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General Decay Synchronization of Competitive Fuzzy Neural Networks Involving Time Delays and Right-Hand Discontinuous Activation

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Abstract

This paper discusses the general decay synchronization problem for a class of fuzzy competitive neural networks with time-varying delays and discontinuous activation functions. Firstly, based on the concept of Filippov solutions for right-hand discontinuous systems, some sufficient conditions for general decay synchronization of the considered system are obtained via designing a nonlinear feedback controller and applying discontinuous differential equation theory, Lyapunov functional methods and some inequality techniques. Finally, one numerical example is given to verify the effectiveness of the proposed theoretical results. The general decay synchronization considered in this article can better estimate the convergence rate of the system, and the exponential synchronization and polynomial synchronization can be seen as its special cases.

Keywords

Competitive Neural Network, Fuzzy, General Decay Synchronization, Discontinuous Activation Function

1. Introduction

Neural networks are divided into biological neural networks and artificial neural networks. Biological neural networks refer to the network systems formed by the interconnection of neuronal cells in organisms that exist objectively in nature according to certain laws. Artificial neural networks are developed by simulating the structure and function of biological neural networks. As a special mathematical model, artificial neural networks are widely used in pattern recognition, combinatorial optimization, data-driven, machine learning, deep learning and other fields [1]-[5]. In practical applications, the uncertainty or fuzziness of nonlinear dynamic systems is inevitable. In order to explain fuzziness, Yang *et al.* further introduced the so-called fuzzy cellular neural network in 1996 and have achieved many results in this area since then [6] [7]. In addition, competitive neural networks can also solve many practical problems, the most typical of which is solving scheduling problems. Competitive learning rules can not only reduce the time consumed in obtaining coefficients, but also obtain an effective and sound solution [8].

Currently, most of the theoretical results on the stability and synchronization of neural networks are based on continuous activation functions, but in specific applications, the transmission of signals between neurons is usually discontinuous. For example, the frequent switching between states of a neural network may be discontinuous. Other, in some application areas, the analysis of artificial neural network systems often requires the use of right-hand discontinuous generalized differential equations to model them mathematically and research in this area is rare. Therefore, in-depth analysis of the dynamic behavior of the right-hand discontinuous neural networks is of great theoretical significance [9]-[11].

The synchronization of chaotic nonlinear systems plays an important role in many scientific fields, such as climatology, biology, sociology, etc. The synchronization of neural networks also has a large background of applications, such as signal processing, image encryption and secure communication. In recent years, research results on the synchronization control of neural networks have been successfully applied to the field of time series analysis. The study of synchronization in neural networks not only provides a good understanding of the nature of artificial neural networks, but also helps to understand the synchronization phenomena in nature more clearly. Even though there have been many studies on the synchronization of neural networks [12]-[15], there are almost no research results on the right-hand discontinuous fuzzy competitive neural networks. Therefore, it is of great theoretical significance and application value to explore the synchronization of right-hand discontinuous fuzzy competitive neural networks.

On the other hand, time delay is a common phenomenon in nature and in human society, such as the flow of steam and fluids in pipes and the transmission of electrical signals in networks. The time delay of a system implies that the evolution of a real system depends not only on the current state, but also on the past state at a certain moment or period of time, which makes the dynamic behavior of the system richer, but also often leads to oscillations, chaos and other complex behaviors that are not conducive to the stability of the system [16]-[18]. Therefore, how to study the stability of nonlinear systems with time delay and how to design control strategies to achieve the stability of the system has been a hot and difficult problem in the field of complex systems.

In recent decades, various types of neural networks, such as time-varying delay neural networks, fuzzy competitive neural networks, cellular neural networks, and

BAM neural networks, have been widely studied and applied in many fields, such as pattern recognition, parallel computing, and associative memory. Among them, the competitive neural networks are generalization of the classical Hopfield neural network and Cohen-Grossberg neural network, which takes into account both long-term and short-term memory variables. Due to the feature of combining activity and dynamic weights, competitive neural networks have received attention from scholars in recent years and have been widely used in image processing, optimization, and confidential communication [19]-[21]. Taking into account these characteristics of competitive neural networks, this paper investigates the generalized decay synchronization problem of right-hand discontinuous fuzzy competitive neural networks for the first time.

When studying the synchronization of chaotic systems, the estimation of the convergence rate is very important and challenging. Because in some cases, although we can determine that a system is asymptotically stable, we may not be able to estimate its convergence rate to the zero solution. Therefore, scholars have tried to overcome this difficulty by introducing a generalized rate of convergence. Inspired by the above analysis, this paper investigates the generalized decay synchronization of a kind of discontinuous activation function. To sum up, the innovations of this paper are listed as follows:

1) Due to practical applications, signal transmission between neurons is usually discontinuous and competitive neural network is a generalization of Hopfield neural network and Cohen Grossberg neural network, which has the characteristics of combining activity and dynamic weight. Therefore, this article first investigates the generalized decay synchronization problem of right-hand discontinuous fuzzy competitive neural networks by constructing a suitable Lyapunov extension function and applying some inequality techniques.

2) Due to time delay being a common phenomenon in nature and human society. So, in order to deeply analyze the dynamic behavior of right-hand discontinuous delayed neural networks, this paper investigates the generalized decay synchronization of right-hand discontinuous competitive neural networks by designing a novel nonlinear feedback controller with time-varying delays.

3) When the generalized decay function takes $\psi(t)=\mathrm{e}^{\alpha t}$ and $\psi(t)=(1+t)^{\alpha}$, the exponential and asymptotic synchronization can be seen as special cases. From this perspective, the generalized decay synchronization studied in this paper has good application prospects.

The remaining parts of this article are structured as follows: The relevant definitions, assumptions and important lemmas are given in Section 2. The main research process and results of this paper are given in Section 3. In Section 4, a numerical example is used to prove the correctness of the theoretical results. Finally, Section 5 summarizes this paper and gives the future research direction.

Notations. The symbols $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of all real numbers and all n-dimensional real vectors. STM denotes short-term memory, LTM denotes long-term memory; $\wedge$ and $\vee$ are expressed as fuzzy AND operations and fuzzy OR operations, respectively. $\mathrm{co}(\cdot)$ represents the closed convex package.

2. Notations and Preliminaries

In this paper, we consider the model of fuzzy competitive neural networks given by following differential equation:

$$\begin{cases} \text{STM}: \dot{p}_k(t) = -c_k p_k(t) + \sum_{\ell=1}^{n} \tilde{a}_{k\ell} g_\ell(p_\ell(t)) + \sum_{\ell=1}^{n} \tilde{b}_{k\ell} g_\ell(p_\ell(t-\tau_{k\ell}(t))) \\ \qquad\qquad + \bigwedge_{\ell=1}^{n} \alpha_{k\ell} g_\ell(p_\ell(t-\tau_{k\ell}(t))) + \bigvee_{\ell=1}^{n} \beta_{k\ell} g_\ell(p_\ell(t-\tau_{k\ell}(t))) \\ \qquad\qquad + B_k \sum_{\ell=1}^{n} m_{k\ell}(t) h_\ell, \\ \text{LTM}: \dot{m}_{k\ell}(t) = -d_k m_{k\ell}(t) + E_k h_\ell g_k(p_k(t)), \quad k,\ell = 1,2,\cdots,n, \end{cases} \tag{1}$$

where $\tilde{a}_{k\ell}, \tilde{b}_{k\ell}$ denote the connection weight between the k neuron and the ℓ neuron; $\alpha_{k\ell}, \beta_{k\ell}$ are the elements of the fuzzy feedback min and max templates, respectively; c_k, d_k are constants; n indicates the number of neurons. E_k is a positive constant; $p(t) = (p_1(t), \cdots, p_n(t))^{\mathrm{T}}$, $p_k(t)$ denotes the activation level of the k neurons; $m_{k\ell}(t)$ is the salience efficiency, B_k is the intensity of the external stimulus; $g(p(t)) = (g_1(p_1(t)), \cdots, g_n(p_n(t)))^{\mathrm{T}}$; h_ℓ is the constant external stimulus; $\tau_{k\ell}(t)$ corresponds to the time-varying delays and satisfies $0 \le \tau_{k\ell}(t) \le \tau$.

Let $S_k = \sum_{\ell=1}^{n} m_{k\ell}(t) h_\ell = H^{\mathrm{T}} m_k(t)$, where $m_k = (m_{k1}, m_{k2}, \cdots, m_{kn})^{\mathrm{T}}$, $H = (h_1, h_2, \cdots, h_n)^{\mathrm{T}}$. Normalize the magnitude of H, *i.e.* let $|H|^2 = 1$, then the following driving system can be obtained based on the above model:

$$\begin{cases} \text{STM}: \dot{p}_k(t) = -c_k p_k(t) + \sum_{\ell=1}^{n} \tilde{a}_{k\ell} g_\ell(p_\ell(t)) + \sum_{\ell=1}^{n} \tilde{b}_{k\ell} g_\ell(p_\ell(t-\tau_{k\ell}(t))) \\ \qquad\qquad + \bigwedge_{\ell=1}^{n} \alpha_{k\ell} g_\ell(p_\ell(t-\tau_{k\ell}(t))) + \bigvee_{\ell=1}^{n} \beta_{k\ell} g_\ell(p_\ell(t-\tau_{k\ell}(t))) \\ \qquad\qquad + B_k S_k(t), \\ \text{LTM}: \dot{S}_k(t) = -d_k S_k(t) + E_k g_k(p_k(t)), \quad k,\ell = 1,2,\cdots,n, \end{cases} \tag{2}$$

where the initial condition of the system (2) satisfies:

$$p_k(\epsilon) = \varphi_k^{\epsilon}(\epsilon), \quad \epsilon \in [-\tau, 0],$$
$$S_k(\epsilon) = \varphi_k^{s}(\epsilon), \quad \epsilon \in [-\tau, 0].$$

Remark 1. The competitive neural networks studied in this paper is a right-hand discontinuous dynamical system, so an important theoretical problem to overcome is to give a suitable definition of the solution of the right-hand discontinuous differential equation.

Remark 2. The definition of a solution in the Filippov sense is used to analyze the existence of solutions of discontinuous systems arising in engineering applications, due to the fact that the Filippov solution is defined as the limit of a series of continuous solutions. It provides us with a way to approximate the solution of a discontinuous system by the limit of the solution of a continuous system, so that we can use the solution trajectory in the Filippov sense to approximate the solution trajectory of a discontinuous system arising in practical applications. This method has important implications in the fields of sliding mode control, non-smooth analysis, etc.

Definition 1 [22]. Let $E \subset \mathbb{R}^n$, if for $\forall p \in E$, corresponding to a non-empty set $F(p) \subset \mathbb{R}^n$, then we say that $p \mapsto F(p)$ is a set-valued map defined on $E \mapsto \mathbb{R}^n$.

Definition 2 [23]. For the right-hand discontinuous system $\dot{Z}(t) = h(t, Z_t)$, $Z_t(s) = Z(t+s)$, $s \in [-\tau, 0]$, define the following set-valued mapping:

$$\mathbb{H}(t, Z_t) \triangleq \bigcap_{\tau>0} \bigcap_{\mu(N)=0} K\left[h\left(\mathfrak{B}(Z_t, \tau) \setminus N\right)\right],$$

where $h: \mathbb{R} \times C \to \mathbb{R}^n$ is a measurable and eventually bounded function.

$$\mathfrak{B}(Z_t, \tau) := \left\{Z_t^* : \left\|Z_t^* - Z_t\right\| \le \tau\right\}$$

where $\mu(N)$ is the Leberg measure of the set N.

Definition 3 [24]. Suppose the function $V: \mathbb{R}^n \to R$ satisfies the local Lipschitz condition and defines the following operator:

$$\partial V(Z) \triangleq \mathrm{co}\left\{\lim_{k\to\infty} \nabla V\left(Z_k \to Z, Z_k \notin \Omega_V \cup N\right)\right\},$$

where Ω_V is the set of all integrable points of V, and N is the set of measure 0. Then, ∂V is the Clarke generalized gradient of V corresponding to Z.

From the above definition, the solution $Z(t)$ of the system (1) in the Filippov sense has to satisfy the following differential inclusion of the fuzzy contention neural network:

$$\begin{cases} \mathrm{STM}: \dot{p}_k(t) \in -c_k p_k(t) + \sum_{\ell=1}^{n} \tilde{a}_{k\ell} K\left[g_\ell\left(p_\ell(t)\right)\right] + \sum_{\ell=1}^{n} \tilde{b}_{k\ell} K\left[g_\ell\left(p_\ell\left(t-\tau_{k\ell}(t)\right)\right)\right] \\ \qquad + \bigwedge_{\ell=1}^{n} \alpha_{k\ell} K\left[g_\ell\left(p_\ell\left(t-\tau_{k\ell}(t)\right)\right)\right] \\ \qquad + \bigvee_{\ell=1}^{n} \beta_{k\ell} K\left[g_\ell\left(p_\ell\left(t-\tau_{k\ell}(t)\right)\right)\right] + B_k S_k(t), \\ \mathrm{LTM}: \dot{S}_k(t) \in -d_k S_k(t) + E_k K\left[g_k\left(p_k(t)\right)\right], \quad k, \ell = 1, 2, \cdots, n. \end{cases} \tag{3}$$

There exists $\gamma_\ell(t) \in K\left[g_\ell\left(p_\ell(t)\right)\right]$, such that:

$$\begin{cases} \mathrm{STM}: \dot{p}_k(t) = -c_k p_k(t) + \sum_{\ell=1}^{n} \tilde{a}_{k\ell} \gamma_\ell(t) + \sum_{\ell=1}^{n} \tilde{b}_{k\ell} \gamma_\ell\left(t-\tau_{k\ell}(t)\right) \\ \qquad + \prod_{\ell=1}^{n} \alpha_{k\ell} \gamma_\ell\left(t-\tau_{k\ell}(t)\right) + \bigvee_{\ell=1}^{n} \beta_{k\ell} \gamma_\ell\left(t-\tau_{k\ell}(t)\right) + B_k S_k(t), \\ \mathrm{LTM}: \dot{S}_k(t) = -d_k S_k(t) + E_k \gamma_k(t), \quad k, \ell = 1, 2, \cdots, n. \end{cases} \tag{4}$$

In the following, we take systems (3) and (4) as the drive system, with the following response system:

$$\begin{cases} \mathrm{STM}:\dot{q}_k(t)\in -c_k q_k(t)+\sum_{\ell=1}^{n}\tilde{a}_{k\ell}K\left[g_\ell\left(q_\ell(t)\right)\right]+\sum_{\ell=1}^{n}\tilde{b}_{k\ell}K\left[g_\ell\left(q_\ell\left(t-\tau_{k\ell}(t)\right)\right)\right] \\ \qquad\qquad +\bigwedge_{\ell=1}^{n}\alpha_{k\ell}K\left[g_\ell\left(q_\ell\left(t-\tau_{k\ell}(t)\right)\right)\right] \\ \qquad\qquad +\bigvee_{\ell=1}^{n}\beta_{k\ell}K\left[g_\ell\left(q_\ell\left(t-\tau_{k\ell}(t)\right)\right)\right]+B_k W_k(t)+u_k(t) \\ \mathrm{LTM}:\dot{W}_k(t)\in -d_k W_k(t)+E_k K\left[g_k\left(q_k(t)\right)\right]+\tilde{u}_k(t), \quad k,\ell=1,2,\cdots,n, \end{cases} \tag{5}$$

where $q(t)\in R^n$ is the state variable of the response system, and $u_k(t)$, $\tilde{u}_k(t)$ are the controllers to be designed.

Similarly, there exists $\eta_\ell(t)\in K\left[g_\ell\left(q_\ell(t)\right)\right]$, such that:

$$\begin{cases} \mathrm{STM}:\dot{q}_k(t)=-c_k q_k(t)+\sum_{\ell=1}^{n}\tilde{a}_{k\ell}\eta_\ell(t)+\sum_{\ell=1}^{n}\tilde{b}_{k\ell}\eta_\ell\left(t-\tau_{k\ell}(t)\right) \\ \qquad\qquad +\prod_{\ell=1}^{n}\alpha_{k\ell}\eta_\ell\left(t-\tau_{k\ell}(t)\right)+\bigvee_{\ell=1}^{n}\beta_{k\ell}\eta_\ell\left(t-\tau_{k\ell}(t)\right) \\ \qquad\qquad +B_k W_k(t)+u_k(t), \\ \mathrm{LTM}:\dot{W}_k(t)=-d_k W_k(t)+E_k\eta_\ell q_k(t)+\tilde{u}_k(t), \quad k=1,2,\cdots,n. \end{cases} \tag{6}$$

For the sake of proof, let us make the following assumptions:

Assumption 1 [25]: For each k, $h_k(\cdot)$ is continuous in $\mathbb{R}$ except for a countable number of interruptions ρ_k^x; the right limit $h_k^+\left(\rho_x^k\right)$ and the left limit $h_k^-\left(\rho_x^k\right)$ on ρ_k^x exist, and h_k has a finite number of discontinuities in any tight interval in $\mathbb{R}$.

Assumption 2 [22]: For every $\ell=1,2,\cdots,n$, there exist non-negative constants L_ℓ and N_ℓ such that:

$$\sup\left|\gamma_\ell-\eta_\ell\right|\le L_\ell\left|p_\ell-q_\ell\right|+N_\ell, \quad \forall p_\ell,q_\ell\in\mathbb{R},$$

where

$$\gamma_\ell\in K\left[g_\ell\left(p_\ell\right)\right], \quad \eta_\ell\in K\left[g_\ell\left(q_\ell\right)\right]$$
$$K\left[g_\ell(\cdot)\right]=\left[\min\left\{g_\ell^-(\cdot),g_\ell^+(\cdot)\right\},\max\left\{g_\ell^-(\cdot),g_\ell^+(\cdot)\right\}\right].$$

Assumption 3 [26]: $\tau_{k\ell}(t)$ is differentiable and there exists a real number $0\le K_{k\ell}<1$ such that:

$$0\le\dot{\tau}_{k\ell}(t)\le K_{k\ell}.$$

The initial condition of the system (6) is $q(s)=\left(q_1(s),\cdots,q_n(s)\right)^{\mathrm{T}}=\phi(s)\in C\left(\left[-\tau,0\right),\mathbb{R}^n\right)$. a.e. $t\ge 0$, $k\in I$. where q_k is the state of the response system, $c_k(t),a_{k\ell}(t)$ and $b_{k\ell}(t)$ depend on the initial conditions of the system (6). $\phi(s),u_k(t)$ are the control inputs for the design to be determined.

Let $e_k(t)=q_k(t)-p_k(t)$, $Z_k(t)=W_k(t)-S_k(t)$, then the error system is:

$$\begin{cases} \mathrm{STM}: \dot{e}_k(t) = -c_k e_k + \sum_{\ell=1}^{n} \tilde{a}_{k\ell} \lambda_\ell(t) + \sum_{\ell=1}^{n} \tilde{b}_{k\ell} \lambda_\ell(t-\tau_{k\ell}(t)) \\ \qquad\qquad + \bigwedge_{\ell=1}^{n} \alpha_{k\ell} \lambda_\ell(t-\tau_{k\ell}(t)) + \bigvee_{\ell=1}^{n} \beta_{k\ell} \lambda_\ell(t-\tau_{k\ell}(t)) \\ \qquad\qquad + B_k Z_k(t) + u_k(t), \\ \mathrm{LTM}: \dot{Z}_k(t) = -d_k Z_k(t) + E_k \lambda_k(t) + \tilde{u}_k(t), \quad k,\ell = 1,2,\cdots,n, \end{cases} \tag{7}$$

where $\lambda_\ell(t) = \eta_\ell(t) - \gamma_\ell(t)$, $\lambda_\ell(t-\tau_{k\ell}(t)) = \eta_\ell(t-\tau_{k\ell}(t)) - \gamma_\ell(t-\tau_{k\ell}(t))$.

Lemma 1 [23]. Let p and q be the two state variables of the systems (4) and (6), then we have the following inequalities:

$$\left| \bigwedge_{\ell=1}^{n} \alpha_{k\ell} g_\ell(q_\ell) - \bigwedge_{\ell=1}^{n} \alpha_{k\ell} g_\ell(p_\ell) \right| \le \sum_{\ell=1}^{n} |\alpha_{k\ell}| |g_\ell(q_\ell) - g_\ell(p_\ell)|,$$

$$\left| \bigvee_{\ell=1}^{n} \beta_{k\ell} g_\ell(q_\ell) - \bigvee_{\ell=1}^{n} \beta_{k\ell} g_\ell(p_\ell) \right| \le \sum_{\ell=1}^{n} |\beta_{k\ell}| |g_\ell(q_\ell) - g_\ell(p_\ell)|.$$

where the initial conditions for $p_k(t)$ and $S_k(t)$ satisfy:

$$\bar{\varphi}_k^p(\epsilon) = \begin{cases} \varphi_k^p(\epsilon), & -\tau \le \epsilon \le 0, \\ \varphi_k^p(-\tau), & -\tau-\sigma \le \epsilon \le -\tau, \end{cases}$$

$$\bar{\varphi}_k^s(\epsilon) = \begin{cases} \varphi_k^s(\epsilon), & -\tau \le \epsilon \le 0, \\ \varphi_k^s(-\tau), & -\tau-\sigma \le \epsilon \le -\tau. \end{cases}$$

Initial conditions for system (7) satisfies $e_k(\epsilon) = \phi_k^q(\epsilon) - \bar{\varphi}_k^p(\epsilon)$, $Z_k(\epsilon) = \phi_k^w(\epsilon) - \bar{\varphi}_k^s(\epsilon)$, ($-\tau \le \epsilon \le 0 \quad k \in I$).

Definition 4 [26]. A function $\psi: R^+ \to [1,+\infty)$ is called a ψ-type function if it satisfies the following four conditions:

1) it is differentiable and non-decreasing;

2) $\psi(0) = 1$ and $\psi(+\infty) = +\infty$;

3) $\tilde{\psi}(t) = \dot{\psi}(t)/\psi(t)$ is non-decreasing and $\psi^* = \sup_{t\ge0} \tilde{\psi}(t) < +\infty$;

4) for $\forall t,s \ge 0$, $\psi(t+s) \le \psi(t)\psi(s)$.

Since for $\forall \alpha > 0$, the exponential function $\psi(t) = \mathrm{e}^{\alpha t}$ and the polynomial function $\psi(t) = (1+t)^\alpha$ satisfy the four conditions above, and thus these two functions are ψ-type functions.

Assumption 4 [26]: There exists a function $\varrho(t) \in C(R, R^+)$ and a scalar $\varepsilon > 0$, such that for any $t \ge 0$, there $\tilde{\psi}(t) \le 1$, $\sup_{t\in[t,+\infty)} \int_0^t \psi^\varepsilon(s) \varrho(s) \mathrm{d}s < +\infty$.

Lemma 2 [27]. Assuming that the basic conditions (3) of the hypothesis are satisfy, and the synchronization error $e_k(t) = q_k(t) - p_k(t)$ between the drive-response systems (4) and (6) satisfies the differential equation $\dot{e}_k(t) = f(t, e_t)$, where $e_t(s) = e(t+s)$, for $s \in [-\tau, 0]$ the function $f(t, e_t)$ is locally bounded. If there exist differentiable functions $V(t, e_t): R^+ \times C \to R^+$, and positive constants λ_1, λ_2 such that for any $(t, e_t) \in R^+ \times C$, there is:

$$\left(\lambda_1 \left\|e(t)\right\|^2\right) \le V\left(t, e_t\right), \tag{8}$$

$$\frac{\mathrm{d}V\left(t, e_t\right)}{\mathrm{d}t} \le -\epsilon V\left(t, e_t\right) + \lambda_2 \varrho(t), \tag{9}$$

where $p(t)$ and $f(t)$ are the solutions of systems (4) and (6), respectively, $\epsilon > 0$ and $\varrho(t)$ defined in Assumption 4. The drive-response systems (4) and (6) will achieve generalized decay synchronization and converge at ε.

3. Main Results

In this section, we will obtain sufficient conditions for the drive-response systems (4) and (6) to achieve generalized decay synchronization. Design the controller as follows:

$$\begin{cases} u_k(t) = -\zeta_k \operatorname{sign}\left(e_k(t)\right) - \dfrac{\eta_k \left\|e(t)\right\|^2 e_k(t)}{\left\|e(t)\right\|^2 + \varrho(t)}, \\ \tilde{u}_k(t) = -\Pi_k \operatorname{sign}\left(Z_k(t)\right) - \dfrac{H_k \left\|Z(t)\right\|^2 Z_k(t)}{\left\|Z(t)\right\|^2 + \varrho(t)}, \end{cases} \tag{10}$$

where $k \in I$, ζ_k, η_k, Π_k and H_k positive control gain satisfies:

$$\begin{cases} -c_k - \eta_k + \dfrac{L_k |B_k|}{2} + \dfrac{L_k |E_k|}{2} \\ +\dfrac{1}{2}\displaystyle\sum_{\ell=1}^{n}\left(|\tilde{a}_{k\ell}| L_\ell + |\tilde{a}_{k\ell}| L_k + |\tilde{b}_{k\ell}| L_\ell + |\alpha_{k\ell}| L_\ell + |\beta_{k\ell}| L_\ell + 2\omega_{k\ell} + 2\tau_{k\ell}\varpi_{k\ell}\right) < 0, \\ -\zeta_k + \displaystyle\sum_{\ell=1}^{n}\left(|\tilde{a}_{k\ell}| + |\tilde{b}_{k\ell}| + |\alpha_{k\ell}| + |\beta_{k\ell}| + |B_k|\right) N_\ell < 0, \\ -d_k - H_k + \dfrac{L_k |E_k|}{2} + \dfrac{L_k |B_k|}{2} < 0, \\ -\Pi_k + \displaystyle\sum_{\ell=1}^{n} |E_k| N_\ell < 0, \\ -\omega_{k\ell} + |\tilde{b}_{\ell k}| + |\alpha_{\ell k}| + |\beta_{\ell k}| < 0. \end{cases} \tag{11}$$

Using the nonlinear feedback controller (10), the following theorem is obtained.

Theorem 1. If Assumptions 1 - 3 hold, and the control gain of controller (10) satisfies Equation (11), then the drive-response systems (4) and (6) achieve generalized decay synchronization under controller (10).

Proof. Construct the following Lyapunov-Krasovskii function:

$$\begin{aligned} V(t) = &\sum_{k=1}^{n} \frac{1}{2} e_k^2(t) + \sum_{k=1}^{n} \frac{1}{2} Z_k^2(t) + \sum_{k=1}^{n}\sum_{\ell=1}^{n} \int_{t-\tau_{k\ell}(t)}^{t} \omega_{k\ell} e_k^2(s)\,\mathrm{d}s \\ &+ \sum_{k=1}^{n}\sum_{\ell=1}^{n} \int_{-\tau_{k\ell}}^{0} \int_{t+s}^{t} \varpi_{k\ell} e_k^2(\varsigma)\,\mathrm{d}\varsigma\,\mathrm{d}s, \end{aligned} \tag{12}$$

then there exists a positive constant $\xi > 1, r > 1$ such that:

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^{n}e_k^2(t)+\frac{1}{2}\sum_{k=1}^{n}Z_k^2(t)\le V(t)\\&\le\xi\sum_{k=1}^{n}e_k^2(t)+r\sum_{k=1}^{n}Z_k^2(t)+\frac{\xi}{\sigma}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\varpi_{k\ell}\int_{t-\tau_{k\ell}}^{t}e_k^2(s)\mathrm{d}s,\end{aligned}\tag{13}$$

where $\sigma=\min_{k\in I}\{\sigma_k\}$, $\varsigma=\min_{k\in I}\{\varsigma_k\}$.

$$\begin{aligned}\sigma_k&\triangleq -c_k-\eta_k+\frac{L_k|B_k|}{2}+\frac{L_k|E_k|}{2}\\&\quad+\frac{1}{2}\sum_{\ell=1}^{n}\left(|\tilde{a}_{k\ell}|L_\ell+|\tilde{a}_{\ell k}|L_k+|\tilde{b}_{k\ell}|L_\ell+|\alpha_{k\ell}|L_\ell+|\beta_{k\ell}|L_\ell+2\omega_{k\ell}+2\tau_{k\ell}\varpi_{k\ell}\right)\\&<0\\\varsigma_k&\triangleq d_k+H_k-\frac{L_k|E_k|}{2}-\frac{L_k|B_k|}{2}>0.\end{aligned}$$

Calculating the derivative of $V(t)$ gives:

$$\begin{aligned}\dot{V}(t)=\sum_{k=1}^{n}\Bigg\{&e_k(t)\Bigg[-c_ke_k(t)+\sum_{\ell=1}^{n}\tilde{a}_{k\ell}\lambda_\ell(t)+\sum_{\ell=1}^{n}\tilde{b}_{k\ell}\lambda_\ell(t-\tau_{k\ell}(t))\\&+\bigwedge_{\ell=1}^{n}\alpha_{k\ell}\lambda_\ell(t-\tau_{k\ell}(t))+\bigvee_{\ell=1}^{n}\beta_{k\ell}\lambda_\ell(t-\tau_{k\ell}(t))+B_\ell Z_\ell(t)\\&-\zeta_k\,\mathrm{sign}(e_k(t))-\frac{\eta_k\|e(t)\|^2e_k(t)}{\|e(t)\|^2+\varrho(t)}\Bigg]+\sum_{\ell=1}^{n}\omega_{k\ell}\left(e_k^2(t)-e_k^2(t-\tau_{k\ell}(t))\right)\\&+\sum_{\ell=1}^{n}\varpi_{k\ell}(\tau_{k\ell})\left[e_k^2(t)-\int_{t-\tau_{k\ell}}^{t}e_k^2(s)\mathrm{d}s\right]\Bigg\}\\&+\sum_{k=1}^{n}\left\{Z_k(t)\left[-d_kZ_k(t)+E_k\lambda_k(t)-\Pi_k\,\mathrm{sign}(Z_k(t))-\frac{H_k\|Z(t)\|^2Z_k(t)}{\|Z(t)\|^2+\varrho(t)}\right]\right\}.\end{aligned}$$

By Assumption 2 and the inequality $2ab\le a^2+b^2$, for any $a>0,b>0$, there is:

$$\begin{aligned}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\tilde{a}_{k\ell}e_k(t)\lambda_\ell(t)&\le\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\tilde{a}_{k\ell}||e_k(t)||\lambda_\ell(t)|\\&\le\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\tilde{a}_{k\ell}||e_k(t)|\left(L_\ell|e_\ell(t)|+N_\ell\right)\\&\le\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\tilde{a}_{k\ell}|\left[\frac{L_\ell}{2}\left(e_k^2(t)+e_\ell^2(t)\right)+N_\ell|e_k(t)|\right],\end{aligned}$$

similarly,

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{\ell=1}^{n}\tilde{b}_{k\ell}e_k(t)\lambda_\ell(t-\tau_{k\ell}(t))\\&\le\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\tilde{b}_{k\ell}||e_k(t)||\lambda_\ell(t-\tau_{k\ell}(t))|\\&\le\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\tilde{b}_{k\ell}||e_k(t)|\left(L_\ell|e_\ell(t-\tau_{k\ell}(t))|+N_\ell\right)\\&\le\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\tilde{b}_{k\ell}|\left[\frac{L_\ell}{2}\left(e_k^2(t)+e_\ell^2(t-\tau_{k\ell}(t))\right)+N_\ell|e_k(t)|\right]\end{aligned}$$

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n}\alpha_{k\ell}e_k(t)\lambda_\ell(t-\tau_{k\ell}(t))$$
$$\leq\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\alpha_{k\ell}||e_k(t)||\lambda_\ell(t-\tau_{k\ell}(t))|$$
$$\leq\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\alpha_{k\ell}||e_k(t)|\left(L_\ell|e_\ell(t-\tau_{k\ell}(t))|+N_\ell\right)$$
$$\leq\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\alpha_{k\ell}|\left[\frac{L_\ell}{2}\left(e_k^2(t)+e_\ell^2(t-\tau_{k\ell}(t))\right)+N_\ell|e_k(t)|\right]$$
$$\sum_{k=1}^{n}\sum_{\ell=1}^{n}\beta_{k\ell}e_k(t)\lambda_\ell(t-\tau_{k\ell}(t))$$
$$\leq\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\beta_{k\ell}||e_k(t)||\lambda_\ell(t-\tau_{k\ell}(t))|$$
$$\leq\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\beta_{k\ell}||e_k(t)|\left(L_\ell|e_\ell(t-\tau_{k\ell}(t))|+N_\ell\right)$$
$$\leq\sum_{k=1}^{n}\sum_{\ell=1}^{n}|\beta_{k\ell}|\left[\frac{L_\ell}{2}\left(e_k^2(t)+e_\ell^2(t-\tau_{k\ell}(t))\right)+N_\ell|e_k(t)|\right]$$
$$\sum_{k=1}^{n}B_ke_k(t)Z_k(t)\leq\sum_{k=1}^{n}|B_k|\left[\frac{L_k}{2}\left(e_k^2(t)+Z_k^2(t)\right)+N_\ell|e_k(t)|\right]$$
$$\sum_{k=1}^{n}E_k(t)Z_k(t)\lambda_k(t)\leq\sum_{k=1}^{n}|E_k|\left[\frac{L_k}{2}\left(Z_k^2(t)+e_k^2(t)\right)+N_\ell|Z_k(t)|\right]$$

Introducing the above inequality for the derivative of $V(t)$, we have:

$$\begin{aligned}
\dot{V}(t)\leq&\sum_{k=1}^{n}\Bigg[-c_k+\frac{L_k|B_k|}{2}+\frac{L_k|E_k|}{2}\\
&+\frac{1}{2}\sum_{\ell=1}^{n}\left(|\tilde{a}_{k\ell}|L_\ell+|\tilde{a}_{\ell k}|L_k+|\tilde{b}_{k\ell}|L_\ell+|\alpha_{k\ell}|L_\ell+|\beta_{k\ell}|L_\ell+2\omega_{k\ell}+2\tau_{k\ell}\varpi_{k\ell}\right)\Bigg]e_k^2(t)\\
&+\sum_{k=1}^{n}\left[-\zeta_k+\sum_{\ell=1}^{n}\left(|\tilde{a}_{k\ell}|+|\tilde{b}_{k\ell}|+|\alpha_{k\ell}|+|\beta_{k\ell}|+|B_k|\right)N_\ell\right]e_k(t)\\
&-\sum_{k=1}^{n}\frac{\eta_k\|e(t)\|^2e_k^2(t)}{\|e(t)\|^2+\varrho(t)}-\sum_{k=1}^{n}\sum_{\ell=1}^{n}\varpi_{k\ell}\int_{t-\tau_{k\ell}}^{t}e_k^2(s)\mathrm{d}s\\
&+\sum_{k=1}^{n}\left(-d_k+\frac{L_k|E_k|}{2}+\frac{L_k|B_k|}{2}\right)Z_k^2(t)+\sum_{k=1}^{n}\left[-\Pi_k+\sum_{\ell=1}^{n}|E_k|N_\ell\right]Z_k(t)\\
&-\sum_{k=1}^{n}\frac{H_k\|Z(t)\|^2Z_k^2(t)}{\|Z(t)\|^2+\varrho(t)}+\sum_{k=1}^{n}\sum_{\ell=1}^{n}\left(-\omega_{k\ell}+|\tilde{b}_{\ell k}|+|\alpha_{\ell k}|+|\beta_{\ell k}|\right)e_k^2(t-\tau_{k\ell}(t))\\
\leq&\sum_{k=1}^{n}\Bigg[-c_k-\eta_k+\frac{L_k|B_k|}{2}+\frac{L_k|E_k|}{2}\\
&+\frac{1}{2}\sum_{\ell=1}^{n}\left(|\tilde{a}_{k\ell}|L_\ell+|\tilde{a}_{\ell k}|L_k+|\tilde{b}_{k\ell}|L_\ell+|\alpha_{k\ell}|L_\ell+|\beta_{k\ell}|L_\ell+2\omega_{k\ell}+2\tau_{k\ell}\varpi_{k\ell}\right)\Bigg]e_k^2(t)\\
&+\sum_{k=1}^{n}\left[-\zeta_k+\sum_{\ell=1}^{n}\left(|\tilde{a}_{k\ell}|+|\tilde{b}_{k\ell}|+|\alpha_{k\ell}|+|\beta_{k\ell}|+|B_k|\right)N_\ell\right]e_k(t)\\
&-\sum_{k=1}^{n}\frac{\eta_k\|e(t)\|^2e_k(t)}{\|e(t)\|^2+\varrho(t)}-\sum_{k=1}^{n}\sum_{\ell=1}^{n}\varpi_{k\ell}\int_{t-\tau_{k\ell}}^{t}e_k^2(s)\mathrm{d}s+\sum_{k=1}^{n}\eta_ke_k^2(t)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\left(-d_k-H_k+\frac{L_k\left|E_k\right|}{2}+\frac{L_k\left|B_k\right|}{2}\right)Z_k^2(t)+\sum_{k=1}^{n}\left[-\Pi_k+\sum_{\ell=1}^{n}\left|E_k\right|N_\ell\right]Z_k(t)\\
&+\sum_{k=1}^{n}H_kZ_k^2(t)-\sum_{k=1}^{n}\frac{H_k\left\|Z(t)\right\|^2Z_k(t)}{\left\|Z(t)\right\|^2+\varrho(t)}\\
&+\sum_{k=1}^{n}\sum_{\ell=1}^{n}\left(-\omega_{k\ell}+\left|\tilde{b}_{\ell k}\right|+\left|\alpha_{\ell k}\right|+\left|\beta_{\ell k}\right|\right)e_k^2\left(t-\tau_{k\ell}(t)\right)\\
\le&-\sum_{k=1}^{n}\sigma_ke_k^2(t)+\max_{k\in I}\left\{\eta_k\right\}\frac{\left\|e(t)\right\|^2\varrho(t)}{\left\|e(t)\right\|^2+\varrho(t)}-\sum_{k=1}^{n}\sum_{\ell=1}^{n}\varpi_{k\ell}\int_{t-\tau_{k\ell}}^{t}e_k^2(s)\mathrm{d}s\\
&-\sum_{k=1}^{n}\zeta_kZ_k^2(t)+\max_{k\in I}\left\{H_k\right\}\frac{\left\|Z(t)\right\|^2\varrho(t)}{\left\|Z(t)\right\|^2+\varrho(t)}.
\end{aligned}
$$

By making $\eta=\max_{k\in I}\left\{\eta_k\right\}>0$ and $H=\max_{k\in I}\left\{H_k\right\}>0$, using the inequality $0\le\frac{ab}{a+b}\le a$, arbitrary $a>0,b>0$, we have:

$$
\begin{aligned}
\dot{V}(t)\le&-\sum_{k=1}^{n}\sigma_ke_k^2(t)+\eta\varrho(t)-\sum_{k=1}\varsigma_kZ_k^2(t)+H\varrho(t)\\
&-\sum_{k=1}^{n}\sum_{\ell=1}^{n}\varpi_{k\ell}\int_{t-\tau_{k\ell}}^{t}e_k^2(s)\mathrm{d}s.
\end{aligned}
\tag{14}
$$

Taking a sufficiently small δ, so that $\delta\xi\le\sigma,\delta r\le\varsigma$, follows from (13) and (14):

$$
\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t}+\delta V(t)\le&-\sum_{k=1}^{n}\sigma_ke_k^2(t)+\eta\varrho(t)-\sum_{k=1}^{n}\varsigma_kZ_k^2(t)+H\varrho(t)\\
&-\sum_{k=1}^{n}\sum_{\ell=1}^{n}\varpi_{k\ell}\int_{t-\tau_{k\ell}}^{t}e_k^2(s)\mathrm{d}s+\delta\Bigg(\xi\sum_{k=1}^{n}e_k^2(t)\\
&+\frac{\xi}{\sigma}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\varpi_{k\ell}\int_{t-\tau_{k\ell}}^{t}e_k^2(s)\mathrm{d}s+r\sum_{k=1}^{n}Z_k^2(t)\Bigg)\\
\le&(\delta\xi-\sigma)\sum_{k=1}^{n}e_k^2(t)+\sum_{k=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\delta\xi}{\sigma}-1\right)\int_{t-\tau_{k\ell(t)}}^{t}\varpi_{k\ell}e_k^2(s)\mathrm{d}s\\
&+\eta\varrho(t)+(\delta r-\varsigma)\sum_{k=1}^{n}Z_k^2(t)+H\varrho(t)\\
\le&(\eta+H)\varrho(t).
\end{aligned}
$$

Namely,

$$
\frac{\mathrm{d}V(t)}{\mathrm{d}t}+\delta V(t)\le(\eta+H)\varrho(t).
\tag{15}
$$

Thus, the drive-response systems (4) and (6) achieve generalized decay synchronization under the controller (10) with a convergence rate of δ, proof complete.

The system (2) can be degenerated to:

$$
\begin{cases}
\mathrm{STM}:\dot{p}_k(t)=-c_kp_k(t)+\sum_{\ell=1}^{n}\tilde{a}_{k\ell}g_\ell\left(p_\ell(t)\right)+B_kS_k(t)\\
\mathrm{LTM}:\dot{S}_k(t)=-d_kS_k(t)+E_kg_k\left(p_k(t)\right),
\end{cases}
\tag{16}
$$

then the response system is as follows:

$$\begin{cases} \text{STM}: \dot{q}_k(t) = -c_k q_k(t) + \sum_{\ell=1}^{n} \tilde{a}_{k\ell} g_\ell(q_\ell(t)) + B_k W_k(t) + u_k(t), \\ \text{LTM}: \dot{W}_k(t) = -d_k W_k(t) + E_k g_k(q_k(t)) + \tilde{u}_k(t), \end{cases} \tag{17}$$

where the controllers $u_k(t)$ and $\tilde{u}_k(t)$ are as follows:

$$\begin{cases} u_k(t) = -\overline{\zeta}_k \operatorname{sign}(e_k(t)) - \dfrac{\overline{\eta}_k \|e(t)\|^2 e_k(t)}{\|e(t)\|^2 + \varrho(t)} \\ \tilde{u}_k(t) = -\overline{\Pi}_k \operatorname{sign}(Z_k(t)) - \dfrac{\overline{H}_k \|Z(t)\|^2 Z_k(t)}{\|Z(t)\|^2 + \varrho(t)}, \end{cases} \tag{18}$$

where $k \in I$. $\overline{\zeta}_k, \overline{\Pi}_k, \overline{\eta}_k$ and $\overline{H}_k$ are positive control gains:

$$\begin{cases} \overline{\sigma}_k \triangleq c_k + \overline{\eta}_k - \dfrac{L_k|E_k|}{2} - \dfrac{L_k|B_k|}{2} - \sum_{\ell=1}^{n} \left(\dfrac{|\tilde{a}_{k\ell}| L_\ell + |\tilde{a}_{\ell k}| L_k}{2} > 0 \right) \\ \overline{\varsigma}_k \triangleq d_k + \overline{H}_k - \dfrac{L_k|E_k|}{2} - \dfrac{L_k|B_k|}{2} > 0. \end{cases} \tag{19}$$

Corollary 1. If Assumption 2 and Assumption 3 hold, and the control gain of controller (18) satisfies inequality (19), then the drive-response system (16) and (17) achieve generalized decay synchronization under controller (18).

Remark 3. In this paper, by designing a nonlinear controller and applying some inequality methods, we investigate the generalized decay synchronization problem for competing neural networks. Clearly, the exponential and asymptotic synchronization in previous work can be seen as special cases when the generalized decay function takes $\psi(t) = \mathrm{e}^{\alpha t}$ and $\psi(t) = (1+t)^\alpha$. From this point of view, the generalized decay synchronization studied in this paper is of better application.

4. Numerical Simulations

In this section, a numerical example is given to verify the validity of the results obtained. Consider the following fuzzy competitive neural networks:

$$\begin{cases} \text{STM}: \dot{p}_k(t) = -c_k p_k(t) + \sum_{\ell=1}^{2} \tilde{a}_{k\ell} g_\ell(p_\ell(t)) + \sum_{\ell=1}^{2} \tilde{b}_{k\ell} g_\ell(p_\ell(t - \tau_{k\ell}(t))) \\ \qquad + \bigwedge_{\ell=1}^{2} \alpha_{k\ell} g_\ell(p_\ell(t - \tau_{k\ell}(t))) + \bigvee_{\ell=1}^{2} \beta_{k\ell} g_\ell(p_\ell(t - \tau_{k\ell}(t))) \\ \qquad + B_k S_k(t) \\ \text{LTM}: \dot{S}_k(t) = -d_k S_k(t) + E_k g_k(p_k(t)), \quad k, \ell = 1, 2, \end{cases} \tag{20}$$

where $f_1(u) = f_2(u) = 0.5[|u+1| - |u-1|] + 0.01\operatorname{sign}(u)$, $c_{11} = 1.52$, $c_{22} = 1.32$, $d_{11} = 2.4$, $d_{22} = 2.3$, $a_{11} = 1.75$, $a_{12} = -0.1$, $a_{21} = -1.6$, $a_{22} = 2$, $b_{11} = -1.7$, $b_{12} = -0.1$, $b_{21} = 0.1$, $b_{22} = -1.5$, $\alpha_{11} = -0.395$, $\alpha_{12} = -0.23$, $\alpha_{21} = -0.21$, $\alpha_{22} = -0.37$, $\beta_{11} = 0.495$, $\beta_{12} = 0.236$, $\beta_{21} = 0.026$, $\beta_{22} = 0.53$, $E_1 = E_2 = 1$, $B_1 = B_2 = 1$, $\tau_{k\ell}(t) = 1$, $(k = 1, 2)$.

The system (20) corresponds to the initial conditions $x_1(\theta) = 0.2$, $x_2(\theta) = 0.6$,

$S_1(\theta) = -0.1$ and $S_2(\theta) = 0.2$. The chaotic diagram of $\theta \in [-1, 0]$ is shown in **Figure 1**.

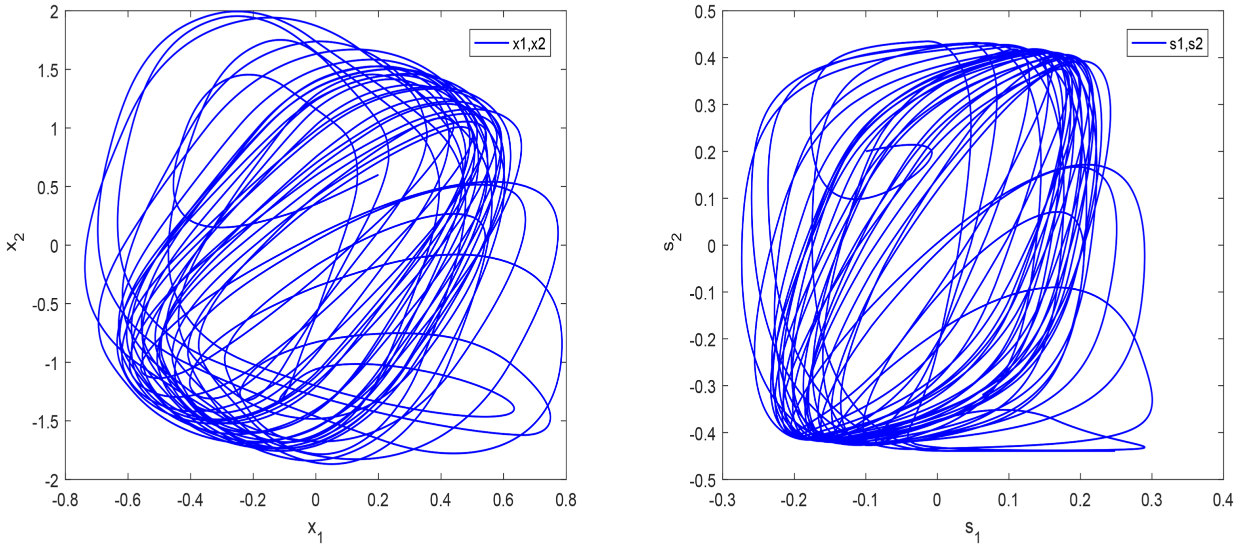

Figure 1. Chaotic diagram of the system (20).

Its corresponding response system is:

$$\begin{cases} \text{STM}: \dot{q}_k(t) = -c_k q_k(t) + \sum_{\ell=1}^{2} \tilde{a}_{k\ell} g_\ell(q_\ell(t)) + \sum_{\ell=1}^{2} \tilde{b}_{k\ell} g_\ell(q_\ell(t - \tau_{k\ell}(t))) \\ \qquad + \bigwedge_{\ell=1}^{2} \alpha_{k\ell} g_\ell(q_\ell(t - \tau_{k\ell}(t))) + \bigvee_{\ell=1}^{2} \beta_{k\ell} g_\ell(q_\ell(t - \tau_{k\ell}(t))) \\ \qquad + B_k W_k(t) + u_k(t), \\ \text{LTM}: \dot{W}_k(t) = -d_k W_k(t) + E_k g_k(q_k(t)) + \tilde{u}_k(t), \end{cases} \tag{21}$$

where $c_k, \tilde{a}_{k\ell}, \tilde{b}_{k\ell}, \alpha_{k\ell}, \beta_{k\ell}, g_\ell$ and $\tau_{k\ell}(t)$ as in system (20), the nonlinear controller $u_k(t)$ is designed as:

$$\begin{cases} u_k(t) = -\zeta_k \operatorname{sign}(e_k(t)) - \dfrac{\eta_k \|e(t)\|^2 e_k(t)}{\|e(t)\|^2 + \varrho(t)}, \\ \tilde{u}_k(t) = -\Pi_k \operatorname{sign}(Z_k(t)) - \dfrac{H_k \|Z(t)\|^2 Z_k(t)}{\|Z(t)\|^2 + \varrho(t)}, \end{cases} \tag{22}$$

where $k \in I$, ζ_k, η_k, Π_k and H_i positive control gain, $e_k(t) = q_k(t) - p_k(t)$, $Z_k(t) = W_k(t) - S_k(t)$ $(k = 1, 2)$.

A simple calculation shows that $L_\ell = 1$, $N_\ell = 0.01$, $K_{k\ell} = 1$, therefore, the Assumption 1 and Assumption 2 of this paper are valid. By choosing $\zeta_1 = \Pi_1 = 0.6$, $\zeta_2 = \Pi_2 = 0.2$, $\zeta_1 = H_1 = 4.6$, $\zeta_2 = H_2 = 3.2$, $\epsilon(t) = e^{-0.2t}$, it is not difficult to verify that the inequality (11) in the conditions of Assumption 3 and Theorem 1 also holds. Therefore, according to Theorem 1, the drive-response systems (20) and (21) can be synchronized by generalized decay under the controller (22). The time evolution of the error system is shown in **Figure 2** and **Figure 3**. The evolution of the synchronisation curves between the drive-response systems (20) and (21) are shown in **Figure 4** and **Figure 5**.

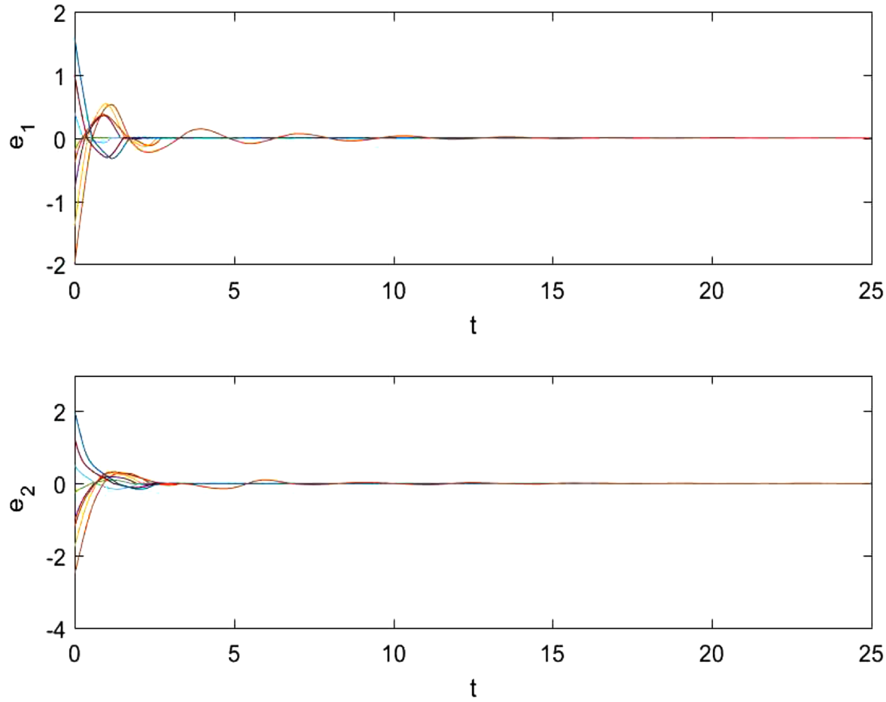

Figure 2. Synchronous evolution diagram of error e_i.

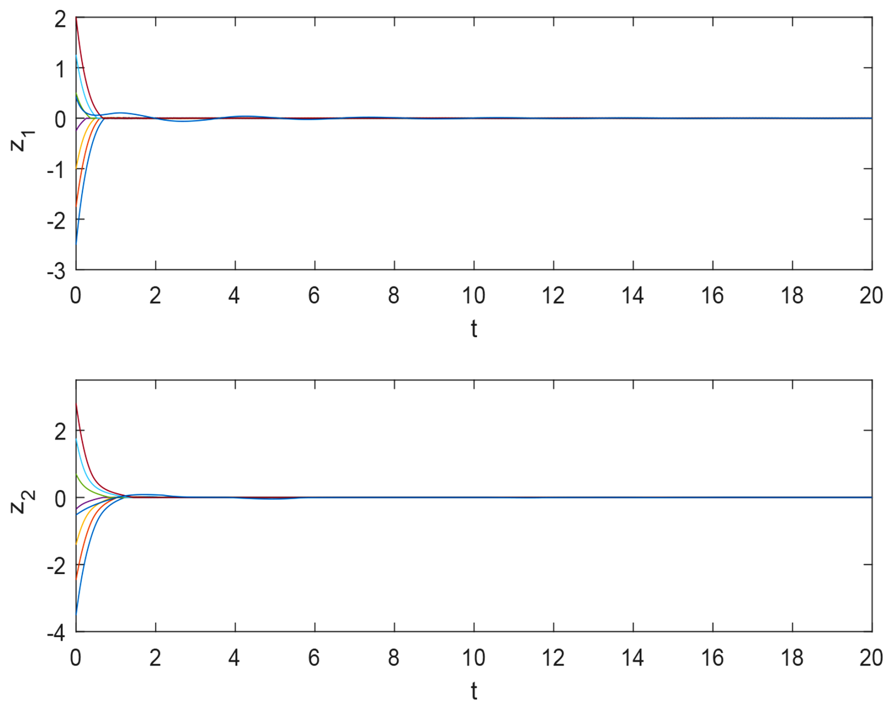

Figure 3. Synchronous evolution diagram of error Z_i.

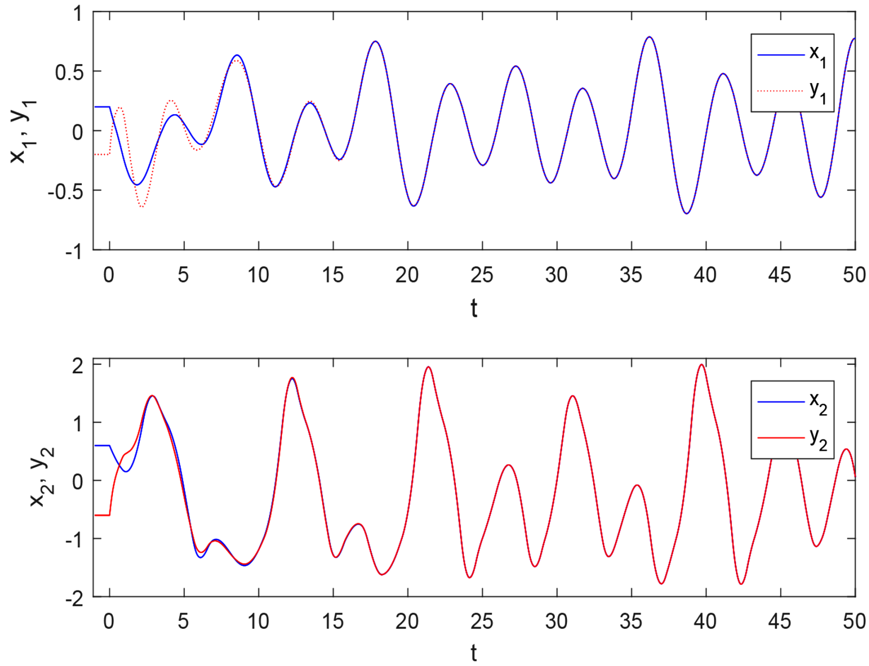

Figure 4. Synchrograms of x_i and y_i.

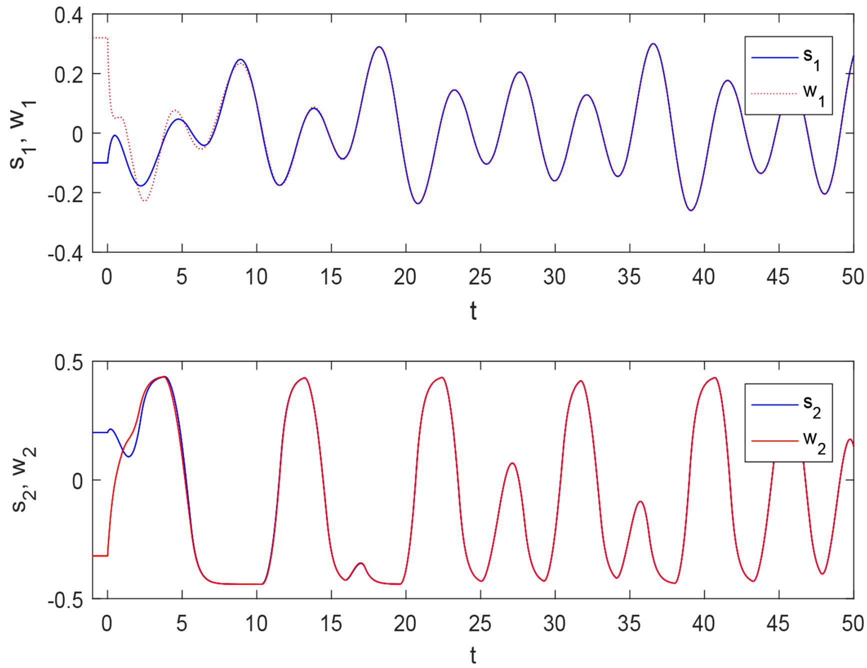

Figure 5. Synchrograms of S_i and W_i.

Remark 3. The main research findings and their impacts on this article can be summarized as follows: 1) From **Figure 1**, it can be seen that the system does not reach synchronization state without the controller. Therefore, the controllers designed in this paper play an important role in achieving generalized decay synchronization. 2) From **Figure 2** and **Figure 3**, it can be seen that under the nonlinear controllers (10) and (18) designed in this paper, the generalized decay synchronization can better estimate the convergence rate of the system. 3) Due to the fact that the commonly used exponential synchronization and polynomial synchronization can be seen as special cases. Therefore, the research results in this paper have good theoretical significance and application prospects.

5. Conclusion

In this paper, a generalized decay synchronization problem with a right-hand discontinuous fuzzy competing neural network is investigated. Firstly, a novel nonlinear feedback controller is designed. Secondly, by using the theory of right-hand discontinuous differential equations, Lyapunov function method and inequality techniques, sufficient conditions are given for the generalized decay synchronization of the considered fuzzy competing neural networks. Finally, a numerical example is given to verify the validity and feasibility of the theoretical results.

6. Prospect

Quaternary neural networks, rather than competitive neural networks, have more extensive applications, such as color image classification and forensics, human motion recognition, etc. Therefore, in the future, we will continue to study the fixed-time synchronization of quaternion neural networks with impulsive effects.

Authors' Contributions

Mairemunisa Abudusaimaiti: Writing original draft and methodology, funding acquisition, and review. Abuduwali Abudukeremu: Writing, visualization, review, editing, and supervision.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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