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The Study of Root Subspace Decomposition between Characteristic Polynomials and Minimum Polynomial

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Abstract

Let $\mathcal{A}$ be the linear transformation on the linear space V in the field P, V^{λ_i} be the root subspace corresponding to the characteristic polynomial of the eigenvalue λ_i, and W^{λ_i} be the root subspace corresponding to the minimum polynomial of λ_i. Consider the problem of whether V^{λ_i} and W^{λ_i} are equal under the condition that the characteristic polynomial of $\mathcal{A}$ has the same eigenvalue as the minimum polynomial (see Theorem 1, 2). This article uses the method of mutual inclusion to prove that $V^{\lambda_i} = W^{\lambda_i}$. Compared to previous studies and proofs, the results of this research can be directly cited in related works. For instance, they can be directly cited in Daoji Meng's book "Introduction to Differential Geometry."

Keywords

Characteristic Polynomial, Minimum Polynomial, Root Subspace

1. Introduction

Let V be a vector space over a field P. Let $\mathcal{A}$ be a linear transformation on V, and let A be the matrix corresponding to $\mathcal{A}$ with respect to some basis. The eigenvalues of both the characteristic polynomial and the minimal polynomial of A are the same. Let the characteristic polynomial of the linear transformation $\mathcal{A}$ be $f(\lambda)=(\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_s)^{r_s}$. Wang Efang demonstrated that the eigenspaces corresponding to the distinct eigenvalues of the characteristic polynomial can decompose V into a direct sum, *i.e.*,

$$V = V^{\lambda_1} \oplus V^{\lambda_2} \oplus \cdots \oplus V^{\lambda_s}, \text{ where } V^{\lambda_i} = \left\{ v \in V \mid (\mathcal{A}-\lambda_i\mathcal{E})^{r_i} v = \mathbf{0} \right\}.$$

The minimal polynomial of $\mathcal{A}$ is $m(\lambda)=(\lambda-\lambda_1)^{l_1}(\lambda-\lambda_2)^{l_2}\cdots(\lambda-\lambda_s)^{l_s}$. Meng Daoji demonstrated that the eigenspaces corresponding to the distinct eigenvalues of the minimal polynomial can decompose V into a direct sum, *i.e.*, decompose as $V=W^{\lambda_1}\oplus W^{\lambda_2}\oplus\cdots\oplus W^{\lambda_s}$, and $W^{\lambda_i}=\left\{v\in V\,|\,(\mathcal{A}-\lambda_i\mathcal{E})^{l_i}v=\mathbf{0}\right\}$.

Meng Daoji and Wang Efang respectively studied the eigenspaces corresponding to the eigenvalues of the minimal polynomial and the characteristic polynomial. Upon investigation, there are many references introducing the decomposition of eigenspaces corresponding to the characteristic polynomial and the minimal polynomial, but there are very few papers discussing the relationship between them. Is it possible to consider that the eigenspaces corresponding to the same eigenvalue in both the characteristic polynomial and the minimal polynomial are the same? Based on this question, this paper proves the equivalence of the eigenspace decompositions corresponding to the same eigenvalue, *i.e.*, $V^{\lambda_i}=W^{\lambda_i}$.

For this problem, the paper will be divided into two parts. The first part will provide the necessary background, introducing the fundamental concepts of matrix characteristic polynomials and minimal polynomials. The second part will present the proof of the equivalence of the eigenspaces corresponding to the same eigenvalue under the characteristic polynomial and the minimal polynomial, accompanied by illustrative examples.

2. Preliminary Knowledge

The following mainly introduces the basic knowledge related to characteristic polynomials and minimal polynomials, as well as the decomposition of eigenspaces corresponding to characteristic polynomials and minimal polynomial.

Definition 1 [1]. Let V be a non-empty set and P be a field. If the following conditions hold true, then the set V is termed as a linear space over the field P.

(1) A rule of correspondence is defined between any two elements α and β in V, such that there exists a unique element γ in V corresponding to them. This rule of correspondence is termed as addition, and the element γ is termed as the sum of α and β, denoted as $\gamma=\alpha+\beta$.

(2) A rule of correspondence is defined between any element k in the field P and any element α in the set V, such that there exists a unique element γ in V corresponding to k and α. This rule of correspondence is termed as scalar multiplication or simply multiplication, and γ is termed as the product of k and α, denoted as $\gamma=k\alpha$.

(3) The addition and scalar multiplication defined above satisfy the eight axioms:

i. $\alpha+\beta=\beta+\alpha$.

ii. $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$.

iii. There exists a zero element, denoted as $\mathbf{0}$, such that for any element α in V, $\mathbf{0}+\alpha=\mathbf{0}$.

iv. For any element α in V, there exists a corresponding additive inverse element β such that $\alpha+\beta=0$.

v. $1\cdot\alpha=\alpha$.

vi. $k(l\alpha)=(kl)\alpha$.

vii. $(k+l)\alpha=k\alpha+l\alpha$.

viii. $k(\alpha+\beta)=k\alpha+k\beta$,

where α, β and γ are arbitrary elements in the set V, and k and l are arbitrary numbers in the field P.

Definition 2 [1]. Let $\mathcal{A}$ be a mapping from the linear space V over the field P to the linear space W. For any two vectors α and β in V, if the following conditions are satisfied:

1. $\mathcal{A}(\alpha+\beta)=\mathcal{A}(\alpha)+\mathcal{A}(\beta)$.
2. $\mathcal{A}(k\alpha)=k\mathcal{A}(\alpha), k\in P$.

Then $\mathcal{A}$ is termed as a linear mapping, and $\mathcal{A}(\alpha)$ is called the image of α under the linear mapping $\mathcal{A}$.

A linear mapping $\mathcal{A}$ from the vector space V to itself is called a linear transformation of the vector space V.

Definition 3 [2]. The set of all univariate polynomials over the field P is denoted by $P[x]$. Addition and multiplication operations can be defined in $P[x]$ as follows: Let

$$f(x)=\sum_{i=0}^{n}a_ix^i,\quad g(x)=\sum_{i=0}^{m}b_ix^i,$$

assuming $n\geq m$. Define

$$f(x)+g(x):=\sum_{i=0}^{n}(a_i+b_i)x^i,$$

$$f(x)g(x):=\sum_{i=0}^{n}\sum_{j=0}^{m}(a_ib_j)x^{i+j}=\sum_{s=0}^{n+m}\left(\sum_{i+j=s}a_ib_j\right)x^s.$$

The term $f(x)+g(x)$ is called the sum of $f(x)$ and $g(x)$, and the term $f(x)g(x)$ is called the product of $f(x)$ and $g(x)$.

Definition 4 [3]. Let $f(x),g(x)\in P[x]$. If $(f(x),g(x))=1$, then $f(x)$ and $g(x)$ are called coprime.

Lemma 1 [4]. Two polynomials $f(x)$ and $g(x)$ in $P[x]$ are coprime if and only if there exist polynomials $u(x),v(x)\in P[x]$ such that $u(x)f(x)+v(x)g(x)=1$.

Proposition 1 [4]. In $P[x]$, if $(f(x),h(x))=1$, $(g(x),h(x))=1$, then $(f(x)g(x),h(x))=1$.

Proof. By Lemma 1, there exist $u_1(x),u_2(x),v_1(x),v_2(x)$ such that $u_1(x)f(x)+v_1(x)h(x)=1$ and $u_2(x)g(x)+v_2(x)h(x)=1$. Multiplying the two equations gives

$u_1(x)f(x)(u_2(x)g(x)+v_2(x)h(x))+v_1(x)h(x)(u_2(x)g(x)+v_2(x)h(x))=1$.

Furthermore, we obtain $u_3(x)(f(x)g(x))+v_3(x)h(x)=1$, where

$u_3(x)=u_1(x)u_2(x)$,

$v_3(x)=u_1(x)v_2(x)f(x)+u_2(x)v_1(x)g(x)+v_1(x)v_2(x)h(x)$.

Remark 1. Using mathematical induction, Lemma 1 can be generalized to: in $P[x]$, if $(f_i(x),h(x))=1, i=1,2,\cdots,s$, then $(f_1(x)f_2(x)\cdots f_s(x),h(x))=1$.

Lemma 2 [4]. Let $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ be a basis of the vector space V, and let

$\alpha_1, \alpha_2, \cdots, \alpha_n$ be arbitrary vectors in *V*. There exists a unique linear transformation $\mathcal{A}$ such that $\mathcal{A}\varepsilon_i = \alpha_i, i = 1, \cdots, n$.

Definition 5 [5]. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be a basis of the *n*-dimensional vector space *V* over the field *P*, and let $\mathcal{A}$ be a linear transformation on *V*. The image of the basis vectors can be expressed as a linear combination of the basis vectors, *i.e.*,

$$f(x) = \begin{cases} A\varepsilon_1 = a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots a_{n1}\varepsilon_n \\ A\varepsilon_2 = a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + \cdots a_{n2}\varepsilon_n \\ \vdots \\ A\varepsilon_1 = a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots a_{nn}\varepsilon_n \end{cases}.$$

In terms of matrices, this can be represented as

$$\mathcal{A}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = (\mathcal{A}\varepsilon_1, \mathcal{A}\varepsilon_2, \cdots, \mathcal{A}\varepsilon_n) = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)A,$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

The matrix *A* is called the matrix of $\mathcal{A}$ with respect to the basis $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$.

Lemma 3 [6]. For a given basis, a linear transformation and its representing matrix are in one-to-one correspondence.

Lemma 4 [7]. Let *V* be a vector space over the field *P*, and let $\mathcal{A}$ be a linear transformation on *V*. Given that $f_1(x), f_2(x) \in P[x]$ and $(f_1(x), f_2(x)) = 1$, if we let $f(x) = f_1(x) f_2(x)$, then $Kerf(\mathcal{A}) = Kerf_1(\mathcal{A}) \cdot Kerf_2(\mathcal{A})$.

Proof. Step 1: Prove $Kerf(\mathcal{A}) = Kerf_1(\mathcal{A}) + Kerf_2(\mathcal{A})$.

For any $\alpha_1 \in Kerf_1(\mathcal{A})$, we have $f_1(\mathcal{A})\alpha_1 = \mathbf{0}$. Therefore, $f_2(\mathcal{A}) f_1(\mathcal{A})\alpha_1 = \mathbf{0}$.

Since $f(x) = f_1(x) f_2(x)$, we have $f(\mathcal{A}) = f_1(\mathcal{A}) f_2(\mathcal{A})$.

Therefore, $f(\mathcal{A})\alpha_1 = f_1(\mathcal{A}) f_2(\mathcal{A})\alpha_1 = f_2(\mathcal{A}) f_1(\mathcal{A})\alpha_1 = \mathbf{0}$.

Consequently, $\alpha_1 \in Kerf(\mathcal{A})$, and thus $Kerf_1(\mathcal{A}) \subseteq Kerf(\mathcal{A})$.

Similarly, $Kerf_1(\mathcal{A}) \subseteq Kerf(\mathcal{A})$. Therefore, $Kerf_1(\mathcal{A}) + Kerf_2(\mathcal{A}) \subseteq Kerf(\mathcal{A})$. For any $\alpha \in Kerf(\mathcal{A})$, and since $(f_1(x), f_2(x)) = 1$, there exists $\mu(x), \nu(x) \in P[x]$ such that

$$\mu(x) f_1(x) + \nu(x) f_2(x) = 1. \tag{1}$$

Substituting *x* into $\mathcal{A}$ in equation (1), we obtain

$$\mu(\mathcal{A}) f_1(\mathcal{A}) + \nu(\mathcal{A}) f_2(\mathcal{A}) = \mathcal{E}. \tag{2}$$

Hence $\alpha = \mathcal{E}\alpha = \mu(\mathcal{A}) f_1(\mathcal{A})\alpha + \nu(\mathcal{A}) f_2(\mathcal{A})\alpha$.

Let $\alpha_1 = \nu(\mathcal{A}) f_2(\mathcal{A})\alpha, \alpha_2 = \mu(\mathcal{A}) f_1(\mathcal{A})\alpha$, then $\alpha = \alpha_1 + \alpha_2$.

Since $f_1(\mathcal{A})\alpha_1 = f_1(\mathcal{A})\nu(\mathcal{A}) f_2(\mathcal{A})\alpha = \nu(\mathcal{A}) f(\mathcal{A})\alpha = \nu(\mathcal{A})\mathbf{0} = \mathbf{0}$, it follows that $\alpha_1 \in Kerf_1(\mathcal{A})$. Similarly, it can be proved that $\alpha_2 \in Kerf_2(\mathcal{A})$. Thus, it follows that $Kerf(\mathcal{A}) = Kerf_1(\mathcal{A}) + Kerf_2(\mathcal{A})$.

Step 2: Prove $Kerf_1(\mathcal{A}) \cap Kerf_2(\mathcal{A}) = \mathbf{0}$.

Given $\beta \in Kerf_1(\mathcal{A}) \cap Kerf_2(\mathcal{A})$, then $f_1(\mathcal{A})\beta = \mathbf{0}, f_2(\mathcal{A})\beta = \mathbf{0}$. Using

equation (2), we have
$\beta = \mathcal{E}\beta = \mu(\mathcal{A}) f_1(\mathcal{A})\beta + \nu(\mathcal{A}) f_2(\mathcal{A})\beta = \mu(\mathcal{A})\mathbf{0} + \nu(\mathcal{A})\mathbf{0} = \mathbf{0}$. Therefore,
$Kerf_1(\mathcal{A}) \cap Kerf_2(\mathcal{A}) = \mathbf{0}$.

Proposition 2 [7]. Let $\mathcal{A}$ be a linear transformation on the linear space V over the field *P*, where $f_1(x), f_2(x), \cdots, f_s(x) \in P[x]$ and they are pairwise coprime. Let

$$f(x) = f_1(x) f_2(x) \cdots f_s(x), \tag{3}$$

then $Kerf(\mathcal{A}) = Kerf_1(\mathcal{A}) \oplus Kerf_2(\mathcal{A}) \oplus \cdots \oplus Kerf_s(\mathcal{A})$.

Proof. Apply mathematical induction on the number of polynomials *s* on the right-hand side of equation (3).

When $s = 2$, Lemma 4 has been proven, and the proposition holds.

Assume that the proposition holds when the number of polynomials on the right-hand side of equation (3) is $s-1$.

Now consider the case where $f(x) = f_1(x) f_2(x) \cdots f_s(x)$. Since $f_1(x), f_2(x), \cdots, f_s(x)$ are pairwise coprime, we have $\left(f_1(x) f_2(x) \cdots f_{s-1}(x), f_s(x)\right) = 1$.

Let $g(x) = f_1(x) f_2(x) \cdots f_{s-1}(x)$. According to Lemma 4, we have

$$Kerf(\mathcal{A}) = Kerg(\mathcal{A}) \oplus Kerf_s(\mathcal{A}). \tag{4}$$

By the inductive hypothesis, we conclude that

$$Kerg(\mathcal{A}) = Kerf_1(\mathcal{A}) \oplus Kerf_2(\mathcal{A}) \oplus \cdots \oplus Kerf_{s-1}(\mathcal{A}). \tag{5}$$

From equations (4) and (5), we have
$Kerf(\mathcal{A}) = Kerf_1(\mathcal{A}) \oplus \cdots \oplus Kerf_{s-1}(\mathcal{A}) \oplus Kerf_s(\mathcal{A})$. By the principle of mathematical induction, Proposition 2 is proven.

If we aim to ensure that $Kerf(\mathcal{A}) = V$, given $Ker\mathcal{O} = V$, we seek a $f(x) \in P[x]$ such that $f(\mathcal{A}) = \mathcal{O}$. This leads us to introduce the following concept:

Definition 6 [8]. Let *V* be a linear space over the field *P*, and $\mathcal{A}$ be a linear transformation on *V*. If there exists a univariate polynomial $f(x)$ over the field *P* such that $f(\mathcal{A}) = \mathcal{O}$, then $f(x)$ is called a nullifying polynomial of $\mathcal{A}$.

Definition 7 [9]. Among all non-zero nullifying polynomials of $\mathcal{A}$, the polynomial with the lowest degree and leading coefficient of 1 is called the minimal polynomial of $\mathcal{A}$.

Proposition 3 [9]. The minimal polynomial of a matrix is unique.

Proposition 4 [10]. Let $\mathcal{A}$ be a linear transformation on a linear space V over the field *P*, and $m(\lambda)$ be the minimal polynomial of $\mathcal{A}$. Then, $g(\lambda) \in P[\lambda]$ is a nullifying polynomial of $\mathcal{A}$ if and only if $m(\lambda) | g(\lambda)$.

Proof. Necessity: Suppose $g(\lambda)$ is a nullifying polynomial of $\mathcal{A}$. Perform polynomial division in $P[\lambda]$:

$$g(\lambda) = h(\lambda)m(\lambda) + r(\lambda), degr(\lambda) < degm(\lambda).$$

Substituting the indeterminate $\mathcal{A}$ with λ in the above equation, we obtain

$$g(\mathcal{A}) = h(\mathcal{A})m(\mathcal{A}) + r(\mathcal{A}).$$

Since $g(\mathcal{A})=\mathcal{O}$, $m(\mathcal{A})=\mathcal{O}$, we have $r(\mathcal{A})=\mathcal{O}$. Thus, $r(\lambda)$ is also a nullifying polynomial of $\mathcal{A}$. Since $degr(\lambda)<degm(\lambda)$ and $r(\lambda)=0$, it follows that $m(\lambda)|g(\lambda)$.

Sufficiency: Suppose $m(\lambda)|g(\lambda)$. Then there exists $h(\lambda)\in P[\lambda]$ such that $g(\lambda)=h(\lambda)m(\lambda)$. Substituting λ with $\mathcal{A}$, we get $g(\mathcal{A})=h(\mathcal{A})m(\mathcal{A})=\mathcal{O}$. Therefore, $g(\lambda)$ is a nullifying polynomial of $\mathcal{A}$.

Remark 2 [11]. Let the minimal polynomial $m(\lambda)$ of the linear transformation $\mathcal{A}$ have a standard factorization in $P[\lambda]$ as $m(\lambda)=(\lambda-\lambda_1)^{l_1}(\lambda-\lambda_2)^{l_2}\cdots(\lambda-\lambda_s)^{l_s}$. Since $(\lambda-\lambda_1)^{l_1},(\lambda-\lambda_2)^{l_2},\cdots,(\lambda-\lambda_s)^{l_s}$ are pairwise coprime, it follows from Proposition2 that $V=Ker(\mathcal{A}-\lambda_1\mathcal{E})^{l_1}\oplus Ker(\mathcal{A}-\lambda_2\mathcal{E})^{l_2}\oplus\cdots\oplus Ker(\mathcal{A}-\lambda_s\mathcal{E})^{l_s}$.

Let $W^{\lambda_i}=\{v\in V\,|\,(\mathcal{A}-\lambda_i\mathcal{E})^{l_i}v=0\}$. Then $V=W^{\lambda_1}\oplus W^{\lambda_2}\oplus\cdots\oplus W^{\lambda_s}$.

Definition 8 [10]. Let A be an $n\times n$ matrix over the field P. If there exist a scalar λ and a non-zero n-dimensional column vector β over P such that $A\beta=\lambda\beta$, then λ is called an eigenvalue of A, and β is called an eigenvector of A corresponding to the eigenvalue λ. The polynomial $f(\lambda)=|\lambda E-A|$ is called the characteristic polynomial of A.

Lemma 4 [12]. Let $\mathcal{A}$ be a linear transformation on an n-dimensional vector space V over a field P. Then the characteristic polynomial $f(\lambda)$ of $\mathcal{A}$ is an annihilating polynomial of $\mathcal{A}$.

Proposition 5 [13]. Let A be an $n\times n$ matrix over the field P. If $f(\lambda)=|\lambda E-A|$ is the characteristic polynomial of A, then $f(A)=A^n-(a_{11}+a_{22}+\cdots+a_{nn})A^{n-1}+\cdots+(-1)^n|A|E=\mathrm{O}$.

Proof. Let $B(\lambda)$ be the companion matrix of $\lambda E-A$. By the properties of determinants, we have $B(\lambda)(\lambda E-A)=|\lambda E-A|E=f(\lambda)E$. Because the elements of the matrix $B(\lambda)$ are the cofactors of $|\lambda E-A|$, which are polynomials in λ of degree at most $n-1$, by the properties of matrix operations, $B(\lambda)=\lambda^{n-1}B_0+\lambda^{n-2}B_1+\cdots+B_{n-1}$, where $B_0,B_1,\cdots,B_{n-1}$ are all $n\times n$ numeric matrices.

Let $f(\lambda)=\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n$, then

$$f(\lambda)E=\lambda^nE+a_1\lambda^{n-1}E+\cdots+a_{n-1}\lambda E+a_nE, \tag{6}$$

and

$$\begin{aligned}B(\lambda)(\lambda E-A)&=(\lambda^{n-1}B_0+\lambda^{n-2}B_1+\cdots+B_{n-1})(\lambda E-A)\\&=\lambda^nB_0+\lambda^{n-1}(B_1-B_0A)+\lambda^{n-2}(B_2-B_1A)\\&\quad+\cdots+\lambda(B_{n-1}-B_{n-2}A)-B_{n-1}A.\end{aligned} \tag{7}$$

Comparing (6) and (7), we obtain

$$\begin{cases}B_0=E,\\B_1-B_0A=a_1E,\\B_2-B_1A=a_2E,\\\vdots\\B_{n-1}-B_{n-2}A=a_{n-1}E,\\-B_{n-1}A=a_nE.\end{cases} \tag{8}$$

By multiplying the first, second, $\cdots$, n-th equations of (8) by $A^n, A^{n-1}, \cdots, A, E$ respectively from the right, we get

$$\begin{cases} B_0A^n = EA^n = A^n, \\ B_1A^{n-1} - B_0A^n = a_1EA^{n-1} = a_1A^{n-1}, \\ B_2A^{n-2} - B_1A^{n-1} = a_2EA^{n-2} = a_2A^{n-2}, \\ \vdots \\ B_{n-1}A - B_{n-2}A^2 = a_{n-1}EA = a_{n-1}A, \\ -B_{n-1}A = a_nE. \end{cases} \tag{9}$$

Adding the $n+1$ equations of (9) together, the left side is zero, and the right side equals $f(A)$. Thus, $f(A) = O$.

Definition 9 [14]. Let $\mathcal{A}$ be a linear transformation on a linear space V over the field P, and let W be a subspace of V. If for any vector ω in W, we have $\mathcal{A}\omega \in W$, then W is called an invariant subspace of $\mathcal{A}$, or simply an $\mathcal{A}$-subspace.

Lemma 5 [14]. Let the characteristic polynomial of the linear transformation $\mathcal{A}$ be $f(\lambda)$, which can be factored into a product of linear factors $f(\lambda) = (\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_s)^{r_s}$. Then V can be decomposed into a direct sum of invariant subspaces $V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$, where $V_i = \left\{v \in V \mid (\mathcal{A}-\lambda_i\mathcal{E})^{r_i} v = \mathbf{0}\right\}$.

Proof. Let $f_i(\lambda) = \dfrac{f(\lambda)}{(\lambda-\lambda_i)^{r_i}} = (\lambda-\lambda_1)^{r_1}\cdots(\lambda-\lambda_{i-1})^{r_{i-1}}(\lambda-\lambda_{i+1})^{r_{i+1}}\cdots(\lambda-\lambda_s)^{r_s}$ and $V_i = f_i(\mathcal{A})V$. Then V_i is the range of $f_i(\mathcal{A})$, and V_i is an invariant subspace of $\mathcal{A}$. Clearly, V_i satisfies $(\mathcal{A}-\lambda_i\mathcal{E})^{r_i} V_i = f(\mathcal{A})V = \{\mathbf{0}\}$.

Now, let's prove $V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$.

To this end, we need to prove two points. First, we need to show that every vector α in V can be expressed as $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_s, \alpha_i \in V_i, i = 1, 2, \cdots, s$. Secondly, this representation of the vector is unique. Clearly, $(f_1(\lambda), f_2(\lambda), \cdots, f_s(\lambda)) = 1$, thus there exist polynomials $\mu_1(\lambda), \mu_2(\lambda), \cdots, \mu_s(\lambda)$ such that $\mu_1(\lambda)f_1(\lambda) + \mu_2(\lambda)f_2(\lambda) + \cdots + \mu_s(\lambda)f_s(\lambda) = 1$.

Thus, $\mu_1(\mathcal{A})f_1(\mathcal{A}) + \mu_2(\mathcal{A})f_2(\mathcal{A}) + \cdots + \mu_s(\mathcal{A})f_s(\mathcal{A}) = \mathcal{E}$. In this way, for every vector α in V, we have $\alpha = \mu_1(\mathcal{A})f_1(\mathcal{A})\alpha + \mu_2(\mathcal{A})f_2(\mathcal{A})\alpha + \cdots + \mu_s(\mathcal{A})f_s(\mathcal{A})\alpha$, where $\mu_i(\mathcal{A})f_i(\mathcal{A})\alpha \in f_i(\mathcal{A})V = V_i, i = 1, 2, \cdots, s$. This proves the first point.

To prove the second point, suppose there is

$$\beta_1 + \beta_2 + \cdots + \beta_s = \mathbf{0}, \tag{10}$$

where β_i satisfies

$$(\mathcal{A}-\lambda_i\mathcal{E})^{r_i}\beta_i = \mathbf{0}, i = 1, 2, \cdots, s. \tag{11}$$

Now, we need to prove that any $\beta_i = \mathbf{0}$.

Since $(\lambda-\lambda_j)^{r_j} \mid f_i(\lambda)(j \neq i)$, we have $f_i(\mathcal{A})\beta_j = 0(j \neq i)$. Applying $f_i(\mathcal{A})$ to both sides of equation (10), we obtain $f_i(\mathcal{A})\beta_i = \mathbf{0}$.

Additionally, $\left(f_i(\lambda), (\lambda-\lambda_i)^{r_i}\right) = 1$. Thus, there exist polynomials $\mu(\lambda)$ and $\nu(\lambda)$ such that $\mu(\lambda)f_i(\lambda) + \nu(\lambda)(\lambda-\lambda_i)^{r_i} = 1$.

Thus, $\beta_i = \mu(\mathcal{A}) f_i(\mathcal{A})\beta_i + \nu(\mathcal{A})(\mathcal{A}-\lambda_i\mathcal{E})^{r_i}\beta_i = \mathbf{0}$.

Now, suppose $\alpha_1 + \alpha_2 + \cdots + \alpha_s = \mathbf{0}$, where $\alpha_i \in V_i$.

Clearly, α_i satisfies $(\mathcal{A}-\lambda_i)^{r_i}\alpha_i = \mathbf{0}, i = 1,2,\cdots,s$. Therefore, $\alpha_i = 0, i = 1,2,\cdots,s$. This implies that the representation in the first point is unique.

Now, suppose there is a vector $\alpha \in (\mathcal{A}-\lambda_i id)^{r_i}$ in the kernel. Express α as $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_s$, $\alpha_i \in V_i(i = 1,2,\cdots,s)$, that is,
$\alpha_1 + \alpha_2 + \cdots + (\alpha_i - \alpha) + \cdots + \alpha_s = \mathbf{0}$.

Let $\beta_j = \alpha_j, j \neq i, \beta_i = \alpha_i - \alpha$. Then $\beta_1, \beta_2, \cdots, \beta_s$ are vectors satisfying (10) and (11). Therefore, $\beta_1 = \beta_2 = \cdots = \beta_i = \cdots = \beta_s = \mathbf{0}$, and thus $\alpha = \alpha_i \in V_i$. This proves that $(\mathcal{A}-\lambda_i\mathcal{E})^{r_i}$ is the kernel of V_i, *i.e.*, $V_i = \left\{v \in V \mid (\mathcal{A}-\lambda_i\mathcal{E})^{r_i} v = \mathbf{0}\right\}$.

Definition 9 [14]. Let V, $\mathcal{A}$, and $f(\lambda)$ be as in Lemma 4.

We call $V_i = \left\{v \in V \mid (\mathcal{A}-\lambda_i\mathcal{E})^{r_i} v = \mathbf{0}\right\}$ the eigenspace of $\mathcal{A}$ corresponding to the eigenvalue λ_i, often denoted by V^{λ_i}.

Proposition 6 [14]. The root subspace of a linear transformation $\mathcal{A}$ on an n-dimensional vector space V over a field P is a nontrivial invariant subspace of $\mathcal{A}$.

3. The Equivalence of Root Subspace Decompositions

The following introduces the relationship between the root subspace decomposition of the characteristic polynomial and the root subspace decomposition of the minimal polynomial.

Theorem 1. Let $\mathcal{A}$ be a linear transformation on an n-dimensional linear space V over the field P. The minimal polynomial $m(\lambda)$ of $\mathcal{A}$ has a standard factorization in $P[x]$ as

$$m(\lambda) = (\lambda-\lambda_1)^{l_1}(\lambda-\lambda_2)^{l_2}\cdots(\lambda-\lambda_s)^{l_s}. \tag{12}$$

Denote $W^{\lambda_i} = Ker(\mathcal{A}-\lambda_i\mathcal{E})^{l_i}$.

To prove: For $t \geq l_i$, we have $Ker(\mathcal{A}-\lambda_i\mathcal{E})^t = W^{\lambda_i}$.

Proof. We use the method of mutual inclusion to prove that two sets are equal.

First, prove that $W^{\lambda_i} \subseteq Ker(\mathcal{A}-\lambda_i\mathcal{E})^t$.

Let $g(\lambda) = (\lambda-\lambda_1)^{l_1}\cdots(\lambda-\lambda_{i-1})^{l_{i-1}}(\lambda-\lambda_i)^t(\lambda-\lambda_{i+1})^{l_{i+1}}\cdots(\lambda-\lambda_s)^{l_s}$. Then $m(\lambda) \mid g(\lambda)$, so by Property 4, we have $g(\mathcal{A}) = \mathcal{O}$. Thus,

$$V = Ker(\mathcal{A}-\lambda_1\mathcal{E})^{l_1} \oplus \cdots \oplus Ker(\mathcal{A}-\lambda_{i-1}\mathcal{E})^{l_{i-1}} \oplus Ker(\mathcal{A}-\lambda_i\mathcal{E})^t \oplus \cdots \oplus Ker(\mathcal{A}-\lambda_s\mathcal{E})^{l_s}.$$

From equation (12), it follows that

$$\begin{aligned} V &= Ker(\mathcal{A}-\lambda_1\mathcal{E})^{l_1} \oplus \cdots \oplus Ker(\mathcal{A}-\lambda_{i-1}\mathcal{E})^{l_{i-1}} \oplus Ker(\mathcal{A}-\lambda_i\mathcal{E})^{l_i} \oplus \cdots \oplus Ker(\mathcal{A}-\lambda_s\mathcal{E})^{l_s} \\ &= W^{\lambda_1} \oplus \cdots \oplus W^{\lambda_{i-1}} \oplus W^{\lambda_i} \oplus \cdots \oplus W^{\lambda_s}. \end{aligned}$$

Take any $\alpha_i \in W^{\lambda_i}$.

Then $(\mathcal{A}-\lambda_i\mathcal{E})^t\alpha_i = (\mathcal{A}-\lambda_i\mathcal{E})^{t-l_i}(\mathcal{A}-\lambda_i\mathcal{E})^{l_i}\alpha_i = (\mathcal{A}-\lambda_i\mathcal{E})^{t-l_i}\mathbf{0} = \mathbf{0}$, hence $\alpha_i \in Ker(\mathcal{A}-\lambda_i\mathcal{E})^t$. Therefore, $W^{\lambda_i} \subseteq Ker(\mathcal{A}-\lambda_i\mathcal{E})^t$.

Next, prove that $Ker(\mathcal{A}-\lambda_i\mathcal{E})^t \subseteq W^{\lambda_i}$.

Choose a basis in W^{λ_i}, and extend it to a basis in $Ker(\mathcal{A}-\lambda_i\mathcal{E})^t$. Then extend it to a basis in W^{λ_1}, $\cdots$, and extend it to a basis in $W^{\lambda_{i-1}}$. Combining a basis of $Ker(\mathcal{A}-\lambda_i\mathcal{E})^t$, $\cdots$, and a basis of W^{λ_s}, we obtain a basis of *V*. Similarly, combining a basis of W^{λ_1}, $\cdots$, and a basis of $W^{\lambda_{i-1}}$, along with a basis of W^{λ_i}, $\cdots$, and a basis of W^{λ_s}, we also obtain a basis of *V*. Therefore, the aforementioned basis of W^{λ_i} is a basis of $Ker(\mathcal{A}-\lambda_i\mathcal{E})^t$.

Hence, $W^{\lambda_i}=Ker(\mathcal{A}-\lambda_i\mathcal{E})^t$.

Theorem 2. Given the conditions and notations as in Theorem 1, and the characteristic polynomial of $f(\lambda)=(\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_s)^{r_s}$ the linear transformation $\mathcal{A}$, then $W^{\lambda_i}=V^{\lambda_i}, i=1,2,\cdots,s$, where $V^{\lambda_i}=\left\{v\in V\mid(\mathcal{A}-\lambda_i\mathcal{E})^{r_i}v=\mathbf{0}\right\}$.

Proof. From Proposition 4 and Lemma 4, it follows that the characteristic polynomial $f(\lambda)$ of a linear transformation $\mathcal{A}$ on an *n*-dimensional vector space *V* over a field *P* is a multiple of the minimal polynomial $m(\lambda)$ of $\mathcal{A}$, *i.e.*, $m(\lambda)\mid g(\lambda)$. Since $m(\lambda)\mid g(\lambda)$, it follows that $l_i\le r_i, i=1,2,\cdots,s$. According to Theorem 1, we have $W^{\lambda_i}=V^{\lambda_i}$.

4. Examples

In this section, we will provide clear and concise examples to illustrate the three cases where the characteristic polynomial and the minimal polynomial are completely the same, partially the same, and completely different under the condition of having the same eigenvalues. This will aid in understanding.

Example 1. Given the matrix

$$A=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix},$$

proves that the eigenspaces corresponding to the eigenvalues of its characteristic polynomial and minimal polynomial are the same.

Proof. From *A*, we have the characteristic polynomial and the minimal polynomial are respectively $f(\lambda)=|\lambda E-A|=(\lambda-1)^2(\lambda-4)$ and $m(\lambda)=(\lambda-1)(\lambda-4)$. Therefore, 1 is a double root and 4 is a single root, so $\lambda_1=1$, $\lambda_2=4$.

The root subspaces corresponding to the eigenvalue λ_1 are given by $V^{\lambda_1}=\left\{v\in V\mid(A-E)^2v=0\right\}$ and $W^{\lambda_1}=\left\{v\in V\mid(A-E)v=0\right\}$.

Therefore $dimV^{\lambda_1}=2$, $dimW^{\lambda_1}=2$.

The root subspaces corresponding to the eigenvalue λ_2 are given by $V^{\lambda_2}=\left\{v\in V\mid(A-4E)v=0\right\}$ and $W^{\lambda_2}=\left\{v\in V\mid(A-4E)v=0\right\}$.

Therefore $dimV^{\lambda_2}=1$, $dimW^{\lambda_2}=1$.

In summary, $V^{\lambda_1}=W^{\lambda_1}$ and $V^{\lambda_2}=W^{\lambda_2}$.

Example 2. Given the matrix

$$A=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},$$

proves that the eigenspaces corresponding to the eigenvalues of its characteristic polynomial and minimal polynomial are the same.

Proof. From *A*, we have the characteristic polynomial and the minimal polynomial are respectively $f(\lambda)=|\lambda E-A|=(\lambda-1)^3$ and $m(\lambda)=(\lambda-1)^2$. Therefore, 1 is a triple root, so $\lambda=1$.

The root subspaces corresponding to the eigenvalue λ are given by
$V^{\lambda}=\{v\in V\mid (A-E)^3 v=0\}$ and $W^{\lambda}=\{v\in V\mid (A-E)^2 v=0\}$.

Therefore $dimV^{\lambda}=3$, $dimW^{\lambda}=3$.

In summary, $V^{\lambda}=W^{\lambda}$.

Example 3. Given the matrix

$$A=\begin{pmatrix}1 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix},$$

proves that the eigenspaces corresponding to the eigenvalues of its characteristic polynomial and minimal polynomial are the same.

Proof. From *A*, we have the characteristic polynomial and the minimal polynomial are respectively $f(\lambda)=|\lambda E-A|=(\lambda-1)(\lambda-2)(\lambda-3)$ and $m(\lambda)=(\lambda-1)(\lambda-2)(\lambda-3)$. Therefore, 1, 2 and 3 are all simple roots, so $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=3$.

The root subspaces corresponding to the eigenvalue λ_1 are given by
$V^{\lambda_1}=\{v\in V\mid (A-E)v=0\}$ and $W^{\lambda_1}=\{v\in V\mid (A-E)v=0\}$.

Therefore $dimV^{\lambda_1}=1$, $dimW^{\lambda_1}=1$.

The root subspaces corresponding to the eigenvalueare given by
$V^{\lambda_2}=\{v\in V\mid (A-2E)v=0\}$ and $W^{\lambda_2}=\{v\in V\mid (A-2E)v=0\}$.

Therefore $dimV^{\lambda_2}=1$, $dimW^{\lambda_2}=1$.

The root subspaces corresponding to the eigenvalue λ_2 are given by
$V^{\lambda_3}=\{v\in V\mid (A-3E)v=0\}$ and $W^{\lambda_3}=\{v\in V\mid (A-3E)v=0\}$.

Therefore $dimV^{\lambda_3}=1$, $dimW^{\lambda_3}=1$.

In summary, $V^{\lambda_1}=W^{\lambda_1}$, $V^{\lambda_2}=W^{\lambda_2}$ and $V^{\lambda_3}=W^{\lambda_3}$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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