$ *with* $P_i = 0$ *for* $i < 0$, *together with a map* $\epsilon : P_0 \to M$ *so that the augmented complex:*

$$\cdots \to P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M \to 0 \tag{26}$$

is exact. It is a projective resolution if each P_i is projective.

Theorem 3.6. *Every R-module M has a projective resolution. More generally, if an abelian category A has enough projectives, then every object M in A has a projective resolution.*

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & & \\
& M_3 \arrow[ur] \arrow[dr] & & & & M_1 \arrow[ur] \arrow[dr] & & & & & \\
\ldots \arrow[ur] \arrow[rr] & & P_3 \arrow[rr, "d"] \arrow[dr] & & P_2 \arrow[rr, "d"] \arrow[ur] & & P_1 \arrow[rr, "d"] \arrow[dr] & & P_0 \arrow[r, "\epsilon"] & M \arrow[r] & {} \\
& & & M_2 \arrow[ur] \arrow[dr] & & & & M_0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Forming a resolution by splicing.

Forming a resolution by splicing.

Theorem 3.7. *Let* $P_\bullet \xrightarrow{\epsilon} M$ *be a projective resolution of M and* $f: M \to N$ *a map in A. Then for every resolution* $Q_\bullet \xrightarrow{\eta} N$ *of N there is a chain map* $f: P_\bullet \to Q_\bullet$ *lifting* f' *in the sense that* $\eta \circ f_0 = f' \circ \epsilon$. *The chain map f is unique up to chain homotopy equivalence.*

$$\begin{tikzcd}
\ldots \arrow[r] & P_2 \arrow[r] \arrow[d, "\exists"] & P_1 \arrow[r] \arrow[d, "\exists"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "\exists"] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\ldots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

Theorem 3.8. *Suppose given a commutative diagram*

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\ldots \arrow[r] & P'_2 \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r] & A' \arrow[r] \arrow[d] & 0 \\
 & & & & A \arrow[d] & \\
\ldots \arrow[r] & P''_2 \arrow[r] & P''_1 \arrow[r] & P''_0 \arrow[r] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the column is exact and the rows are projective resolutions. Set $P_n = P'_n \oplus P''_n$ Then the P_n assemble to form a projective resolution P of A, and the right-hand column lifts to an exact sequence of complexes $0 \to P' \to P \to P'' \to 0$ where are the natural inclusion and projection, respectively.

Definition 3.9 *Let M be an object of A. A right resolution of M is a cochain complex* $I^\bullet$ *with* $I^i = 0$ *for* $i < 0$ *and a map* $M \to I^0$ *such that the augmented complex*:

$$0 \to M \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots \tag{27}$$

is exact. This is the same as a cochain map $M \to I^\bullet$, where M is considered as a complex concentrated in degree 0. It is called an injective resolution if each I^i is injective.

The other theorems mentioned above are all in dual form in injective.

What Is Homological Algebra?

Now we have met some injective and projective module and also found the continuity with algebraic topology. It's appropriate to ask, what is homological algebra?

I'd like to the view that homological algerbra is a tool used to prove some theorems in algebra and it can show us how far away from the good structure. At the same time, it can also provide enough details in arithmetic. From a historical point of view, homological algebra is actually a metaphysical weapon. Kant used to say: There can be no doubt that all our knowledge begins with experience. Whether or not there is such knowledge, which does not rely on experience, or even on any sensory impressions, is at least a question that needs to be examined more carefully, and one that cannot be answered immediately and lightly. Therefore, it is easy to see that the weapon we use is actually derived from algebraic topology, but it is beyond the category of algebraic topology and has become a kind of prior knowledge. This provides us with a philosophical reflection on how to understand mathematics, Copernicus, a philosopher born 300 years ago, said that the intellectual category can only be used empirically, not transcendentally. Empirical knowledge such as arithmetic, is difficult to be as transcendent as such knowledge. However, this kind of prior knowledge is difficult to understand and acquire, and it still requires a lot of empirical knowledge as a background refinement and the field is still breeding lilacs out of the dead land.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Hungerford, T.W. (1974) Algebra. Springer.

[2] Milne, J.S. (2020) Class Field Theory. https://www.jmilne.org/math/CourseNotes/CFTc.pdf

[3] Weibel, C.A. (1994) An Introduction to Homological Algebra. Cambridge University Press. https://doi.org/10.1017/cbo9781139644136

[4] Grothendieck, A. (1957) Sur quelques points d'algèbre homologique, II. *Tohoku Mathematical Journal*, **9**, 119-221. https://doi.org/10.2748/tmj/1178244774

[5] Gelfand, S.I. and Manin, Y.I. (2003) Methods of Homological Algebra. Springer.

[6] Hartshorne, R. (1977) Algebraic Geometry. Springer.

[7] Kashiwara, M. and Schapira, P. (1990) Sheaves on Manifolds. Springer.

[8] Kashiwara, M. and Schapira, P. (2006) Categories and Sheaves. Springer.