Open Journal of Applied Sciences, 2024, 14, 3165-3182
https://www.scirp.org/journal/ojapps
ISSN Online: 2165-3925
ISSN Print: 2165-3917

New Approaches of Theoretical Astrophysics for Application to Some Astronomical Objects: I. An Application of Non-Classical Equation Mathematical Physics to the Magneto-Hydrodynamic Equilibrium (in Case of Mixed Magnetic Field) of Magnetic Stars

Mahammad A. Nurmammadov

Department of Theoretical Astrophysics and Cosmology of the Shamakhy Astrophysical Observatory Named after N. Tusi of Ministry of Science and Education of Azerbaijan Republic, Shamakhy, Azerbaijan
Email: nurmamedov_55@mail.ru

How to cite this paper: Nurmammadov, M.A. (2024) New Approaches of Theoretical Astrophysics for Application to Some Astronomical Objects: I. An Application of Non-Classical Equation Mathematical Physics to the Magneto-Hydrodynamic Equilibrium (in Case of Mixed Magnetic Field) of Magnetic Stars. *Open Journal of Applied Sciences*, **14**, 3165-3182.
https://doi.org/10.4236/ojapps.2024.1411208

Received: October 12, 2024
Accepted: November 18, 2024
Published: November 21, 2024

Abstract

In this paper, we consider the application of the equation of non-classical mathematical physics to magneto-hydrodynamic equilibrium (in the case of a mixed magnetic field) for magnetic stars. First, we give the necessary concepts about the equation of non-classical mathematical physics and the possibility of their applicability to astrophysical problems. The conditions of magneto-hydrodynamic equilibrium are determinate, and self-consistence provides the means to derive the corresponding partial differential equations describing this equilibrium in a magnetosphere magnetic star. Namely, this process is to the non-classical equations of mathematical physics in cases of types. Keldysh-Tricomi, a common case equation of non-classical type, is at first introduced by the author. Using the two main physical efficiencies of MHD. A mathematical model of a poloidal-toroidal mixed magnetic field for magnetic stars is constructed, and this model is classified with respect to degenerating case equations. According to Hopf's theorem, Maxwell's equation and the magnetic force balance equation constructed equilibrium conditions of the poloidal-toroidal of the magnetic field for a magnetic star. At the same time, the taken example, which is the self-consistency of this model by observation dates, is investigated. At first, in an application, the method of straight lines for recurrent formulas of calculation of magnetic flux and stream functions is used. The physical means, the corresponding singular point of the sonic line, cutoff, and resonance phenomena are considered. In this case, a general solution equation is found, which is interpreted by this phenomenon as a cutoff, resonance. Finally, this obtained solution gives the conditions of magneto-hydrodynamic equilibrium on the magnetosphere of magnetic stars. Methodology and obtained equations are new approaches that are at first considered.

Keywords

Magnetic Star, MHD, Equilibrium, Keldysh and Tricomi Type, Plasma, Non-Classical Equations of Mathematical Physics

1. Introduction

First of all, let's give a few words to explain the title of the article, since it contains such concepts as terms, non-classical equations of mathematical physics, hydrodynamics, and equilibrium magneto-hydrodynamics. In fluid dynamics, the Hicks equation, sometimes also called the Bragg-Hawthorne equation (or Squire-Long equation) (see [1]), is a partial differential equation that describes the distribution of the stream function for an axisymmetric fluid residue. However, the Grad-Shafranov equation [2], which arises in plasma physics, also has the same form as the Hicks equation (see [3]-[5]). Therefore, the equilibrium in ideal magneto-hydrodynamics (MHDs) for a two-dimensional plasma, such as the discovered axisymmetric toroidal plasma equation in a tokamak. This equation reflects the form of the Hicks equation (see [3] [4]) from fluid dynamics and at the same time, it is a two-dimensional nonlinear elliptic partial differential equation obtained by simplifying the ideal MHD equations to two dimensions, usually in the context of toroidal rotational symmetry, as is relevant for tokamaks. Hence, including directly that the Hicks equation from hydrodynamics, also belongs to the category of equations of non-classical mathematical physics. Now, we need to clarify what does it means in equations of non-classical mathematical physics.

Non-classical equations of mathematical physics:

Definition (non-classical equation in mathematical physics) and applicable cases: Non-classical models of mathematical physics are those whose representations in the form of equations or systems of partial differential equations do not fit within the framework of one of the classical types: elliptic, parabolic, or hyperbolic (1) $\Delta U = U_{xx} + U_{yy} = 0$ or equation of Poisson: $\Delta U = U_{xx} + U_{yy} = -f(x, y)$, Δ -operators Laplace, 2) $U_t = a\Delta U$ and 3) $U_{tt} = aU_{xx}$). In particular, non-classical models include those described by mixed-type equations (for example, the Tricomi equation [6] $K(z)u_{xx}(x,z) + u_{zz} = 0$, $K(z) = z, z^2, z^m$, degenerate equations (for example, the generalized Keldysh equation $K(z)u_{zz}(x,z) + u_{xx}(x,z) + a(x,z)u_z = 0$ [7]), or Sobolev-type equations (for example, hydrodynamical equations: the Barenblatt-Zheltov-Kochina equation (see [8] [9])).

The term "non-classical equations of mathematical physics" at first was introduced by V. N. Vragov (see [10]), and his students (for example, in case of model mixed type equation with changing time direction by M. A. Nurmamedov (see [11])), and at the same time, applicable cases to astrophysical fluid dynamics are a modern branch of astronomy that includes fluid mechanics, which deals with the movement of fluids such as the gases that make up stars or any fluid that is found in outer space (see based on the point of view of this definition, applicable indicted problems of astronomy and astrophysics which is noted by M. A. Nurmammadov (see [12] [13])).

2. Concept of MHD Equilibrium and Field of Basic Applications of MHD

Magnetism is found throughout the universe. Magnetic fields are known to exist in planets, stars, accretion disks, the interstellar medium, galaxies, and active galactic nuclei (see the article on galactic magnetic fields). Often, these magnetic fields are generated and maintained by the action of a magneto-hydrodynamic dynamo and in many cases, the magnetic field dynamically dominates, determining the evolution of the object.

Note that the nature of the equilibrium, be it a tokamak, a reversed field pinch, etc., is largely determined by the choice of two functions that depend on the arguments of the flow functions Ψ and how, as well as the boundary conditions. It must be borne in mind that the fundamental condition for equilibrium is that the forces are equal to zero at all points. For the simplest, assuming no plasma resistance (*i.e.* ideal MHD conditions), we require that the magnetic and pressure forces are balanced at all points, $\boldsymbol{J}\times\boldsymbol{B}=\nabla P$, where, $\boldsymbol{J}$ is the current density, $\boldsymbol{B}$ is the magnetic field and P is the pressure. Frankly, note that the Grad-Shafranov equation is directly derived from the magneto-hydrodynamic (MHD) equilibrium equations, which are given by the following cases:

1) The momentum equation for a plasma in equilibrium:

$$\boldsymbol{J}\times\boldsymbol{B}=\nabla P \tag{1}$$

No time variation (υ is velocity), therefore must satisfy equality: $\frac{\mathrm{d}\upsilon}{\mathrm{d}t}=0$.

2) Amperes law relating the current density $\boldsymbol{J}$ to the curl of the magnetic flux density $\boldsymbol{B}$.

$$\mu_0\boldsymbol{J}=\boldsymbol{B}=\nabla\times\boldsymbol{B} \tag{2}$$

where μ_0 is the magnetic permeability, $P=P(\psi)$ is the pressure.

3) The divergence-free postulate, stating that there are no sources of magnetic flux, that is, no magnetic monopoles:

$$\nabla\times\boldsymbol{B}=0 \tag{3}$$

Since Equation (1) establishes the force balance needed for equilibrium, the pressure gradient (expansion force) needs to be equal to the magnetic force (confinement force). In this way, the plasma is in equilibrium. It is important to note that the plane is defined by $\boldsymbol{J}$ and everywhere tangent to the surfaces of *P*.

3. Main of Two Physical Effects in MHD for Presented Model

Note that the above-mentioned two physical effects are individually expressed as follows:

1) The first physical effect occurs when a good conductor is placed in a magnetic field and an electric current is induced in the conductor. Then, according to Lenz's law, this current creates its own magnetic field. Consequently, this induced magnetic field tends to neutralize the original, and externally supported, field. Thus, effectively eliminating the magnetic field line from the conductor. Then, the result is that the lines of force seem to stretch along with the conductor and they "float with the flow". Moreover, if the conductor is like a liquid with complex movements, the resulting distribution of the magnetic field can become very complex. In this case, the current will increase until its growth is balanced by Ohmic dissipation.

2) The second key physical effect is dynamic. That is, the Lorentz force (or $\boldsymbol{J} \times \boldsymbol{B}$) will act on the fluid only when the currents are induced by the movement of a conducting fluid through a magnetic field. This will allow it to change the movement of the fluid through the magnetic field. In order to derive the basic MHD equations, first of all, we should always consider the choice of the main variables.

Therefore, in the theory of electromagnetism, we define spatial and temporal changes in either the electromagnetic field or its source, electric charge density and current density (for example, in Figure 1). However, it can be calculated from another, namely using Maxwell's equations, supplemented by suitable boundary conditions.

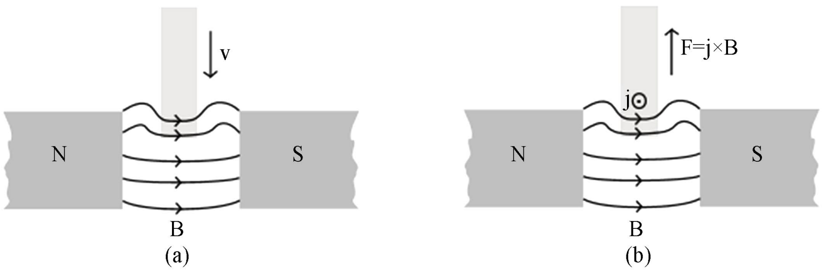

Figure 1. The two physical effects that occur in MHD: (a) A moving conductor changes the magnetic field, as if dragging the magnetic field lines along with it; the lines of force appear to be frozen into the moving conductor; (b) When the electric current flowing in the conductor crosses the magnetic field lines, the Lorentz force is generated, which accelerates the fluid.

Finally, in terminology MHD and plasma note that, the fluid is electrically conductive and moves in a magnetic field, which is a type of flow known as magnetohydrodynamics or MHD for short. However, the main application of MHD is in plasma physics (plasma is a hot ionized gas containing free electrons and ions). It is not at all obvious that plasma can be considered a liquid, since the mean free paths for Coulomb collisions between electrons and ions are macroscopically long. Now we need to give magneto-hydrodynamics concept and for this equilibrium:

Definition (about of magneto-hydrodynamic): Magneto-hydrodynamics (MHDs) (magneto-hydrodynamics or hydro magnetics) is the study of the dynamics of electrically conductive fluids. Examples of such fluids include plasma, liquid metals, and salt water or electrolytes. The word magneto-hydrodynamics comes from the words: "magneto"—magnetic field, "hydro"—fluid and "dynamic"—motion. The field of MHD was initiated by Hannes Alfven, for which he received the Nobel Prize in Physics in 1970. The fundamental concept of MHD is that magnetic fields can induce currents in a moving conducting fluid, which in turn creates forces on the fluid and changes the magnetic field itself. Currently, there is no other global solution to the system of equations, which includes a complex of equations: describing the steady flow of an inviscid magnetic fluid with high electrical conductivity, the equation Faraday's law of electromagnetic induction, the equation of momentum, the equation of conservation of magnetic flux, with the addition of an equation of state, a detailed energy balance equation, and Poisson's law of gravity. An important class of problems studied so far is the magnetostatic equilibrium of plasma in magnetic and gravitational fields.

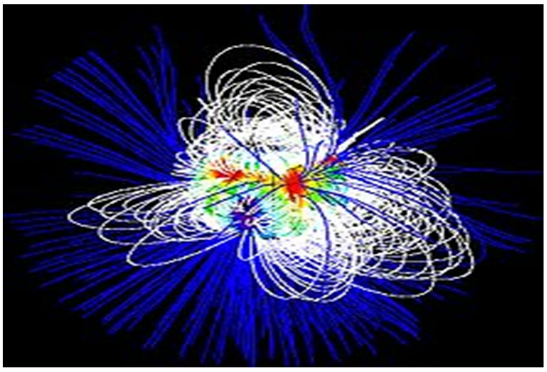

Figure 2. Surface magnetic field of the star SU Aurigae (a young T Tauri star), reconstructed using the Zeeman-Doppler effect (see [14] [15]).

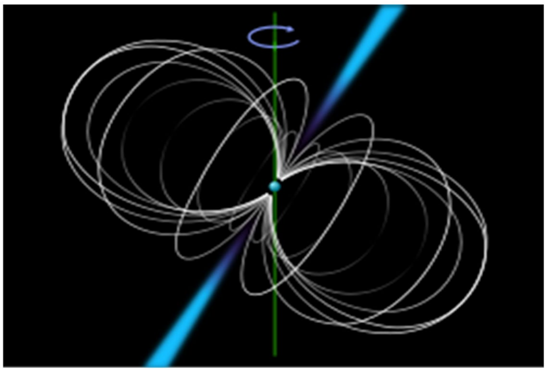

Figure 3. Schematic representation of the pulsar.

The sphere in the center of the image is a neutron star, curved lines indicate the magnetic field lines of the pulsar (see **Figure 2** and **Figure 3**) decrease by many orders of magnitude, and since the magnetic moment of the star is completely preserved, the strength of the magnetic field of the neutron star increases proportionally by many orders of magnitude. The rapid rotation of neutron stars turns them into a pulsar, which emits a narrow beam of energy. It is important to express magneto-hydrodynamic conditions and mixed magnetic field equations in the following.

4. Magneto-Hydrodynamic Conditions and Mixed Magnetic Field Equations

In this model, the hydrostatic equilibrium equation is modified to include a magnetic term due to the Lorentz force as:

$$\nabla P\rho + \nabla\Phi = \left(\nabla \times \boldsymbol{B}\right) \times \boldsymbol{B} 4\pi\rho = 4\pi\rho \boldsymbol{F}_L, \tag{4}$$

$$\nabla\Phi = 4\pi\rho G \tag{5}$$

$$\Phi = -\frac{GM}{r} \tag{6}$$

where M is the mass of the star, r is the distance from the center, P is pressure, ρ is density, G is constant of gravity, Φ is gravitational potential, $\boldsymbol{B}$ is the vector magnetic field, and is the Lorentz force. We refer to Equation (4) as the hydromagnetic equilibrium equation. In this case, magnetic field configurations satisfying Equation (4) must satisfy an additional constraint determined by taking the rotor of the hydromagnetic equilibrium equation. Since the gradient of the vortex is zero, *i.e.* $\nabla \times \nabla f = 0$, where f is any scalar function, the left side of Equation (4) and the $\nabla P\rho + \nabla\Phi_g = f$ disappears with such an operation, and we arrive at the following restriction:

$$\nabla \times \boldsymbol{B} \times \left(\nabla \times \boldsymbol{B}\right) \times \rho = 0, \tag{7}$$

where ρ is the barotropic equation of state. Since the density ρ is present in this constraint, Equation (6) must be solved together with Equation (4), and mixed magnetic fields, including both poloidal and toroidal components, have stable configurations (see **Figure 4**).

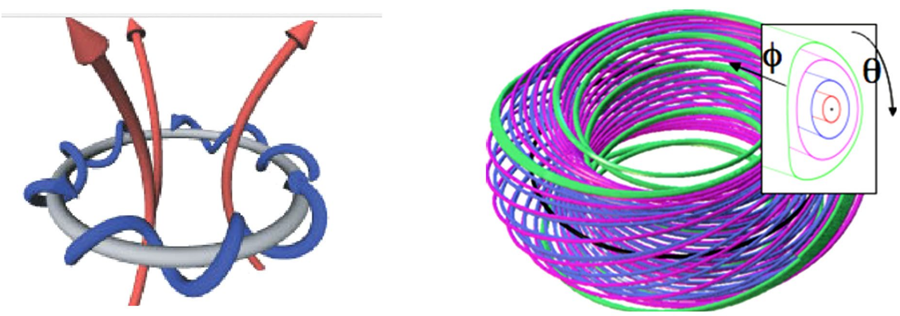

Figure 4. A very simple depiction of a combination of poloidal (red) and toroidal (blue) field lines.

Magnetic field lines lie on nested surfaces of constant magnetic flux. Stellar winds and jets from accretion discs are examples of outflows in which rotation and magnetic fields have important or essential roles [16]-[18]. Using cylindrical polar coordinates (r, φ, z), we examine steady ($\partial/\partial t = 0$), axisymmetric ($\partial/\partial\varphi = 0$) models based on the equations of ideal MHD.

$B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}$, $B_z = \frac{1}{r}\frac{\partial\psi}{\partial r}$, where $\psi(r,z)$ is the magnetic flux function (as corresponding similar to Figure 5). This is related to the magnetic vector potential by $\psi(r,z) = rA_\phi$. The magnetic flux contained inside the circle (r = constant, z = constant). The poloidal (meridional) and toroidal (azimuthal) parts of the magnetic field: $\boldsymbol{B} = \nabla\psi\times\nabla\Phi + B_\phi\boldsymbol{e}_\phi = \boldsymbol{B}_p + B_\phi\boldsymbol{e}_\phi = -\frac{1}{r}\boldsymbol{e}_\phi\times\nabla\psi + B_\phi\boldsymbol{e}_\phi$. Note that, in this equation, first addend $-\frac{1}{r}\boldsymbol{e}_\phi\times\nabla\psi$ means poloidal (see Figure 6), but the second addend $B_\phi\boldsymbol{e}_\phi$ means toroidal (see Figure 7) magnetic field of $\boldsymbol{B}$, which is mixed magnetic field.

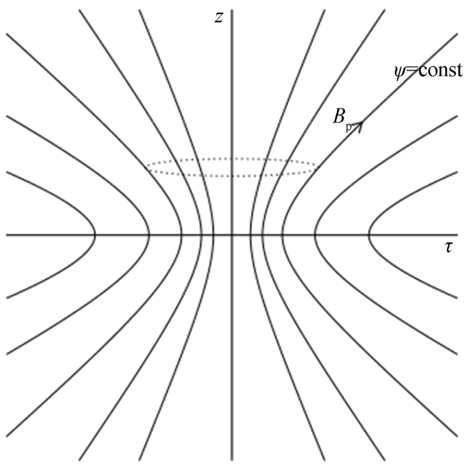

Figure 5. Magnetic flux function and poloidal magnetic field.

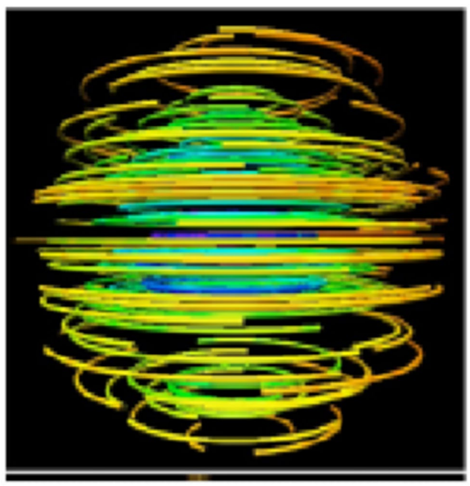

Figure 6. Purely: poloidal field.

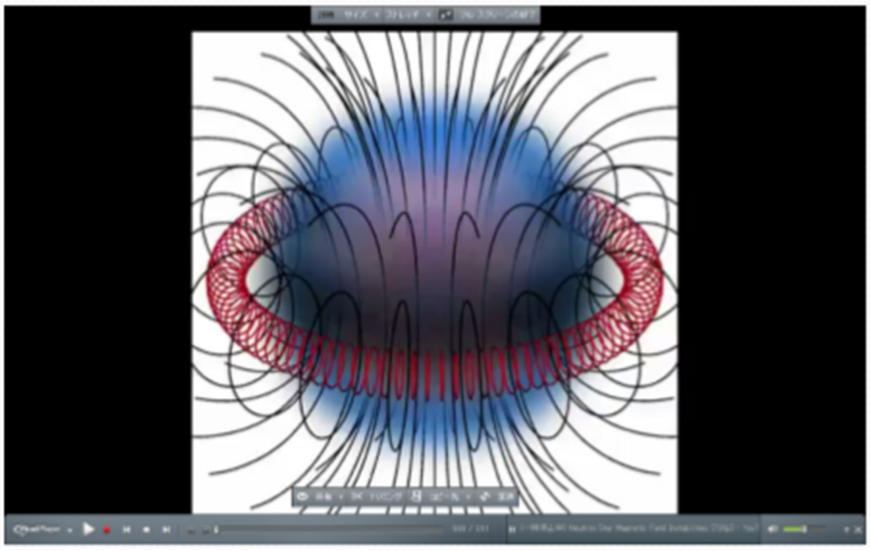

Figure 7. Purely: toroidal field.

5. Being of Axisymmetric Mixed Magnetic Field Model into the Keldysh-Tricomi Type Equations for Magnetic Stars

In this model, the magnetic field is divided into poloidal ($\boldsymbol{B}_p$) and ($\boldsymbol{B}_t$) toroidal components, where $\boldsymbol{B}_p = (B_r, B_\theta, 0)$ and $\boldsymbol{B}_t = (0, 0, B_\Phi)$. is the axisymmetric mixed model of the poloidal-toroidal magnetic field in Tokmak's terminology describes mixed elliptic-hyperbolical, when, for example, the following conditions are met for a star (for example):

$$|\boldsymbol{B}| = B < \begin{cases} 5\times 10^{15}\,G, \text{ purely-poloidal} \\ 2\times 10^{16}\,G, \text{ purely-toriodal} \end{cases} \tag{8}$$

For example, let us take the model that we have considered for the case of rotating pulsars (purpose as strong magnetic field and satisfying conditions of mixed field), where conditions (8) are satisfied. In the physical and mechanical sense, the situation is not so pessimistic, if we consider more slowly rotating pulsars. Consider, for example, the current LIGO upper limit of for the Crab pulsar [19]. For the models we have considered, we must then have. At first glance, these limits may seem less interesting than the limits for a more rapidly rotating system. However, we must remember that young pulsars, such as the Crab Nebula, are believed to have much stronger magnetic fields. Maybe to take data observation of another class of magnetic stars. Along with these additional examples, we will simultaneously solve a specific problem together with a general class of problems based on the studies of the model presented in this paper.

The ideal MHD equilibrium equation for an axisymmetric toroidal plasma can be written as a differential equation for the poloidal magnetic flux function Ψ and is known as the Grad-Shafranov equation. It contains two arbitrary poloidal magnetic flux functions, $P(\psi)$ and $g(\psi)$ respectively associated with the pressure and current profiles (see **Appendix A**),

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi-\frac{1}{R_0+r\cos\theta}\left(\frac{\partial\psi}{\partial r}\cos\theta-\frac{\partial\psi}{\partial\theta}\frac{1}{r}\sin\theta\right)$$
$$=-\mu_0\left(R_0+r\cos\theta\right)^2\frac{\partial P}{\partial\psi}-\frac{\partial g}{\partial\psi}g\left(\psi\right) \quad (9)$$

Equilibrium is partly determined by external conditions such as the total current, the applied toroidal magnetic field and by notation, we have:

$$K_1\left(r\right)=r^3, K_2\left(\theta\right)=\sin^2\theta, a\left(r,\theta\right)=r\left(r-\frac{1}{R_0+r\cos\theta}\cos\theta\right), b\left(r,\theta\right)=-\sin\theta,$$

$$K_1\left(r\right)\frac{\partial^2\psi}{\partial r^2}+K_2\left(\theta\right)\frac{\partial^2\psi}{\partial\theta^2}+a\left(r,\theta\right)\frac{\partial\psi}{\partial r}+b\left(r,\theta\right)\frac{\partial\psi}{\partial\theta}=rf\left(r,\theta,\psi\right)=f_1\left(r,\theta,\psi\right), \quad (10)$$

$$K_1\left(r\right)\psi_{rr}+K_2\left(\theta\right)\psi_{\theta\theta}+a\left(r,\theta\right)\psi_\theta+b\left(r,\theta\right)_\theta=rf\left(r,\theta,\psi\right)=f_1\left(r,\theta,\psi\right), \quad (11)$$

$$K_1\left(x\right)u_{xx}\left(x,z\right)+K_2\left(z\right)u_{zz}+a\left(x,z\right)u_x+b\left(x,z\right)u_z=0, \quad (12)$$

$$K_1\left(x\right)u_{xx}\left(x,z\right)+K_2\left(z\right)u_{zz}+a\left(x,z\right)u_x+b\left(x,z\right)u_z+c\left(x,z\right)=f\left(x,z\right), \quad (13)$$

When $xK_1\left(x\right)>0$, for $x\neq 0$, $zK_1\left(z\right)<0$, for $z\neq 0$. Equation (13) is called mixed elliptic-hyperbolic type of M. V. Keldysh in two-dimensional problems (see [11]-[13]). Two-dimensional inhomogeneities of the kind represented by equations of (11) and (12), (13) can be expected to arise in toroidal fields, such as those created in tokamaks. Finally, frankly speaking, the cold plasma and toroidal-poloidal plasma, which in tokamaks, describe mathematical models corresponding to elliptical-hyperbolic equation of mixed type, both Keldysh and Tricomi or Keldysh-Trikomi types (see **Figure 8**, in case of sign of coefficients).

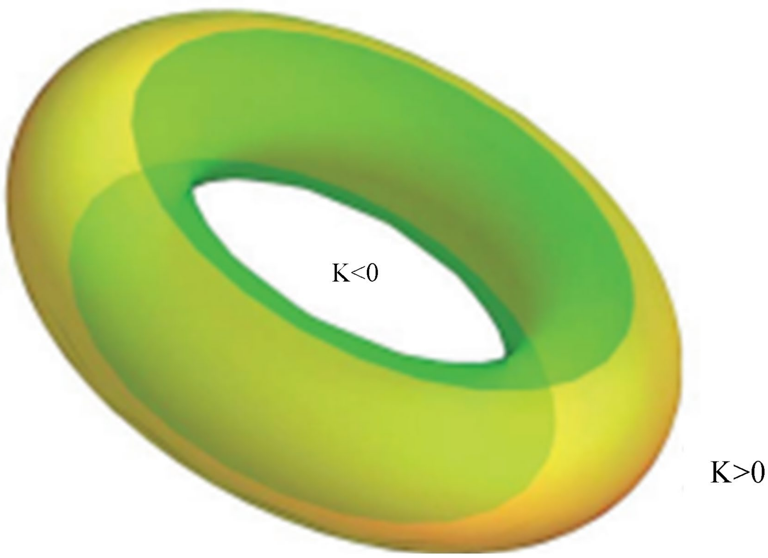

Figure 8. The sign of sign of coefficients of equations elliptical-hyperbolic types.

In **Figure 8**, using sign of coefficients for Equations (12) and (13), the physical, hydrodynamical, aerodynamical and gas dynamic means in terminology number Mache is described (see **Figure 9**).

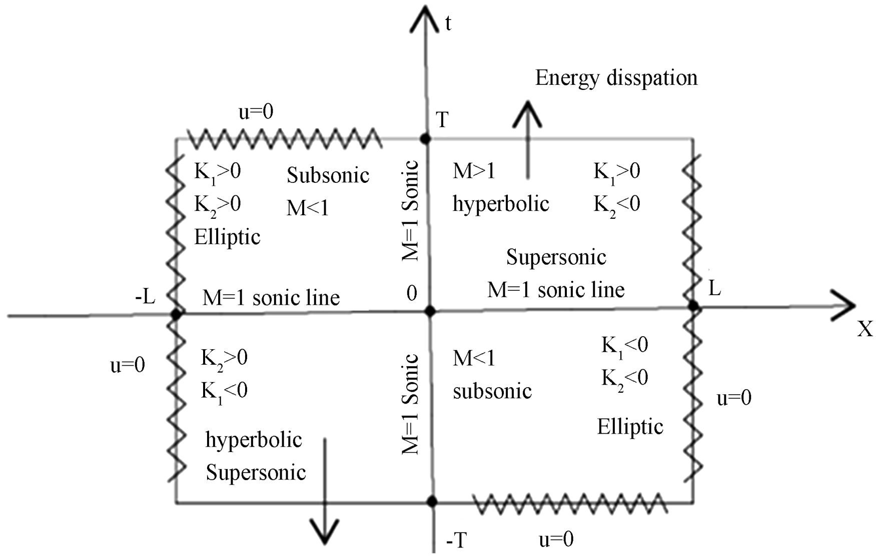

Figure 9. The physical, hydro dynamical, aero-dynamic and gas dynamic described (see [12]).

In this sense, as a study from a mathematical point of view, the presented model, described by the use of non-classical equations of mathematical physics (physical mean see **Figure 9**), which included the Tricomi and Keldysh equations, was studied for the first time. In order to have motivation for non-classical models, we first illustrate the Tricomi, Keldysh, and Tricomi-Keldysh type model equations:

$$zu_{xx}(x,z)+u_{zz}=0,\ K(z)u_{xx}(x,z)+u_{zz}=0, K(z)=z,z^2,z^m$$

Equations (10)-(12) are the particular case of Equation (13) by M. A. Nurmammadov (see [11]-[13]). Therefore, in any case, pure poloidal or pure toroidal magnetic field similar to the so-called tokamak plasma model are also described by equations in frequent derivatives. Introducing stream function as $S(r,\theta)$ in the following form:

$$B_r=\frac{1}{r^2\sin\theta}\frac{\partial S}{\partial\theta},\ B_\theta=-\frac{1}{r\sin\theta}\frac{\partial S}{\partial r}, \tag{14}$$

and accordance to the Hopf's theorem, from Maxwell's equation and magnetic force balance equation, we get $\boldsymbol{B}_P\cdot\nabla(r\sin(\theta)\boldsymbol{B}_t)=0$, and it follows that $\boldsymbol{B}_\Phi=\frac{\beta(S)}{r\sin(\theta)}$. Where β is some function of the stream function $S(r,\theta)$. This means that the toroidal part of the field is a function of the poloidal part. In this case for steam function, $S(r,\theta)$ elliptical equation can be written as:

$$\Delta S(r,\theta)=\frac{B}{R^2}r^2\sin^2(\theta) \tag{15}$$

where Δ -is Laplace operator respect to independent variables (r,θ). Hence, by means of boundary conditions, the stream function $S(r,\theta)$ can be found and finally, the B_r, B_θ solution may be expressed by analytical formulas. In this case, if consider tokamak poloidal-toroidal instead of $|\boldsymbol{B}| = B$ replacing in (8) we obtain mixed poliadal-toriadal potential equation of magnetic field (which is physical corresponding as mathematical model to the poloidal-toroidal field equation):

$$\Delta S(r,\theta) = \begin{cases} \dfrac{4.5\times 10^{15} G}{R^2} r^2 \sin^2(\theta) \\ \dfrac{1.9\times 10^{16} G}{R^2} r^2 \sin^2(\theta) \end{cases}$$

If take $B \propto r^{-3}$ as chosen ordinary case (see [20]) and by astronomical coordinate with $\mu = \cos(\theta)$, then equation of (15) can be written:

$$\Delta S(r,\theta) = \frac{1}{rR^2}\left(1-\mu^2\right) = f(r,\mu) \quad \text{or} \quad \frac{\partial^2 S}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2 S}{\partial \mu^2} = \frac{1}{rR^2}\left(1-\mu^2\right) = f(r,\mu)$$

$$\frac{\partial^2 S}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2 S}{\partial \mu^2} = \begin{cases} \dfrac{4.5\times 10^{15} G}{R^2} r^2 \left(1-\mu^2\right) \\ \dfrac{1.9\times 10^{16} G}{R^2} r^2 \left(1-\mu^2\right) \end{cases} \tag{16}$$

In this case when $\theta = 0^\circ$ $\mu = 1$, the equation is degenerating and belongs to the class equation Keldysh-Tricomi type:

$$\frac{\partial^2 S}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2 S}{\partial \mu^2} = f(r,\mu) \tag{17}$$

If $r \to \infty$ $\theta = 90^\circ$ this equation belongs to the class of Tricomi type and at the same time for $0 < \theta \le 90^\circ$ having singularity:

$$\frac{\partial^2 S}{\partial r^2} = f(r,\mu), \tag{18}$$

when $\theta = 0^\circ$ $\mu = 1$, then we have the following class equations:

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \mu^2} = f(r,\mu), \tag{19}$$

Note that the Equation (18) has a general solution:

$$S(r,\mu) = \int_{r_0}^{r}\int_{r_0}^{r} f(\xi,\mu)\,\mathrm{d}\xi\,\mathrm{d}\xi + \int_{r_0}^{r} C_1(\mu)\,\mathrm{d}\xi + C_2(\mu)$$

where C_1, C_2 are arbitrary constants. Physical means the corresponding singular point of the sonic line, cutoff, and resonance phenomena having notations—parallel propagation: the principal resonance to be these which occurs at $\theta = 0^\circ$ $\theta = \frac{\pi}{2}$. As an initial condition, it is assumed that the magnetic field is a dipole, described using the notation as:

$$U_{initial} = U_{dipol} = \frac{1-\cos^2(\theta)}{r}, \tag{20}$$

If $\theta = 0^\circ$, then $U_{initial} = U_{dipol} = 0$, this means that beyond the critical radius, the magnetic field may be too weak to support plasma coronation and coronation violations can be calculated for magnetic star from the height of the integral conductivity and external plasma flux.

Finally, common case, Equations (17) and (19) can be solved by means of the following recurrent formulas (see Section 6), which is at first applied in the theory of astrophysics.

6. An Application of the Method of Straight Line for Recurrent Formula of Calculation

$$\frac{\partial^2 U}{\partial r^2}+\frac{1-\cos^2(\theta)}{r^2}\frac{\partial^2 U}{\partial \theta^2}=f(r,\mu,U),\quad K(r,\theta)=\frac{1-\cos^2(\theta)}{r^2} \tag{21}$$

Then, it is easy to see that Equations (18) and (19) are Tricomi type:

$$\frac{\partial^2 U}{\partial r^2}+K_1(r)\frac{\partial^2 U}{\partial \theta^2}=f(r,0,U),\ \frac{\partial^2 U}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}=f(r,0,U),\ K_1(r)=\frac{1}{r^2}. \tag{22}$$

The same analogical as above from equation of (14), we have equation of Keldysh type:

$$r^2\frac{\partial^2 \psi}{\partial r^2}+\frac{\partial^2 \psi}{\partial \mu^2}=f(r,\mu,\psi), \tag{23}$$

Note. The method of straight lines can be based on two concepts. One of them considers this method as a development of the finite difference method. The other concept is based on the ideas of projecting infinite-dimensional spaces of solutions to problems for partial differential equations onto spaces with a finite functional basis.

Problem 1. Find solution of Equation (23) such that satisfies the following conditions:

$$\psi(r,\mu^0)=\phi^0(r),(r,\mu^0+l)=\phi^1(r),(\alpha_0\le r\le\alpha_1),$$

$$\psi(\alpha_0,\mu)=\phi^2(\mu),\psi(\alpha_1,\mu)=\phi^3(r),(\mu^0\le\mu\le\mu^0+l).$$

After application of method of straight lines (see note), we obtain recurrent formulas of Equation (22):

$$r^2U_k''(r)+\frac{1}{h^2}\left[U_{k+1}(r)-2U_k(r)+U_{k-1}(r)\right]=0,\ U_0(r)=U_{n+1}(\alpha_1)=0, \tag{24}$$

Main Result: Finally, we found general solution with control tests:

$$U_k(r)=\sum_{s=1}^{n}\sin\frac{s\pi(\mu^s-\mu^0)}{l}\left[A_s e^{\delta_s r}+B_s e^{-\delta_s r}\right],$$

$$\delta_s'^2=\frac{24\sin^2\frac{\pi(\mu^s-\mu^0)}{2l},}{h^2\left(5+\cos\frac{\pi(\mu^s-\mu^0)}{l}\right)}(s=0,1,2,\cdots,n).$$

where A_i, B_i are arbitrarily constants. Application in case of $n=3, h=0.25$ and

f_n right hand of Equation (23) in case for Equation (23) can be written in the following:

$$r^2U_1''(r)+16\left[U_3(r)-2U_2(r)\right]=f_1, U_0(r)=U_4(r)=0,$$

$$r^2U_2''(r)+16\left[U_3(r)-2U_2(r)+U_1(r)\right]=f_2,$$

$$r^2U_3''(r)+16\left[U_2(r)-2U_3(r)\right]=f_3, U_i(0)=U_i(0,5)=0.$$

Let's the particular solution in the form $U_i(r)=A_i=const, i=1,2,3$. For given $f_k(r)=r^2 f_{1,k}(r)$ -right hand of Equation (14) or Equation (23). Using the value of A_1, A_2, A_3 and obtained formulas of general solution, we have:

$$U_1(r)=\sin 0.25\pi\left(C_1 e^{\delta_1 r}+D_1 e^{-\delta_1 r}\right)+\sin 0.5\pi\left(C_2 e^{\delta_2 r}+D_2 e^{-\delta_2 r}\right)$$
$$+\sin 0.75\pi\left(C_3 e^{\delta_3 r}+D_3 e^{-\delta_3 r}\right)+\frac{f_2-f_1}{48},$$

$$U_2(r)=\sin 0.5\pi\left(C_1 e^{\delta_1 r}+D_1 e^{-\delta_1 r}\right)+\sin \pi\left(C_2 e^{\delta_2 r}+D_2 e^{-\delta_2 r}\right)$$
$$+\sin 1.5\pi\left(C_3 e^{\delta_3 r}+D_3 e^{-\delta_3 r}\right)+\frac{2f_2+f_3}{48},$$

$$U_3(r)=\sin 0.75\pi\left(C_1 e^{\delta_1 r}+D_1 e^{-\delta_1 r}\right)+\sin 1.5\pi\left(C_2 e^{\delta_2 r}+D_2 e^{-\delta_2 r}\right)$$
$$+\sin 2.25\pi\left(C_3 e^{\delta_3 r}+D_3 e^{-\delta_3 r}\right)+\frac{7f_1+2f_3}{48},$$

$$\delta_1^2=4*16\sin^2\frac{\pi}{8}=9.3726; \delta_1=3.0611;$$

$$\delta_2^2=4*16\sin^2\frac{\pi}{4}=32; \delta_2=5.6568,$$

$$\delta_3^2=4*16\sin^2\frac{3\pi}{8}=54.6274; \delta_2=7.3910.$$

Hence, account into boundary value conditions and symmetrically, take $C_i=D_i$ obtain inhomogeneous algebraically system equations respect to C_1, C_2, C_3 and after substituting its value, which is found above as the general solution $U_i(r), i=1,2,3$. Accordance to testing values of $f_1=8$; $f_2=3.5$; $f_3=-10$ finding constants are $C_1=-0.0266$; $C_2=0$; $C_3=-0.00009$.

Thus, we have

$U_1(r)=U_3(r)=0.0938-0.0376Ch(3.0611r)-0.00013Ch(7.3910r)$;

$U_2(r)=0.01250-0.0532Ch(3.0611r)+0.00018Ch(7.3910r)$; hence, approximately solution in the center of square will be $U_2(0)=0.0720$; but at the same time, the exact solution is $\psi(r,\mu)=u(0,0)=0.07360$; $error=0.0016$. It means that this method is best and at first applied for the stars model.

After substituting these flux band stream functions, it can be found axisymmetric mixed model of the poloidal-toroidal magnetic field. Hence, the poloidal (meridional) and toroidal (azimuthal) parts of the magnetic field may be expressed by means above general solution and using boundary conditions special solution which choosing near to exact solution. That is, at first, we applied new methods and determined initial boundary conditions in the following: As an initial condition, it is assumed that the magnetic field is a dipole, described using the previous notation as: $U_{initial} = 0$, $U_{initial} = \frac{1-\mu^2}{r}$. Finally, these obtained solution provide the conditions of magneto-hydrodynamic equilibrium on the magnetosphere of magnetic stars.

Accordance equality of force balance in the magnetosphere, standard form defines the plasma and the Alfven Mach number for magneto-hydrodynamic equilibrium can used (**Appendix B**).

Remark (about open question): In almost all cases of numerical modeling, the recreation of real physical conditions in the depths of stars will remain unattainable in the foreseeable future, regardless of the expected increase in computing power and therefore need to create retrospective physical and mathematical models.

Note. However, as already seen by the applications of mathematical models, that in the theory of magnetic field still has many shortcomings, so it is necessary to make more and more new theoretical assumptions and then compare the reliability of the practical applied. Therefore, our duty is to seek and try to find new theoretical assumptions on the basis of the existing justified theory and the facts confirming these theories of observational results. Although, in this work, more or less we tried to observe these necessary only that we noted approaches, which confirms the above all assumptions.

7. Conclusion

In this work, first of all, we give the definition of non-classical equations of mathematical physics and their applicable cases presented, which are used by the author's works. Considering the concept of MHD equilibrium and the field of basic applications of MHD, the model of the MHD equilibrium field of a magnetic star is constructed. Using the main physical effects in MHD to model MHD equilibrium and the field of a magnetic star, we chose a mathematical model. The main results consist of remonstration magneto-hydrodynamic conditions of mixed magnetic field equations. A new equation of model and axisymmetric mixed magnetic field model is obtained into the Keldysh-Tricomi type equations for magnetic stars. Finally, the application of some new methods to the solution of the equation for the hydromagnetic-dynamic equilibrium of magnetic stars is introduced. For the linear and quasilinear (in the case of the right hand of the equation) cases, we apply the straight-line method to obtain a recurrent formula for the calculation of the stream and magnetic flux functions. For this reason, the corresponding problems are solved and given numerical calculations, which are tested.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Grad, H. and Rubin, H. (1958) Hydromagnetic Equilibria and Force-Free Fields. *Proceedings of the 2nd UN Conference on the Peaceful Uses of Atomic Energy*, Geneva, 1-13 September 1958, 190.

[2] Shafranov, V.D. (1975) Reviews of Plasma Physics. Consultants Bureau.

[3] Hicks, W.M. (1898) Researches in Vortex Motion. Part III. On Spiral or Gyrostatic Vortex Aggregates. https://royalsocietypublishing.org/doi/pdf/10.1098/rspl.1897.0119

[4] Hicks, W.M. (1899) Researches in Vortex Motion—Part III. On Spiral or Gyrostatic Vortex Aggregates. Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character, 33-99. https://royalsocietypublishing.org/doi/pdf/10.1098/rsta.1899.0002

[5] Llewellyn Smith, S.G. and Hattori, Y. (2012) Axisymmetric Magnetic Vortices with Swirl. *Communications in Nonlinear Science and Numerical Simulation*, **17**, 2101-2107. https://doi.org/10.1016/j.cnsns.2011.06.001

[6] Tricomi, F. (1923) Sulle equazioni differenziali alle derivate parziali di secondo ordine di tipo misto. In: *Memorie della Reale Accademia dei Lincei*, Vol. XIV, Articolo Digitalizzato nel Quadro del Programma Bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI. http://www.bdim.eu/

[7] Keldysh, M.V. (1951) On Some Cases of Degenerate Elliptic Equations on the Boundary of a Domain. *Doklady Akademii Nauk USSR*, **77**, 181-183.

[8] Polubarinova-Kochina, P. (1977) Theory of Groundwater Movement. 2nd Edition, Nauka. (In Russian)

[9] Barnblatt, G.I., Zheltov, Y.P. and Kochina, I.N. (1960) On the Basic Concepts of the Theory of Filtration of Homogeneous Liquids in Fractured Rocks. *Applied Mathematics and Mechanics*, **25**, 852-864.

[10] Vragov, V.N. (1983) Boundary Value Problems for the Non-Classical Equations of Mathematical Physics Novosibirsk. NSU.

[11] Nurmamedov, M.A. (1985) The First Boundary Value Problems for the Model Equation of Mixed Type. In: *Non-Classical Equations of Mathematical Physics*, Institute of Mathematics of Siberian Branch of the Academy of Sciences, 117-122.

[12] Nurmammadov, M.A. (2022) Applications of Non-Classical Equations and Their Approaches to the Solution of Some of Classes Equations Arise in the Kelvin-Helmholtz Mechanism and Instability. *Open Journal of Applied Sciences*, **12**, 1873-1891. https://doi.org/10.4236/ojapps.2022.1211129

[13] Nurmammadov, M. (2024) New Approaches on the Theory of Planetary Sciences: Applications of Non-Classical Equations of Mathematical Physics for Plasma Models of Jupiter's Magnetosphere. *American Journal of Astronomy and Astrophysics*, **11**, 14-32. https://doi.org/10.11648/j.ajaa.20241101.12

[14] Brainerd, J.J. (2005) X-Rays from Stellar Coronas. The Astrophysics Spectator.

[15] Wade, G.A. (2004) Stellar Magnetic Fields: The View from the Ground and from Space. *Proceedings of the International Astronomical Union*, **2004**, 235-243. https://doi.org/10.1017/s1743921304004612

[16] Basri, G. (2006) Big Fields on Small Stars. *Science*, **311**, 618-619. https://doi.org/10.1126/science.1122815

[17] Sherwood, J. (2002) Dark Edge of Sunspots Reveal Magnetic Melee. University of Rochester.

[18] Braithwaite, J. and Spruit, H.C. (2004) A Fossil Origin for the Magnetic Field in a Stars

and White Dwarfs. *Nature*, **431**, 819-821. https://doi.org/10.1038/nature02934

[19] Abbott, B., *et al.* (2007) Search for Gravitational Waves from Scorpius X-1 with a Hidden Markov Model in O_3 LIGO Data. *Physical Review D*, **106**, Article ID: 062002. https://doi.org/10.1103/PhysRevD.106.062002

[20] Solov'ev, L.S. (1968) The Theory of Hydromagnetic Stability of Toroidal Plasma Configurations. *Soviet Journal of Experimental and Theoretical Physics*, **26**, Article 400.

Appendixes

Appendix A

Define (r,θ,ϕ) coordinates by $R = R_0 + r\cos\phi$, where (R,ϕ,Z) are cylindrical coordinates $Z = r\sin\phi$, where (R,ϕ,Z) are cylindrical coordinates and R_0 (see Table A1) is a constant. The above transformation is shown in Figure 4. The Jacobian of (r,θ,ϕ) coordinate can be calculated using the definition. Using $x = r\cos\phi$, $y = r\sin\phi$, $z = Z$, the Jacobian (with respect to Cartesian coordinates (x,y,z)) is written as $J = -Rr$. The toroidal elliptic operator in (r,θ,ϕ) can be defined by transforming the Grand-Shafronov equation from (R,Z) coordinates to (r,θ) coordinates:

$$r = \sqrt{(R-R_0)^2 + Z^2}, \frac{\partial r}{\partial Z} = \frac{Z}{r} = \sin\theta, \sin\theta = \frac{Z}{\sqrt{(R-R_0)^2 + Z^2}}, \cos\theta\frac{\partial\theta}{\partial Z} = \frac{r - Z\dfrac{Z}{r}}{r^2},$$

$$\frac{\partial\theta}{\partial Z} = \frac{r - Z\dfrac{Z}{r}}{r(R-R_0)} = \frac{\cos^2\theta}{r}, \frac{\partial^2\psi}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) = -\mu_0 R^2\frac{\partial P}{\partial\psi} - \frac{\partial g}{\partial\psi}g(\psi),$$

$$\frac{\partial\psi}{\partial Z} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial Z} = \frac{\partial\psi}{\partial r}\sin\theta + \frac{\partial\psi}{\partial\theta}\frac{\cos^2\theta}{R-R_0}$$

$$\frac{\partial^2\psi}{\partial Z^2} = \sin\theta\left(\frac{\partial^2\psi}{\partial r^2}\sin\theta + \frac{\partial^2\psi}{\partial r\partial\theta}\frac{\cos^2\theta}{R-R_0}\right) + \frac{\partial\psi}{\partial r}\frac{\cos^2\theta}{r}$$
$$+\frac{\cos^2\theta}{R-R_0}\left(\frac{\partial^2\psi}{\partial r\partial\theta}\sin\theta + \frac{\partial^2\psi}{\partial\theta^2}\frac{\cos^2\theta}{R-R_0}\right) - \frac{\partial\psi}{\partial\theta}\frac{1}{R-R_0}\sin 2\theta\frac{\cos^2\theta}{R-R_0}.$$

Account in the following obvious formulas:

$$\sin\theta = \frac{Z}{\sqrt{(R-R_0)^2 + Z^2}}, \cos\theta\frac{\partial\theta}{\partial R} = -Z\frac{R-R_0}{r^3}, \frac{\partial\theta}{\partial R} = -\frac{Z}{r^2}, \frac{\partial}{\partial R}\cos\theta = \frac{\sin^2\theta}{r},$$

We have:

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) = \left(\frac{\partial^2\psi}{\partial r^2}\frac{\partial r}{\partial R} + \frac{\partial^2\psi}{\partial r\partial\theta}\frac{\partial\theta}{\partial R}\right)\cos\theta + \frac{\partial\psi}{\partial r}\frac{\partial}{\partial R}\cos\theta$$
$$-\left(\frac{\partial^2\psi}{\partial r\partial\theta}\frac{\partial r}{\partial R} + \frac{\partial^2\psi}{\partial\theta^2}\frac{\partial\theta}{\partial R}\right)\frac{Z}{r^2} - \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial R}\left(\frac{Z}{r^2}\right)$$
$$-\frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{\partial\psi}{\partial\theta}\frac{Z}{r^2}\right).$$

After some reduced, we have:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi - \frac{1}{R_0 + r\cos\theta}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{\partial\psi}{\partial\theta}\frac{1}{r}\sin\theta\right)$$
$$= -\mu_0(R_0 + r\cos\theta)^2\frac{\partial P}{\partial\psi} - \frac{\partial g}{\partial\psi}g(\psi).$$

where $L\psi = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \psi$ is operator similar form as operator Tricomi-Keldysh in case of mixed magnetic poliadol-toriadol field which analogy equation of mixed elliptic-hyperbolic type.

Table A1. Main TCV technical parameters.

Parameters	Symbol	Value
Major radius	R_0	0.88 m
Minor radius	a	0.25 m
Nominal aspect ratio	$A = R_0/a$	3.5
Vacuum vessel elongation	k_{TCV}	3
Maximum plasma current	I_p	1.2 MA
Maximum central magnetic field	B_0	1.5 T
Discharge duration		<4s
Edge plasma elongation	Ka	0.9 - 2.82
Edge plasma triangularity	δa	(−0.8) - (+0.9)

Appendix B

Now, accordance equality of force balance in the magnetosphere, standard form defines the plasma $\beta = \frac{8\pi\rho}{B^2}$ and the Alfven Mach number by (see [13]):

$$M_A^2 = \frac{1}{2} \beta \frac{\partial \ln \rho}{\partial \ln \omega} + \frac{4\pi}{c} \frac{I_0}{B} \tag{25}$$

It is important note that, in the tail magnetosphere of magnetic star, the Euler potential force is constant, and inside the tail, the Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$ is equal to $\nabla P + \rho \nabla \Phi$, *i.e.* $\boldsymbol{J} \times \boldsymbol{B} = \nabla P + \rho \nabla \Phi$.

For the theoretical justification of magneto-hydrodynamical equilibrium by observation, date was used and considered (Table A1).