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Applying the General Riccati Equation to Construct New Solitary Wave Solutions with Complex Structure of Burgers-Fisher Equation

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Abstract

In the realm of nonlinear physics, it is crucial to establish precise traveling wave solutions and solitary wave solutions for a variety of nonlinear models, as this aids our exploration of these fields. In this paper, we propose a new method to construct precise solitary wave solutions if nonlinear equation with complex structure. As an application, we employ this method to solve the Burgers-Fisher equation, yielding a multitude of new solitary wave solutions. This approach demonstrates a broader applicability in addressing nonlinear evolution equations (NLEEs).

Keywords

Riccati Equation, Hyperbolic Function Solutions, Nonlinear Evolution Equation, Solitary Wave Solution, Auxiliary Equation

1. Introduction

Most physical processes could be expressed as nonlinear evolution equations (NLEEs). The exploration of NLEEs has penetrated into various fields of natural science, such as hydrodynamics, optics, plasma, condensed matter physics, elementary particle physics, material physics, ocean engineering, astrophysics and biology, etc. [1]-[6] The exact solutions of NLEEs could be helpful to the understanding of the mechanism of many nonlinear phenomena and the processes in these different fields of natural science. It is very important to search for the analytical and numerical solutions of NLEEs, which has important physical and practical significance to study their characteristics, wave parameter information and their applications.

In recent years, a lot of fruitful work has been carried out in solving NLEEs. In various literatures, many powerful and effective methods have been proposed, for example, F-expansion method [7] [8], tanh-sech method and the extended tanh-coth method [9] [10], Jacobi elliptic function method [11] [12], auxiliary equation method [13]-[15], and so on. Most methods give various types of traveling wave solutions and solitary wave solutions for some special nonlinear equations. Among these methods, the auxiliary equation method is based on these original methods by introducing auxiliary equations to construct exact solutions of NLEEs. A suitable auxiliary equation could greatly simplify the solution process and give various complex forms of exact traveling wave and solitary wave interaction solutions, which has the great potential for most models.

In this research work, we solve the general Riccati equation through several different function transformation and obtain many new types of hyperbolic function solutions, which greatly extend the earlier Riccati equation method [16]. We use this general Riccati equation as an auxiliary equation to solve NLEEs and obtain many new types of solitary wave interaction solutions. As application, the solutions of Burgers-Fisher equation are discussed by this method. The result shows this modified method presents a wider applicability for handling NLEEs with a simplified process.

The manuscript is organized in the following way: in Section 2, the auxiliary equation with Riccati equation has been constructed, which gives abundant hyperbolic function solutions. This part is the theoretic basis of this paper. In Section 3, the main steps of the scheme are described in detail. This method is used to solve Burgers-Fisher equation to prove the wider applicability for handling NLEEs with a simplified process. Finally, the summary is given in Section 4.

2. Construction of Auxiliary Equation and its Abundant Hyperbolic Function Solutions

In Ref. [16], by using the following Riccati equation

$$f'(\xi) = f^2(\xi) + \mu \tag{1}$$

The following hyperbolic function solutions are obtained

$$f(\xi) = -\sqrt{-\mu} \cdot \tanh\left(\sqrt{-\mu} \cdot \xi\right), (\mu < 0) \tag{2}$$

$$f(\xi) = -\sqrt{-\mu} \cdot \coth\left(\sqrt{-\mu} \cdot \xi\right), (\mu < 0) \tag{3}$$

This method is simple and effective, and can be used to solve constant coefficient, variable coefficient, high-order and high-dimensional NLEEs. In this paper, we first consider the Riccati equation in the following general form [17]:

$$f'(\xi) = p_1 \cdot f^2(\xi) + q_1 \tag{4}$$

where p_1 and q_1 are constants to be determined later. From Equations (2) and (3), we can know Equation (4) has the following simple form hyperbolic function solutions:

$$f(\xi) = \tanh(\xi), (p_1 = -1, q_1 = 1) \tag{5}$$

$$f(\xi) = \coth(\xi), (p_1 = -1, q_1 = 1) \tag{6}$$

Next, we construct a new form of auxiliary function $g(\xi)$ to solve Equation (4), which satisfies the following relationship:

$$\left[g'(\xi)\right]^2 = p_2 \cdot g^2(\xi) + q_2 \tag{7}$$

where p_2 and q_2 are constants to be determined later. It is easy to know that Equation (7) has the following hyperbolic function solutions:

$$g(\xi) = \sinh(\xi), (p_2 = 1, q_2 = 1) \tag{8}$$

$$g(\xi) = \cosh(\xi), (p_2 = 1, q_2 = -1) \tag{9}$$

Equation (7) can be generalized to the following solutions:

$$g_1(\xi) = \left[\sinh(\xi) \pm \cosh(\xi)\right]^n, \left(p_2 = n^2, q_2 = 0, n \neq 0\right) \tag{10}$$

$$g_2(\xi) = C_0 \cdot \sinh(\xi) \pm C_1 \cdot \cosh(\xi), \left(p_2 = 1, q_2 = C_0^2 - C_1^2, C_0^2 + C_1^2 \neq 0\right) \tag{11}$$

$$\begin{gathered} g_3(\xi) = \sqrt{C_0 \cdot \sinh(\xi) + \varepsilon \cdot C_1 \cdot \cosh(\xi) \pm \sqrt{C_1^2 - C_0^2}}, \\ \left(p_2 = \frac{1}{4}, q_2 = \mp\frac{1}{2}\sqrt{C_1^2 - C_0^2}, C_0^2 + C_1^2 \neq 0, \varepsilon^2 = 1\right) \end{gathered} \tag{12}$$

Equation (10) is a set of solutions containing a large number of integral, fractional, surd and exponential hyperbolic functions. It is obvious that Equation (11) contains the solutions represented by Equations (8) and (9). Equation (11) is a generalization of Equation (10) in the solution of quadratic root hyperbolic function, which has also not been found in previous studies. When $C_0 = C_1$, Equation (12) degenerates to a set of solutions in Equation (10). In order to simplify the formula as much as possible, in the following, we make:

$$\begin{cases} A = C_0 \cdot \sinh(\xi) \pm C_1 \cdot \cosh(\xi) \\ B = C_0 \cdot \cosh(\xi) \pm C_1 \cdot \sinh(\xi) \\ C = \left[\sinh(\xi) \pm \cosh(\xi)\right]^n \end{cases} \tag{13}$$

Then suppose Equation (4) has the following formal solution:

$$f(\xi) = \frac{g'(\xi)}{g(\xi) + r} \tag{14}$$

where r is a constant to be determined later. Substituting Equation (14) into Equation (4) and using Equation (7), We can obtain the hyperbolic function solutions in the following form

$$f_1(\xi) = \frac{B}{A \pm \sqrt{C_1^2 - C_0^2}}, \left(p_1 = -\frac{1}{2}, q_1 = \frac{1}{2}, C_1^2 > C_0^2\right) \tag{15}$$

$$f_2(\xi)=\frac{B}{2\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\pm\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)},\quad\left(p_1=-\frac{1}{2},q_1=\frac{1}{8},C_1^2>C_0^2,\varepsilon^2=1\right) \tag{16}$$

Again suppose Equation (4) has the following formal solution

$$f(\xi)=\frac{g(\xi)\cdot g'(\xi)}{g^2(\xi)+r} \tag{17}$$

where r is a constant to be determined later. Substituting Equation (17) into Equation (4) and using Equation (7), We can obtain the following hyperbolic function solution:

$$f_3(\xi)=\frac{A\cdot B}{A^2+\frac{C_0^2-C_1^2}{2}},\left(p_1=-2,q_1=2,C_0^2\neq C_1^2\right) \tag{18}$$

It is obvious that $h(\xi)=1/f(\xi)$ is also the solution of Equation (4) in the condition of $p_1'=-q_1$, $q_1'=-p_1$. Equations (5) and (6) are a pair of solutions satisfying this condition. Therefore, the following three equations are also the solutions of Equation (4)

$$f_4(\xi)=\frac{A\pm\sqrt{C_1^2-C_0^2}}{B},\left(p_1=-\frac{1}{2},q_1=\frac{1}{2},C_1^2>C_0^2\right) \tag{19}$$

$$f_5(\xi)=\frac{2\cdot\left(A+\sqrt{C_1^2-C_0^2}\pm\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)}{B},\quad\left(p_1=-\frac{1}{8},q_1=\frac{1}{2},C_1^2>C_0^2,\varepsilon^2=1\right) \tag{20}$$

$$f_6(\xi)=\frac{A^2+\frac{C_0^2-C_1^2}{2}}{A\cdot B},\left(p_1=-2,q_1=2,C_0^2\neq C_1^2\right) \tag{21}$$

Then we introduce a more general Riccati equation in the following

$$z'(\xi)=p_3\cdot z^2(\xi)+q_3\cdot z(\xi)+r_3 \tag{22}$$

where p_3, q_3 and r_3 are constants to be determined later. We first use Equation (7) and the following form solution to solve Equation (22)

Then, we use the following form

$$z(\xi)=\frac{f(\xi)}{f^2(\xi)+k\cdot f(\xi)+r} \tag{23}$$

where k and r are constants to be determined later. Solving this case, we can obtain

Family VI: $p_3=-q_1-p_1k^2,q_3=2p_1k,r_3=-p_1,r=0,k\in R$ (24)

Family VII: $p_3=-4q_1-p_1k^2,q_3=2p_1k,r_3=-p_1,r=-\frac{q_1}{p_1},k\in R$ (25)

So the solutions write as

$$z_1(\xi)=\frac{A\pm\sqrt{C_1^2-C_0^2}}{B+k\cdot\left(A\pm\sqrt{C_1^2-C_0^2}\right)} \tag{26}$$

where $p_3=-\frac{1}{2}+\frac{k^2}{2},q_3=-k,r_3=\frac{1}{2},C_1^2>C_0^2,k\in R$.

$$z_2(\xi)=\frac{2\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\pm\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)}{B+2k\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\pm\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)} \tag{27}$$

where $p_3=-\frac{1}{8}+\frac{k^2}{2},q_3=-k,r_3=\frac{1}{2},C_1^2>C_0^2,\varepsilon^2=1,k\in R$.

$$z_3(\xi)=\frac{A^2+\frac{C_0^2-C_1^2}{2}}{A\cdot B+k\cdot\left(A^2+\frac{C_0^2-C_1^2}{2}\right)} \tag{28}$$

where $p_3=-2+2k^2,q_3=-4k,r_3=2,C_0^2\neq C_1^2,k\in R$.

$$z_4(\xi)=\frac{B}{A+k\cdot B\pm\sqrt{C_1^2-C_0^2}} \tag{29}$$

where $p_3=-\frac{1}{2}+\frac{k^2}{2},q_3=-k,r_3=\frac{1}{2},C_1^2>C_0^2,k\in R$.

$$z_5(\xi)=\frac{B}{2\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\pm\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)+k\cdot B} \tag{30}$$

where $p_3=-\frac{1}{2}+\frac{k^2}{8},q_3=-\frac{k}{4},r_3=\frac{1}{8},C_1^2>C_0^2,\varepsilon^2=1,k\in R$.

$$z_6(\xi)=\frac{A\cdot B}{2A^2+k\cdot A\cdot B+\frac{C_0^2-C_1^2}{2}} \tag{31}$$

where $p_3=-2+2k^2,q_3=-4k,r_3=2,C_0^2\neq C_1^2,k\in R$.

$$z_7(\xi)=\frac{B\cdot\left(A\pm\sqrt{C_1^2-C_0^2}\right)}{B^2+k\cdot B\cdot\left(A\pm\sqrt{C_1^2-C_0^2}\right)+\left(A\pm\sqrt{C_1^2-C_0^2}\right)^2} \tag{32}$$

where $p_3=-2+\frac{k^2}{2},q_3=-k,r_3=\frac{1}{2},C_1^2>C_0^2,k\in R$.

$$z_8(\xi)=\frac{2B\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\pm\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)}{B^2+2k\cdot B\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\pm\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)+\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\pm\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\sqrt{C_1^2-C_0^2}}\right)^2} \tag{33}$$

where $p_3 = -\frac{1}{2} + \frac{k^2}{2}, q_3 = -k, r_3 = \frac{1}{2}, C_1^2 > C_0^2, \varepsilon^2 = 1, k \in R$.

$$z_9(\xi) = \frac{A \cdot B \cdot \left(A^2 + \frac{C_0^2 - C_1^2}{2}\right)}{A^2 B^2 + k \cdot A \cdot B \cdot \left(A^2 + \frac{C_0^2 - C_1^2}{2}\right) + \left(A^2 + \frac{C_0^2 - C_1^2}{2}\right)^2} \tag{34}$$

where $p_3 = -8 + 2k^2, q_3 = -4k, r_3 = 2, C_0^2 \neq C_1^2, k \in R$.

In this case, when $r \neq 0$, the solutions corresponding to $1/f(\xi)$ are the same as that corresponding to $f(\xi)$, so we don't give the solutions for $f_4(\xi)$, $f_5(\xi)$ and $f_6(\xi)$.

3. Main Steps of the Scheme and Application

We consider the following NLEE

$$N(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \cdots) = 0 \tag{35}$$

Then suppose Equation (35) has the following traveling wave solution

$$u(x,t) = u(\xi), \ \xi = \mu x + ct \tag{36}$$

where μ and c are pending wave parameters. Substitute Equation (36) into Equation (46), and Equation (35) becomes an ordinary differential equation with the following form:

$$N(u, u', u'', \cdots) = 0 \tag{37}$$

where u' represents $\mathrm{d}u/\mathrm{d}\xi$. We assume that Equation (37) has the following formal solution:

$$u(\xi) = \sum_{i=0}^{n} a_i z^i(\xi) \tag{38}$$

where a_i are constants to be determined later and $z^i(\xi)$ satisfies Equation (35). The positive integer n can be obtained by the homogeneous balance between the dominant nonlinear term and the highest order derivative of $u(\xi)$ in Equation (37). In our method, the complex formal solution can be put in the solving process of the general Riccati Equation (22), because its solving process is relatively simple. The advantage of this is that on the one hand, it can greatly simplify the solution process of NLEEs, on the other hand, many complex solutions are actually the same group of solutions after simplification, but the simplification process is extremely complex, and we can exclude the same solutions by doing so.

Burgers-Fisher Equation

The following Burgers-Fisher equation [18] is considered

$$u_t + u \cdot u_x + u_{xx} + u \cdot (1 - u) = 0 \tag{39}$$

The traveling wave Equation (36) is substituted into Equation (39) and integrated once, and then the integration constant is set to zero to obtain

$$cu' + \mu uu' + \mu^2 u'' + u(1-u) = 0 \tag{40}$$

Considering the homogeneous balance, the formal solutions of Equation (39) can be expressed as

$$u(\xi) = a_0 + a_1 f(\xi) \tag{41}$$

Then substituting Equation (28) into Equation (40) and using Equation (22) yields a set of algebraic equations for a_0, a_1, μ and c. All terms with the same power of $z(\xi)$ together are collected and equate each coefficient to zero. At the end, solving the algebraic equations, a_0, a_1, μ and c can be obtained as follows:

$$\begin{cases} a_0 = \frac{1}{2} \pm \frac{q_3}{2} \cdot \sqrt{\frac{1}{q_3^2 - 4p_3r_3}} \\ a_1 = \pm p_3 \cdot \sqrt{\frac{1}{q_3^2 - 4p_3r_3}} \\ \mu = \mp \frac{1}{2} \cdot \sqrt{\frac{1}{q_3^2 - 4p_3r_3}} \\ c = \pm \frac{5}{4} \cdot \sqrt{\frac{1}{q_3^2 - 4p_3r_3}} \end{cases} \tag{42}$$

Thus, by selecting different solutions of Equation (22), the new solitary wave solutions of Equation (40) can be written as

$$u_1(\xi) = \frac{1}{2} \mp \frac{k}{2} \pm \left(-\frac{1}{2} + \frac{k^2}{2}\right) \cdot \frac{A + \varepsilon \cdot \sqrt{C_1^2 - C_0^2}}{B + k \cdot \left(A + \varepsilon \cdot \sqrt{C_1^2 - C_0^2}\right)} \tag{43}$$

where $\xi = \mu x + ct, \mu = \mp \frac{1}{2}, c = \pm \frac{5}{4}, C_1^2 > C_0^2, \varepsilon^2 = 1, k \in R$.

$$u_2(\xi) = \frac{1}{2} \mp k \pm \left(-\frac{1}{4} + k^2\right) \cdot \frac{2 \cdot \left(A + \varepsilon \cdot \sqrt{C_1^2 - C_0^2} + \lambda \cdot \sqrt{2\varepsilon \cdot \sqrt{C_1^2 - C_0^2}} \cdot \sqrt{A + \varepsilon \cdot \sqrt{C_1^2 - C_0^2}}\right)}{B + 2k \cdot \left(A + \varepsilon \cdot \sqrt{C_1^2 - C_0^2} + \lambda \cdot \sqrt{2\varepsilon \cdot \sqrt{C_1^2 - C_0^2}} \cdot \sqrt{A + \varepsilon \cdot \sqrt{C_1^2 - C_0^2}}\right)} \tag{44}$$

where $\xi = \mu x + ct, \mu = \mp 1, c = \pm \frac{5}{2}, C_1^2 > C_0^2, \varepsilon^2 = 1, \lambda^2 = 1, k \in R$.

$$u_3(\xi) = \frac{1}{2} \mp \frac{k}{2} \pm \left(-\frac{1}{2} + \frac{k^2}{2}\right) \cdot \frac{A^2 + \frac{C_0^2 - C_1^2}{2}}{A \cdot B + k \cdot \left(A^2 + \frac{C_0^2 - C_1^2}{2}\right)} \tag{45}$$

where $\xi = \mu x + ct, \mu = \mp \frac{1}{8}, c = \pm \frac{5}{16}, C_0^2 \neq C_1^2, k \in R$.

$$u_4(\xi) = \frac{1}{2} \mp \frac{k}{2} \pm \left(-\frac{1}{2} + \frac{k^2}{2}\right) \cdot \frac{B}{A + k \cdot B + \varepsilon \cdot \sqrt{C_1^2 - C_0^2}} \tag{46}$$

where $\xi=\mu x+ct,\mu=\mp\frac{1}{2},c=\pm\frac{5}{4},C_1^2>C_0^2,\varepsilon^2=1,k\in R$.

$$u_5(\xi)=\frac{1}{2}\mp\frac{k}{4}\pm\left(-1+\frac{k^2}{4}\right)\cdot\frac{B}{2\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}+\lambda\cdot\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)+k\cdot B} \tag{47}$$

where $\xi=\mu x+ct,\mu=\mp1,c=\pm\frac{5}{2},C_1^2>C_0^2,\varepsilon^2=1,\lambda^2=1,k\in R$.

$$u_6(\xi)=\frac{1}{2}\mp\frac{k}{2}\pm\left(-\frac{1}{2}+\frac{k^2}{2}\right)\cdot\frac{A\cdot B}{2A^2+k\cdot A\cdot B+\frac{C_0^2-C_1^2}{2}} \tag{48}$$

where $\xi=\mu x+ct,\mu=\mp\frac{1}{8},c=\pm\frac{5}{16},C_0^2\neq C_1^2,k\in R$.

$$u_7(\xi)=\frac{1}{2}\mp\frac{k}{4}\pm\left(-1+\frac{k^2}{4}\right)\cdot\frac{B\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\right)}{B^2+k\cdot B\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\right)+\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}\right)^2} \tag{49}$$

where $\xi=\mu x+ct,\mu=\mp\frac{1}{4},c=\pm\frac{5}{8},C_1^2>C_0^2,\varepsilon^2=1,k\in R$.

$$u_8(\xi)=\frac{1}{2}\mp\frac{k}{4}\pm\left(-1+\frac{k^2}{4}\right)\cdot\frac{2B\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}+\lambda\cdot\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)}{B^2+2k\cdot B\cdot\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}+\lambda\cdot\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)+\left(A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}+\lambda\cdot\sqrt{2\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\cdot\sqrt{A+\varepsilon\cdot\sqrt{C_1^2-C_0^2}}\right)^2} \tag{50}$$

where $\xi=\mu x+ct,\mu=\mp\frac{1}{4},c=\pm\frac{5}{8},C_1^2>C_0^2,\varepsilon^2=1,\lambda^2=1,k\in R$.

$$u_9(\xi)=\frac{1}{2}\mp\frac{k}{4}\pm\left(-1+\frac{k^2}{4}\right)\cdot\frac{A\cdot B\cdot\left(A^2+\frac{C_0^2-C_1^2}{2}\right)}{A^2B^2+k\cdot A\cdot B\cdot\left(A^2+\frac{C_0^2-C_1^2}{2}\right)+\left(A^2+\frac{C_0^2-C_1^2}{2}\right)^2} \tag{51}$$

where $\xi=\mu x+ct,\mu=\mp\frac{1}{16},c=\pm\frac{5}{32},C_0^2\neq C_1^2,k\in R$.

4. Examples of Solitary Wave Solutions in Fluids

In this section, we discuss the dynamic characteristics of some solutions by choosing some special values for the parameters in these solutions. Figure 1 and Figure 2 present two-dimensional and three-dimensional images of two sets of solutions to Burger Fisher equation, representing the propagation of kink solitary wave and traveling wave in fluids. These solitary wave and traveling wave maintain constant amplitude, velocity, wave number, and width during propagation, ensuring the stability of the waveform.

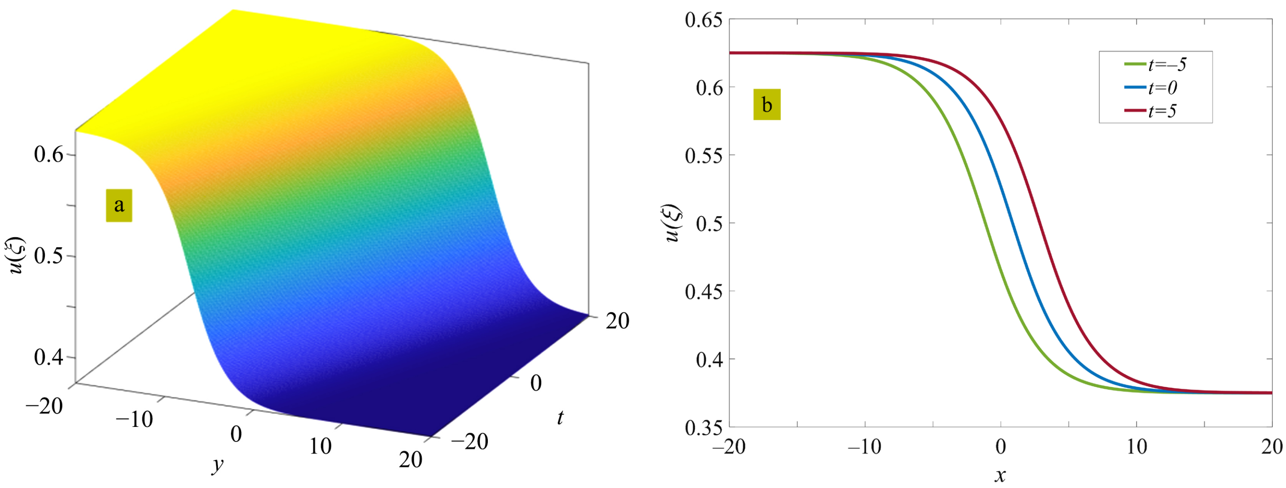

Figure 1. Three-dimensional plot (a) and two-dimensional evolution (b) in (x, t) phase space of a kink-solitary wave represented of Burgers-Fisher equation under the conditions of $\pm$ symbol is taken as −, $\mp$ symbol is taken as +, $\varepsilon = 1$, $\lambda = 1$, $C_1 = 2$ and $C_0 = 1$.

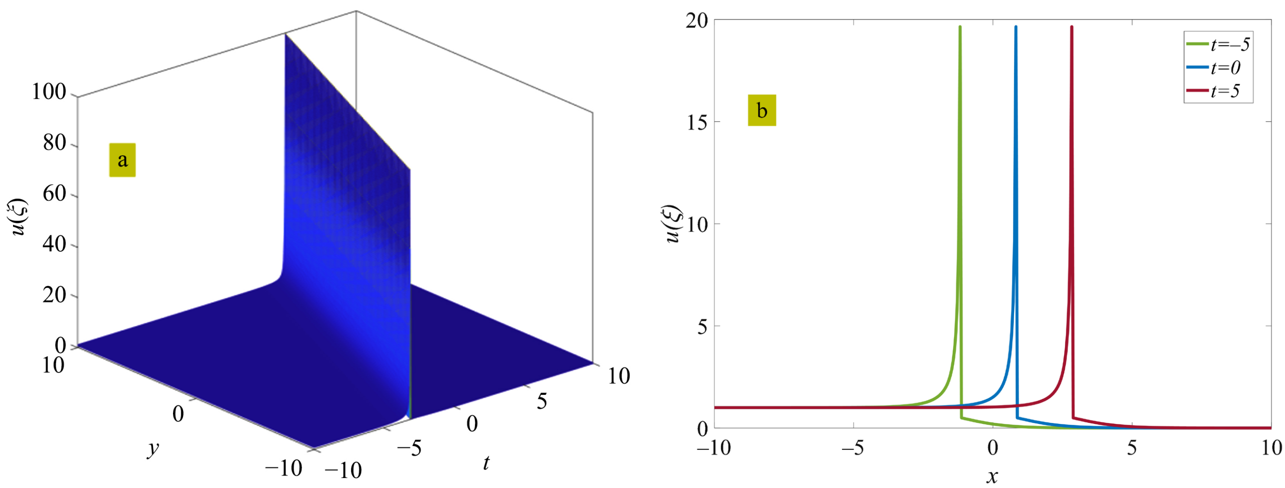

Figure 2. Three-dimensional plot (a) and two-dimensional evolution (b) in (x, t) phase space of a traveling wave represented by of Burgers-Fisher equation under the conditions of $\pm$ symbol is taken as −, $\mp$ symbol is taken as +, $\varepsilon = 1$, $C_1 = 2$ and $C_0 = 1$.

5. Summary

Applying the general Riccati equation, the abundant hyperbolic function solutions are successfully constructed for the complex structure. Using this method, the well-known nonlinear wave equation, Burgers-Fisher equation, is handled easily. As a result, abundant new types of solitary wave solutions are obtained by treating the general Riccati equation differently, many of which have not been found in other documents. The performance of this method is reliable and effective. More solutions are given by this method, which may be related to the mechanism of some nonlinear phenomena and processes in the different nature science, such as hydrodynamics, optics, plasma, etc. This part will be discussed in the further work for the specified physics process. The application of this method to the Burgers-Fisher equation proves that this method has the potential to establish more entirely new solutions for other kinds of nonlinear wave equations, which will be done next step.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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