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Analytical Proof of the Solution to Second Order Linear Homogeneous Differential Equation

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Abstract

There is no gainsaying that the field of differential equations opened up in numerable fields of research which hitherto remains yet to be fully explored in spite of the remarkable achievements made over the centuries. In this study, the subject is given renewed interest with a view to addressing a question that is considered germane to the field of differential equations (DEs). Over the centuries, the field of ordinary differential equations (ODEs) has received varied interest from mathematicians across the globe. One problem that has remained unresolved hitherto is a deductive proof of the solution to linear homogeneous differential equations of order 2, 3 or more. Solutions to this set of equations usually come in form of intuitive assumptions such as let $y = e^{rx}$ or $y = x^r$, which is later confirmed by direct substitution. This study aims to provide a deductive solution leading to the proof that $y = e^{rx}$ and $y = x^r$ are not just intuitive assumptions but are indeed accurate in every sense of the word.

Subject Areas

Ordinary Differential Equation

Keywords

Analytical Solutions, Linear, Homogeneous, Ordinary Differential Equations

1. Introduction

Generally, equations are viewed as mathematical statements that relate one or more physical quantities usually referred to as variables. These variables are so called because they undergo changes. Equations that describe the relationship between variables are known as functions. An example of functions is given by the equation below:

$$y = x^2 \tag{1}$$

In Equation (1), x and y are variables. Functions describing the rates of change of variables are called derivatives. Equation (2) below is a function which is also a derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \tag{2}$$

In Equation (2), $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the derivative of y with respect to x. Differential equations are those which relate one on more functions and their derivatives. As noted earlier, the functions represent physical quantities that are subject to change while the derivatives represent their rates of change.

Differential equations occur very commonly in different phenomena, hence their role in many disciplines including such fields as engineering, physical and chemical sciences, economics and biology [1]. An example of differential equations is given by the equation below.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 3x \tag{3}$$

Equations of the kind above first came into existence through the efforts of Isaac Newton and Gottfried Leibniz who independently pioneered calculus around the 17th century. Others who made significant contributions, to the development of differential equations in the 18th century included Jacob Bernoulli, Jean Le Rond D'Alembert, Leonard Euler, Daniel Bernoulli, Joseph-Louis Legrange, and Laplace. From the 19th century through the 20th and up to the 21st century, outstanding achievements were made by quite a lot of great mathematicians prominent among them were Fourier Maxwell and Pioneare [2].

Reflecting retrospectively on the development of differential equations actually brings to mind the temptation to think that the field of differential equations has been stretched to its elastic limit. In other words, it seems quite likely that the field of differential equations has been exhausted of new topics to explore, leaving nothing to be investigated any further. This probably accounts for why research efforts are grossly directed toward their applications.

Grappling with the foregoing conception, curiosity drives the mind into retrospection of some great works of ancient pioneers of the field of differential equations. As a teacher of differential equations, personal experience in the teaching of second order linear homogeneous differential equations, pioneered by Leonhard Euler [3], always points to students' curiosity to understand why Euler's assumption of the exponential function

$$y = \mathrm{e}^{rx} \tag{4}$$

fits well into the solution of the homogeneous linear differential equation of order 2, *i.e.*

$$y'' + py' + q = 0 \tag{5}$$

with constant coefficients *p* and *q*. Both researchers and students often desire to explore the possibility of solving the Equation (3) above analytically to prove that $y = e^{rx}$ is the resultant solution rather than just intuitively assuming it as the solution and substituting it in order to prove its accuracy. Indeed, the view that $y = e^{rx}$ is an intuitive guess is widely shared by many mathematicians even though there has not been any better way of solving the problem [4]. For example, Nagy noted that the solution to Equation (3) is obtained by trial and error [4]. According to Nagy, $y = e^{rx}$ is first assumed to be the solution to equation (3) because the exponential cancels out of the equation leaving only a condition for "*r*". Frankly, the fact that $y = e^{rx}$ proves to generates the solution to equation (3) only by trial and error has lead us to investigating these linear systems in order to determine their stability and consistency of their solutions.

Again, while studying the Euler linear homogeneous equation with variable coefficients

$$x^2 y'' + pxy' + qy = 0 \tag{6}$$

students are faced with similar questions as with the Equation (3). They often want to understand why the Euler substitution

$$y = x^r \tag{7}$$

fits well as the solution to the equation (6). Nagy also pointed that Equation (7) is obtained by assumption [4]. According to Nagy [4], the solution $y = e^r$ is being sought for because it has the property that

$$y' = re^{r-1} \to xy' = rx^r \quad \text{and that}$$

$$y'' = r(-1)x^{r-2} \to x^2 y'' = r(r-1)x^r .$$

This clearly demonstrates that Euler's solution were obtained by intuition rather than deduction. Any curious minded mathematician would want to derive the functions in Equations (4) and (7) in order to make assurance doubly sure. What can be made of this fact is that there does not seem to be adequate understanding of the deductive processes leading to the solutions to Equations (4) and (7). Indeed, this lack of adequate understanding must have lead Euler into the conclusion that the equation

$$y'' + pxy' + qx^2 y = 0 \tag{8}$$

has no solution in terms of elementary functions. In addition, the area of series solutions to homogeneous linear differential equations of order 2 with variable coefficients came into existence due to the perception that such equations cannot be solved in terms of known elementary functions or at least in terms of integral functions besides the Euler equation [5]. The general form of homogeneous linear ordinary differential equation of order 2 is given by

$$y'' + p(x)y' + q(x)y = 0 \tag{9}$$

The conception that Equation (9) has no known solutions in terms of elementary functions or in terms integral functions besides the Euler equations has been held for centuries and perhaps this is the case because there has not been any better away to handle Equation (9). Even the series solutions are usually obtained by assumption [4] [5].

$$y = \sum_{n-0}^{\infty} c_n \left(x - x_0\right)^n \tag{10}$$

Note here that the intention is not to undermine assumption as an essential procedure in mathematical analysis. The point projected here is that assumptions actually make mathematics more presumptive and less deductive. It is important, therefore, to try the deductive procedures as well in order to provide sufficient ground for acceptance. It becomes pertinent to consider the possibility of proving the assumption in Equation (10) by some deductive method.

In the light of the foregoing, this paper presents a study that sets out to achieve the following objectives.

1) To prove that $y = e^{rx}$ is not just an intuitive assumption by some analytical derivation of it;

2) To prove that $y = e^{r}$ is not just another ansatz but one that can be derived analytically; and

3) To prove that $y = \sum_{n-0}^{\infty} c_n \left(x - x_0\right)^n$ is equally not just a guess but an analytically proven derivation, and

4) To show by these proofs that all linear homogeneous equations of order 2 actually derive the solutions by the same procedure irrespective of whether the coefficients are constants or variables.

2. Theoretical Framework

The framework upon which this study is based is quite simple, familiar and easy enough to comprehend. If we reexamined closely the actual meanings of the derivatives of the second order, third order, etc. then we might be well on the way to determining deductively the derivations of Equations (4), (7), (8) and (9) respectively.

Recall that

$$y' = \frac{dx}{dx}$$

$$y'' = \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dx}{dx}\right)$$

$$y''' = \frac{d^3 y}{dx^3} = \frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right) = \frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} \text{ etc.}$$

applying this logically, we can generate the proofs of statements (i), (ii), (iii) and (iv). The proofs will now be divided into the four sections following afterward.

2.1. Solution to the Second Order Linear Homogeneous Differential Equations with Constant Coefficient

Consider the second order linear homogeneous Equation (4) below

$$y'' + py' + qy = 0 \tag{5}$$

Using actual notations we have

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$$

Reinterpreting the notations, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)+p\frac{dy}{dx}+qy=0$$

Factorizing out $\frac{dy}{dx}$ give us the following

$$\frac{dy}{dx}\left(\frac{d}{dx}+p\right)+qy=0$$

It must be understood here that the expression in parenthesis is a differential operator with a constant. Thus we can denote it by D *i.e.*

$$\frac{d}{dx}+p=D \tag{11}$$

The equation above becomes

$$\frac{d}{dx}D+qy=0 \tag{12}$$

From (12)

$$\frac{dy}{dx}D=-qy$$

$$\rightarrow \quad \frac{dy}{dx}=\frac{-q}{D}y$$

$$\rightarrow \quad \frac{1}{y}dy=\frac{-q}{D}dx$$

Integrating both sides we get

$$\int\frac{1}{y}dx=\int\frac{-q}{D}dx$$

$$\rightarrow \quad \ln y=\frac{-q}{D}x$$

$$y=e^{\frac{-qx}{D}}$$

If we let

$r=\frac{-q}{D}$, then we have

$$y=e^{rx} \tag{4}$$

As the solution to the Equation (5) we can equally determine the values of D and r by going back to Equation (11). Multiplying Equation (11) by y gives

$$\left(\frac{d}{dx}+p\right)y=Dy$$

$$\rightarrow \quad \frac{dy}{dx}+py=Dy-py$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = Dy - py$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = (D-p)y$$

But from (12)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-q}{D}y$$

Therefore it follows that by substitution

$$\frac{-q}{D}y = (D-p)y$$

$$\rightarrow \quad \frac{-q}{D} = D - p$$

$$\rightarrow \quad D^2 - pD = -q$$

$$\therefore \quad D^2 - pD + q = 0$$

Equation (13) is called the characteristics equation of the differential operator D.

From (13)

$$D = \frac{-(-p) \pm \sqrt{(-p)^2 - 4q}}{2}$$

$$\therefore \quad D = \frac{P \pm \sqrt{P^2 - 4q}}{2}$$

Thus D has two values D_1 and D_2 which implies also that r has two values

$$r_1 = \frac{-q}{D_1} \text{ and } r_2 = \frac{-q}{D_2}$$

Therefore, the two linearly independent solutions of the Equation (5) are

$$y_1 = \mathrm{e}^{r_1 x} \text{ and } y_2 = \mathrm{e}^{r_2 x}$$

$$\rightarrow \quad y = C_1 \mathrm{e}^{r_1 x} + C_2 \mathrm{e}^{r_2 x} \tag{15}$$

We can equally find the characteristic equation in terms of r – in fact Euler [5] already demonstrated that even though he was not quite aware that r depends on the differential operator D.

From (4),

$$y = \mathrm{e}^{rx}, \quad y' = r\mathrm{e}^{rx} \quad \text{and}$$

$$y'' = r^2 \mathrm{e}^{rx} \quad \text{which by substitution gives}$$

$$r^2 \mathrm{e}^{ra} + pr\mathrm{e}^{rx} + q\mathrm{e}^{rx} = 0$$

$$\rightarrow \quad (r^2 + pr + q)\mathrm{e}^{rx} = 0$$

$$\therefore \quad r^2 + pr + q = 0 \tag{16}$$

From Equation (16),

$$r = \frac{-P \pm \sqrt{P^2 - 4q}}{2} \tag{17}$$

Notice here that the values of r and D are different. They are, however, related by the formula

$$r = \frac{-q}{D}$$

Another method we can use to find the solution to Equation (5) is by factorizing y out. See the procedure below.

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

Factoring out y gives us

$$\left(\frac{d^2}{dx^2} + p\frac{d}{dx} + q\right)y = 0$$

Note here that $\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right)$, thus if we let $r = \frac{d}{dx}$ then

$$\left[\frac{d}{dx}\left(\frac{d}{dx}\right)p\frac{d}{dx} + q\right]y = 0$$

By substituting

$$\rightarrow \quad (r \cdot r + pr + q)y = 0$$

$$\rightarrow \quad (r^2 + pr + q)y = 0$$

Since $y \neq 0$, it follows that

$$r^2 + pr + q = 0 \tag{16}$$

This is the same characteristics equation that was obtained earlier, but

$$\frac{d}{dx} = r$$

Multiplying both side by y, we get

$$\frac{dy}{dx} = ry$$

$$\rightarrow \quad \frac{dy}{y} = r dx$$

$$\rightarrow \quad \int \frac{dy}{y} = \int r dx$$

$$\therefore \quad \ln y = rx$$

$$\rightarrow \quad y = e^{rx} \tag{4}$$

Notice here too that the second method is much easier than the first

2.2. Solution to the Euler Second Order Linear Homogeneous Equation with Variable Coefficients

Given the Euler equation

$$x^2 y'' + pxy' + qy = 0 \tag{6}$$

using actual notations, we have

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + px\frac{\mathrm{d}y}{\mathrm{d}x} + qy = 0$$

dividing through by x^2,we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + p\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + q\frac{1}{x^2}y = 0$$

Factoring out $\frac{\mathrm{d}y}{\mathrm{d}x}$ as before, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{P}{x}\right) + \frac{q}{x^2}y = 0$$

Now let

$$\frac{\mathrm{d}}{\mathrm{d}x} + \frac{p}{x} = R(x) \tag{17}$$

Then by substitution, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x}R(x) + \frac{q}{x^2}y = 0$$

$$\rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}R(x) = \frac{-q}{x^2}y$$

$$\rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{R(x)} \times \frac{-q}{x^2}y \tag{18}$$

From (17), multiplying by y gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{p}{x}y = R(x)y$$

$$\rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = R(x)y - \frac{p}{x}y$$

By substitution, we get

$$R(x)y - \frac{p}{x}y = \frac{-q}{R(x)e^2}y$$

$$\rightarrow x^2\left[R(x)\right]^2 - px\left[R(x)\right] = -q$$

$$\therefore x^2\left[R(x)\right]^2 - pxR(x) + q = 0 \tag{19}$$

Equation (19) is the characteristic equation of the differential operator $R(x)$.
From Equation (19),

$$R(x) = -\frac{(-px) \pm \sqrt{(-px)^2 - 4x^2 q}}{2x^2}$$

$$= \frac{px \pm \sqrt{p^2 x^2 - 4x^2 q}}{2x^2}$$

$$= \frac{px \pm x\sqrt{p^2 - 4q}}{2x^2}$$

$$= \frac{x\left(p \pm \sqrt{p^2 - 4q}\right)}{2x^2}$$

$$\therefore \quad R(x)=\frac{1}{2x}\left(p\pm\sqrt{p^2-4q}\right) \tag{20}$$

By substituting (20) in (18), we get

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-qy}{\frac{1}{2x}\left(p\pm\sqrt{p^2-4q}\right)x^2}$$

$$\rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-qy}{\frac{x}{2}\left(p\pm\sqrt{p^2-4q}\right)}$$

$$\rightarrow \frac{\mathrm{d}y}{y}=\frac{1}{x}\times\frac{-2q\mathrm{d}x}{p\pm\sqrt{p^2-4q}}$$

Integrating both sides gives

$$\int\frac{\mathrm{d}y}{y}=\frac{-2q}{p\pm\sqrt{p^2-4q}}\int\frac{1}{x}\mathrm{d}x$$

$$\rightarrow \ln y=\frac{-2q}{p\pm\sqrt{p^2-4q}}\ln x$$

Let

$$r=\frac{-2q}{p\pm\sqrt{p^2-4q}}$$

Then

$$\ln y=r\ln x$$

$$\rightarrow \ln y=\ln x^r$$

$$\therefore \quad y=x^r \tag{7}$$

Another method of doing this is to factor out y instead of $\frac{\mathrm{d}y}{\mathrm{d}x}$. Let us consider this method from (6), we get

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+px\frac{\mathrm{d}y}{\mathrm{d}x}+qy=0$$

Factoring out y gives

$$\left(x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+px\frac{\mathrm{d}}{\mathrm{d}x}+q\right)y=0$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}x}=R(x)$$

Then

$$\left\{x^2\left[R(x)\right]^2+pxR(x)+q\right\}y=0$$

Since y ≠ 0, it follows that

$$x^2\left[R(x)\right]^2+pxR(x)+q \tag{21}$$

From

$$\rightarrow \quad R(x)=\frac{-(px)\pm\sqrt{(px)^2-4x^2q}}{2x^2}$$

$$=\frac{-px\pm\sqrt{p^2x^2-4x^2q}}{2x^2}$$

$$=\frac{-px\pm x\sqrt{p^2-4q}}{2x^2}$$

$$=\frac{-p\pm\sqrt{p^2-4q}}{2x}$$

$$=\frac{1}{x}\left(\frac{-px\pm x\sqrt{p^2-4q}}{2}\right)$$

But

$$\frac{\mathrm{d}}{\mathrm{d}x}=R(x)$$

Multiplying both sides by y gives

$$\frac{\mathrm{d}y}{\mathrm{d}x}=R(x)y$$

$$\rightarrow \quad \frac{\mathrm{d}y}{y}=R(x)\mathrm{d}x$$

$$\rightarrow \quad \int\frac{\mathrm{d}y}{y}=\int R(x)\mathrm{d}x$$

$$\ln y=\int\frac{1}{x}\left(\frac{-px\pm x\sqrt{p^2-4q}}{2}\right)\mathrm{d}x$$

$$\rightarrow \quad \ln y=\frac{-px\pm x\sqrt{p^2-4q}}{2}\int\frac{1}{x}\mathrm{d}x$$

$$=\frac{-px\pm x\sqrt{p^2-4q}}{2}\ln x$$

Let $r=\frac{-px\pm x\sqrt{p^2-4q}}{2}$, then

$$\ln y=r\ln x$$

$$y=x^r$$

It is already common knowledge that

$$y=rx^r,$$

and

$$y''=r(r-1)x^{r-2}$$

which by substitution leads to

$$x^2r(r-1)x^{r-2}+pxrx^{r-1}+qx^r=0$$

$$\rightarrow \quad r(r-1)x^r + prx^r + qx^r = 0$$

$$\rightarrow \quad (r^2 - r + pr + q)x^r = 0$$

$$\rightarrow \quad (r^2 + (p-1)r + q)x^r = 0$$

$$r^2 + (p-1)r + q = 0 \tag{22}$$

Note here that Equation (22) is a very familiar characteristics equation and from it

$$r = \frac{-(p-1) \pm \sqrt{(p-1)^2 - 4q}}{2}$$

This gives a different value of r when compared to the previous value.

3. Solution to the Equation $y'' + pxy' + qx^2 y = 0$

Euler's prediction that Equation (7) would be an appropriate solution to Equation (6) was in indeed commendable achievement. However, it seemed likely that Euler did not envisage the possibility the Equation (8)

$$y'' + pxy' + qx^2 y = 0 \tag{8}$$

could equally have a solution in terms of elementary functions. While we cannot claim to have found a solution for Equation (8) with uttermost certainty, it is only fitting to demonstrate at this point that Equation (8) could possibly have a solution in terms of elementary functions. Rewriting Equation (8) with the usual notations, we have the following

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + px\frac{\mathrm{d}y}{\mathrm{d}x} + qx^2 y = 0$$

$$\rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + px\frac{\mathrm{d}y}{\mathrm{d}x} + qx^2 y = 0$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x} + px\right) + qx^2 y = 0$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}x} + px = S(x)$$

Then by substitution, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} S(x) + qx^2 y = 0$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-qx^2 y}{S(x)} \tag{23}$$

From (23), we have

$$\frac{\mathrm{d}y}{y} = \frac{-qx^2}{S(x)}\mathrm{d}x$$

Integrating both sides, we get

$$\int \frac{\mathrm{d}y}{y} = \int \frac{-qx^2}{S(x)}\mathrm{d}x$$

$$\ln y = \int \frac{-qx^2}{S(x)}\mathrm{d}x$$

$$y = \mathrm{e}^{\int \frac{-qx^2}{S(x)}\mathrm{d}x} \tag{24}$$

Recall also that

$$\frac{\mathrm{d}y}{\mathrm{d}x} + pxy = S(x)y$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = S(x)y - pxy$$

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \left(S(x) - px\right)y \tag{25}$$

From both (23) and (25), we have

$$\left(S(x) - px\right)y = \frac{-qx^2 y}{S(x)}$$

$$\rightarrow \quad S(x) - px = \frac{-qx^2}{S(x)}$$

$$\rightarrow \quad \left[S(x)\right]^2 - pxS(x) = -qx^2$$

$$\rightarrow \quad \left[S(x)\right]^2 - pxS(x) + qx^2 = 0 \tag{26}$$

Equation (26) gives us the value of $S(x)$ to establish the solution to Equation (8)

$$\therefore \quad S(x) = \frac{-(p-1) \pm \sqrt{(-px)^2 - 4\left(qx^2\right)}}{2}$$

$$= \frac{px \pm \sqrt{p^2x^2 - 4qx^2}}{2}$$

$$= \frac{px \pm x\sqrt{p^2 - 4q}}{2}$$

$$\therefore \quad S(x) = \left(\frac{px \pm x\sqrt{p^2 - 4q}}{2}\right)x \tag{27}$$

Substituting (27) in (24) gives

$$y = \mathrm{e}^{\int \frac{-qx^2\mathrm{d}x}{\frac{1}{x}\left(p \pm \sqrt{p^2-4q}\right)x}}$$

$$\rightarrow \quad y = \mathrm{e}^{-\int \frac{-2qx\mathrm{d}x}{p \pm \sqrt{p^2-4q}}}$$

$$\rightarrow \quad y = \mathrm{e}^{\frac{-2q}{p \pm \sqrt{p^2-4q}}\int x\mathrm{d}x}$$

$$= \mathrm{e}^{-\frac{2q}{p \pm \sqrt{p^2-4q}}\frac{x^2}{2}}$$

$$= \mathrm{e}^{-\frac{qx^2}{p \pm \sqrt{p^2-4q}}}$$

If we let

$$m = \frac{-q}{P \pm \sqrt{P^2 - 4q}}$$

Then

$$y = e^{mx^2} \tag{28}$$

Equation (28) is the solution to Equation (8). However, we cannot conclude that until we are able to establish more facts about it, so it is proper to take equation (28) as a tentative solution.

Another method we can use to find the solution to Equation (8) is given as shown below.

From Equation (8), we already have

$$\frac{d^2 y}{dx^2} + px\frac{dy}{dx} + qx^2 y = 0$$

If we decide to factor out y, then we have

$$\left(\frac{d^2}{dx^2} + px\frac{d}{dx} + qx^2\right) y = 0$$

Let

$$\frac{d}{dx} = S(x)$$

Then by substitution, we have

$$\left\{\left[S(x)\right]^2 + pxS(x) + qx^2\right\} y = 0$$

$$\rightarrow \ \left(S(x)\right)^2 + pxS(x) + qx^2 = 0 \tag{29}$$

$$\therefore \ S(x) = \frac{-px \pm \sqrt{(px)^2 - 4(qx^2)}}{2}$$

$$= \frac{-px \pm \sqrt{p^2x^2 - 4qx^2}}{2}$$

$$= \frac{-px \pm \sqrt{(p^2 - 4q)x^2}}{2}$$

$$= \frac{-px \pm \sqrt{p^2 - 4q}}{2}$$

$$\therefore \ S(x) = \left(\frac{-px \pm \sqrt{p^2 - 4q}}{2}\right) x \tag{30}$$

But

$$\frac{d}{dx} = S(x)$$

Multiplying both side by y, we get

$$\frac{dy}{dx} = S(x) y$$

$$\rightarrow \ \frac{dy}{y} = S(x) dx$$

Integrating both sides and ignoring constants, gives

$$\int \frac{\mathrm{d}y}{y} = \int S(x)\mathrm{d}x$$

$$\rightarrow \ \ln y = \int S(x)\mathrm{d}x$$

$$\therefore \ y = \mathrm{e}^{\int S(x)\mathrm{d}x}$$

$$= \mathrm{e}^{\int\left(\frac{-p\pm\sqrt{p^2-4q}}{2}\right)x\mathrm{d}x}$$

$$= \mathrm{e}^{\left(\frac{-p\pm\sqrt{p^2-4q}}{2}\right)\frac{x^2}{2}}$$

$$= \mathrm{e}^{\left(\frac{-p\pm\sqrt{p^2-4q}}{4}\right)x^2}$$

If we let,

$$m = \frac{-p \pm \sqrt{p^2 - 4q}}{4}$$

Then by substitution,

$$y = \mathrm{e}^{mx^2} \tag{28}$$

Now, in order to investigate Equation (28), we need to substitute it into the original differential equation. Thus from Equation (28) we have

$$y' = 2mx\mathrm{e}^{mx^2}$$

$$y'' = 4mx^2\mathrm{e}^{mx^2} + 2m\mathrm{e}^{mx^2}$$

Substituting into (8) gives

$$4mx^2\mathrm{e}^{mx^2} + 2m\mathrm{e}^{mx^2} + px\left(2mx\mathrm{e}^{mx^2}\right) + qx^2\mathrm{e}^{mx^2} = 0$$

$$\rightarrow \ \left(4m^2 + 2pm + q\right)x^2\mathrm{e}^{mx^2} + 2m\mathrm{e}^{mx^2} = 0$$

$$\rightarrow \ \left\{\left(4m^2 + 2pm + q\right)x^2 + 2m\right\}\mathrm{e}^{mx^2} = 0$$

From which we get

$$4m^2 + 2pm + q = 0 \tag{31}$$

And $$2m = 0 \tag{32}$$

From (32), $m = 0$

And from (31),

$$m = \frac{-2p \pm \sqrt{(2p)^2 - 4\times 4\times q}}{2\times 4}$$

$$= \frac{-2p \pm \sqrt{4p^2 - 4\times 4q}}{8}$$

$$= \frac{-2p \pm 2\sqrt{p^2 - 4q}}{8}$$

$$\therefore \quad m=\frac{-p\pm\sqrt{p^2-4q}}{4}$$

This value of m is same as that obtained earlier, therefore,

$$m_1=0,\ m_2=\frac{-p+\sqrt{p^2-4q}}{4} \text{ and } m_3=\frac{-p-\sqrt{p^2-4q}}{4}$$

Since the solutions are independent, it follows that

$$y=c_0y_1+c_1y_2+c_2y_3$$

$$\therefore \quad y=c_0\mathrm{e}^{m_1x^2}+c_1\mathrm{e}^{m_2x^2}+c_2\mathrm{e}^{m_3x^2}$$

$$\rightarrow \quad y=c_0\mathrm{e}^{0\times x^2}+c_1\mathrm{e}^{m_2x^2}+c_2\mathrm{e}^{m_3x^2}$$

$$\therefore \quad y=c_0+c_1\mathrm{e}^{m_2x^2}+c_2\mathrm{e}^{m_3x^2}$$

Solution to the General Equation

It is generally agreed that an equation of the form

$$y''+p(x)y'+q(x)y=0 \tag{33}$$

is called the general equation of second order homogenous linear differential equation. The solution to this equation can be found by adopting the same procedures as in the other cases. For instance, re-writing the equation, we get,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+p(x)\frac{\mathrm{d}y}{\mathrm{d}x}+q(x)y=0$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)+p(x)\frac{\mathrm{d}y}{\mathrm{d}x}+q(x)y=0$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}+p(x)\right)+q(x)y=0$$

If we let

$$\frac{\mathrm{d}}{\mathrm{d}x}+p(x)=0$$

Then we get

$$\frac{\mathrm{d}y}{\mathrm{d}x}D(x)+q(x)y=0$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-q(x)y}{D(x)} \tag{34}$$

But

$$\frac{\mathrm{d}}{\mathrm{d}x}+p(x)=D(x)$$

Multiplying it by y gives

$$\frac{\mathrm{d}y}{\mathrm{d}x}+p(x)y=D(x)y$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}=\big(D(x)-p(x)\big)y \tag{35}$$

By comparing Equations (34) and (35), we can see that

$$\left(D(x)-p(x)\right)y=\frac{-q(x)}{D(x)}y$$

$$\rightarrow\quad D(x)-p(x)=\frac{-q(x)}{D(x)}$$

$$\rightarrow\quad \left[D(x)\right]^2-p(x)D(x)+q(x)=0 \tag{36}$$

From Equation (36), we get

$$D(x)=\frac{-\left(-p(x)\right)\pm\sqrt{\left(-p(x)\right)^2-4q(x)}}{2}$$

$$\therefore\quad D(x)=\frac{p(x)\pm\sqrt{\left(p(x)\right)^2-4q(x)}}{2}$$

$$\rightarrow\quad D(x)=\frac{p(x)+\sqrt{\left(p(x)\right)^2-4q(x)}}{2}$$

And

$$D_2(x)=\frac{p(x)-\sqrt{\left(p(x)\right)^2-4q(x)}}{2}$$

If follows that from Equation (34)

$$\begin{aligned}\frac{q(x)}{D(x)}&=\frac{q(x)}{\frac{1}{2}\left\{p(x)+\sqrt{\left(p(x)\right)^2-4q(x)}\right\}}\\&=\frac{2q(x)}{\left\{p(x)+\sqrt{\left(p(x)\right)^2-4q(x)}\right\}}\\&=\frac{2q(x)\left\{p(x)-\sqrt{\left(p(x)\right)^2-4q(x)}\right\}}{\left\{p(x)+\sqrt{\left(p(x)\right)^2-4q(x)}\right\}\left\{p(x)-\sqrt{\left(p(x)\right)^2-4q(x)}\right\}}\\&=\frac{2q(x)\left\{p(x)-\sqrt{\left(p(x)\right)^2-4q(x)}\right\}}{\left(p(x)\right)^2-\left\{\sqrt{\left(p(x)\right)^2-4q(x)}\right\}^2}\\&=\frac{2q(x)\left\{p(x)-\sqrt{\left(p(x)\right)^2-4q(x)}\right\}}{\left(p(x)\right)^2-\left(p(x)\right)^2+4q(x)}\\&=\frac{2q(x)\left\{p(x)-\sqrt{\left(p(x)\right)^2-4q(x)}\right\}}{4q(x)}\end{aligned}$$

$$\therefore\quad \frac{q(x)}{D_1(x)}=\frac{1}{2}\left\{p(x)-\sqrt{p-4q(x)}\right\} \tag{37}$$

By similar procedure, we can equally have

$$\frac{q(x)}{D_2(x)} = \frac{1}{2}\left\{ p(x) + \sqrt{\left(p(x)\right)^2 - 4q(x)} \right\} \tag{38}$$

Now from Equation (34)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-q(x)}{D(x)} y$$

$$\rightarrow \quad \frac{\mathrm{d}y}{y} = \frac{-q(x)}{D(x)} \mathrm{d}x$$

Integrating both sides gives us

$$\rightarrow \quad \ln y = -\int \frac{q(x)}{D(x)} \mathrm{d}x$$

$$\therefore \quad y = \mathrm{e}^{-\int \frac{q(x)}{D(x)} \mathrm{d}x} \tag{39}$$

Where

$$\frac{q(x)}{D(x)} = \frac{1}{2}\left\{ p(x) \pm \sqrt{\left(p(x)\right)^2 - 4q(x)} \right\}$$

By substitution therefore,

$$y = \mathrm{e}^{-\int \frac{1}{2}\left\{ p(x) \pm \sqrt{\left(p(x)\right)^2 - 4q(x)} \right\} \mathrm{d}x}$$

$$\therefore \quad y = \mathrm{e}^{-\frac{1}{2}\int \left\{ p(x) \pm \sqrt{\left(p(x)\right)^2 - 4q(x)} \right\} \mathrm{d}x}$$

Let

$$T(x) = p(x) \pm \sqrt{\left(p(x)\right)^2 - 4q(x)}$$

Then

$$y = \mathrm{e}^{-\frac{1}{2}\int T(x)\mathrm{d}x} \tag{40}$$

Equation (40) may be an appropriate solution to the general Equation (33) it can be seen then that Equation (40) makes it possible for the solution to the general equation of the homogenous linear differential equation of order 2 to come in terms of an integral function.

Another way we can obtain a solution for Equation (33) is by factoring out y from (33)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x) y = 0$$

$$\rightarrow \quad \left\{ \frac{\mathrm{d}^2}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x) \right\} y = 0$$

$$\rightarrow \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x) = 0$$

Let

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} = D(x) = D$$

Then by substitution,

$$D^2 + p(x)D + q(x) = 0 \tag{41}$$

$$\therefore \quad D = \frac{-p(x) \pm \sqrt{(p(x))^2 - 4q(x)}}{2}$$

$$D = -\frac{1}{2}\left\{p(x) \pm \sqrt{(p(x))^2 - 4q(x)}\right\} \tag{42}$$

But

$$\rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x} = D$$

$$\rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = Dy$$

$$\rightarrow \quad \frac{\mathrm{d}y}{y} = D\mathrm{d}x$$

$$\rightarrow \quad \ln y = \int D\mathrm{d}x$$

$$\therefore \quad y = \mathrm{e}^{\int D\mathrm{d}x} \tag{43}$$

Substituting (42) in (43) gives

$$y = \mathrm{e}^{\int -\frac{1}{2}\left\{p(x) \pm \sqrt{(p(x))^2 - 4q(x)}\right\}\mathrm{d}x}$$

$$\therefore \quad y = \mathrm{e}^{-\frac{1}{2}\int\left\{p(x) \pm \sqrt{(p(x))^2 - 4q(x)}\right\}\mathrm{d}x}$$

Let

$$T(x) = p(x) \pm \sqrt{(p(x))^2 - 4q(x)}$$

Then by substitution

$$y = \mathrm{e}^{-\frac{1}{2}\int T(x)\mathrm{d}x} \tag{40}$$

The Equation (40) works well if p and q are constants. It also works well if p and q are not constants in the Euler's equation as well as the Equation (8). It can seen that for other homogenous equations with variable co-efficient, the workability of the Equation (40) depends on whether or not the function $p(x)$ and the discriminant function $\sqrt{(p(x))^2 - 4q(x)}$ are both integrable since $p(x)$ may always be integrable, it is safe to say that if the discriminant function $L(x) = \sqrt{(p(x))^2 - 4q(x)}$ is integrable, then the solution in Equation (40) is an elementary function. If not then Equation (40) is not an elementary function. The case where the function

$L(x) = \sqrt{(p(x))^2 - 4q(x)}$ is not easily integrable leads us to think of possible power series solution. For simplicity, let $p(x) = p$ and $q(x) = q$, then

$$\begin{aligned}
L(x) &= \sqrt{p^2-4q} \\
&= \left\{p^2\left(1-\frac{4q}{p^2}\right)\right\}^{\frac{1}{2}} \\
&= p^{2\frac{1}{2}}\left(1-\frac{4q}{p^2}\right)^{\frac{1}{2}} \\
&= p\left(1-\frac{4q}{p^2}\right)^{\frac{1}{2}} \\
&= p\left\{1+\frac{1}{2}\left(\frac{-4q}{p^2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-4q}{p^2}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{-4q}{p^2}\right)^3+\cdots\right\} \\
&= p\left\{1-\frac{2q}{p^2}+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}\left(\frac{16q^2}{p^4}\right)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!}\left(\frac{-4^3q^3}{p^6}\right)+\cdots\right\} \\
&= p\left\{1-\frac{2q}{p^2}-\frac{1}{2^3}\times\frac{2^4q^2}{p^4}-\frac{1}{3\times 2}\times\frac{3}{2^3}\times\frac{4^3q^3}{p^6}-\cdots\right\} \\
&= p\left\{1-\frac{2q}{p^2}-\frac{2q^2}{p^4}-\frac{4q^3}{p^6}-\cdots\right\} \\
&= p-\frac{2q}{p}-\frac{2q^2}{p^3}-\frac{4q^3}{p^5}-\cdots \\
&= p-\frac{2q}{p}\left(1+\frac{q}{p^2}+\frac{2q^2}{p^4}+\cdots\right)
\end{aligned}$$

But the series in the bracket converges to 1 if $\left|\frac{q}{p^2}\right|<1$

$$\begin{aligned}
\therefore\ L(x) &\cong p-\frac{2q}{p} = p(x)-\frac{2q(x)}{p(x)} \\
&= \frac{\{p(x)\}^2-2q(x)}{p(x)}
\end{aligned}$$

Recall that

$$T(x)=p(x)\pm\sqrt{(p(x))^2-4q(x)}$$

where

$$L(x)=\sqrt{(p(x))^2-4q(x)}$$

$$\begin{aligned}
\rightarrow\ T(x) &= p(x)\pm\left(p(x)-\frac{2q(x)}{p(x)}\right) \\
&= p(x)\pm\frac{(p(x))^2-2q(x)}{p(x)} \\
&= \frac{(p(x))^2\pm\left\{(p(x))^2-2q(x)\right\}}{p(x)}
\end{aligned}$$

$$\rightarrow \int T(x)\mathrm{d}x = \int \left\{ \frac{(p(x))^2 \pm (p(x))^2 - 2q(x)}{p(x)} \right\} \mathrm{d}x$$

$$\rightarrow \frac{1}{2} \int \left\{ \frac{(p(x))^2 \pm (p(x))^2 - 2q(x)}{p(x)} \right\} \mathrm{d}x$$

Let

$$F(x) = -\frac{1}{2} \int T(x)\mathrm{d}x$$

Then

$$F(x) = \frac{-1}{2} \int \left\{ \frac{(p(x))^2 \pm \left[(p(x))^2 - 2q(x) \right]}{p(x)} \right\} \mathrm{d}x \tag{44}$$

This implies that

$$y = \mathrm{e}^{F(x)} \tag{45}$$

From (45)

$$y' = F(x)\mathrm{e}^{F(x)}$$

$$y'' = F'(x)\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{F(x)} + \mathrm{e}^{F(x)}\frac{\mathrm{d}}{\mathrm{d}x}F'(x)$$

$$= F'(x)F'(x)\mathrm{e}^{F(x)} + F''(x)\mathrm{e}^{F(x)}$$

$$y''' = (F'(x))^2 F'(x)\mathrm{e}^{F(x)} + \mathrm{e}^{F(x)} 2(F'(x))F''(x)$$

$$+ F''(x)F'(x)\mathrm{e}^{F(x)} + \mathrm{e}^{F(x)}$$

$$y(x) = \mathrm{e}^{F(x)}$$

$$y'(x_0) = F'(x_0)\mathrm{e}^{F(x)}$$

$$y''(x_0) = (F'(x_0))^2 \mathrm{e}^{F(x_0)} + F''(x_0)\mathrm{e}^{F(x_0)}$$

$$y'''(x_0) = (F'(x_0))^3 \mathrm{e}^{F(x_0)} + 3F'(x_0)F''(x_0)\mathrm{e}^{F(x_0)} + F'''(x_0)\mathrm{e}^{F(x_0)}$$

Thus, the Taylor series expansion of y at an ordinary point is given by

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2$$

$$+ \frac{y'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{y^{(n)}}{n!}(x - x_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{y^{(n)}}{n!}(x - x_0)^n$$

$$y(x) = \sum_{n=0}^{\infty} C_n (x - x_0)^n \tag{46}$$

Equation (46) is therefore the general solution to equation (33)

4. Results and Discussion

The first objective of the study was to determine analytically the Equation (4) as the solution to the Equation (5) which is a homogenous linear differential equation with constant coefficients p and q. This was done through two different methods both of which yielded the same result $y = e^{r}$. Method 1 actually does something fascinating in a way it enables us to have a deeper view of the concept of characteristic equation. It enable us to understand that the r in the characteristic Equation (16) is actually operator D where $D = \frac{d}{dx} + p$. Method 2 does the same thing too but in this case the operator $D = \frac{d}{dx}$.

The second objective of the study was to demonstrate that Equation (7) can be proven deductively to be the solution to Equation (6). By this, we mean that $y = x^{r}$ is the solution to the Euler equation $x^2 y'' + pxy' + qy = 0$. This was equally achieved through two different methods and the procedure is similar to the procedure in the first case.

In a very interesting way again, the third objective of this study was achieved by proving deductively that Equation (8) has a solution of the form in equation (28). By this we mean that $y = e^{mx^2}$ is the solution to the equation $y'' + pxy' + qx^2 y = 0$. Notice that if this Equation (8) *i.e.* $y = e^{mx^2}$ is substituted into Equation (28) above, the resulting characteristic equation has two parts. One part is constant while the other part isobtained by the characteristic equation. $4m^2 + 2pm + q = 0$ the constant part is obtained by the equation $2m = 0$ thus making the complementary solution $y(x)$ to be of the form $y(x) = C_0 + C_1 e^{m_1 x^2} + C_2 e^{m_2 x^2}$. This makes the solution to appear like there are three independent solutions, whereas this is not the case. So what then does the constant stand for? It does in fact seem like there is more to be understood. To understand this, let us take advantage of the fact that the solutions are independent. Then we have to take

$$y_1 = C_1 e^{m_1 x^2}$$

so that

$$y_1' = C_1 e^{m_1 x^2} \frac{d}{dx}\left(m_1 x^2\right)$$
$$= 2C_1 m_1 x e^{m_1 x^2}$$

And

$$y_1'' = 2C_1 m_1 x \frac{d}{dx}\left(m_1 x^2\right) + e^{m_1 x^2} \frac{d}{dx}\left(2C_1 m x\right)$$
$$= 4C_1 m_1^2 x^2 e^{m_1 x^2} + 2C_1 m_1 e^{m_1 x^2}$$

Substituting the above into the left hand side of the original equation

$$y_1'' + pxy_1' + qx^2 y_1 = 0$$
$$\to \quad 4C_1 m_1^2 x^2 e^{m_1 x^2} + 2C_1 m_1 e^{m_1 x^2} + px \cdot 2C_1 m_1 e^{m_1 x^2} + qx^2 \cdot C_1 m_1 e^{m_1 x^2} = 0$$
$$\to \quad \left(4m_1^2 + 2pm_1 + 1\right) C_1 x^2 e^{m_1 x^2} + 2C_1 m_1 e^{m_1 x^2} = 0$$

We understand from here that the expression in the bracket $C_1 x^2 e^{m_1 x^2} \neq 0$.

However, $2C_1m_1 \neq 0$ and $e^{m_1x^2} \neq 0$, $\rightarrow 2C_1m_1e^{m_1x^2} \neq 0$ If this must be taken care of, then we must add an opposite function to make the equation balance. This opposite must as well be non-differentiable, which means that it is constant in a way. This non-differentiable function appears as part of the constant in the solution. There since $2C_1m_1e^{m_1x^2} - 2C_1m_1e^{m_1x^2} = 0$ if follows that $-2C_1m_1e^{m_1x^2}$ is a non-differentiable independent function, this actually represent the non-differentiable part of the same function (8) which serves as the solution to Equation (8) *i.e.* Equation (28). This is largely accurate because such functions in the form $f(n) = e^{m_1x^2}$ usually have both differentiable and non-differentiable regions. The diagram below illustrates this.

From the graphs in **Figure 1** and **Figure 2** above we can see that $y(x)$ has two parts, one part being differentiable while the other is non-differentiable *i.e.*

$$y(x) = y(x)_{\text{diff}} + y(x)_{\text{non-diff}}$$

where

$$y(x)_{\text{diff}} = C_1e^{m_1x^2} + C_2e^{m_2x^2}$$

And

$$y(x)_{\text{non-diff}} = -2C_1m_1e^{m_1x^2} - 2C_2m_2e^{m_2x^2}$$

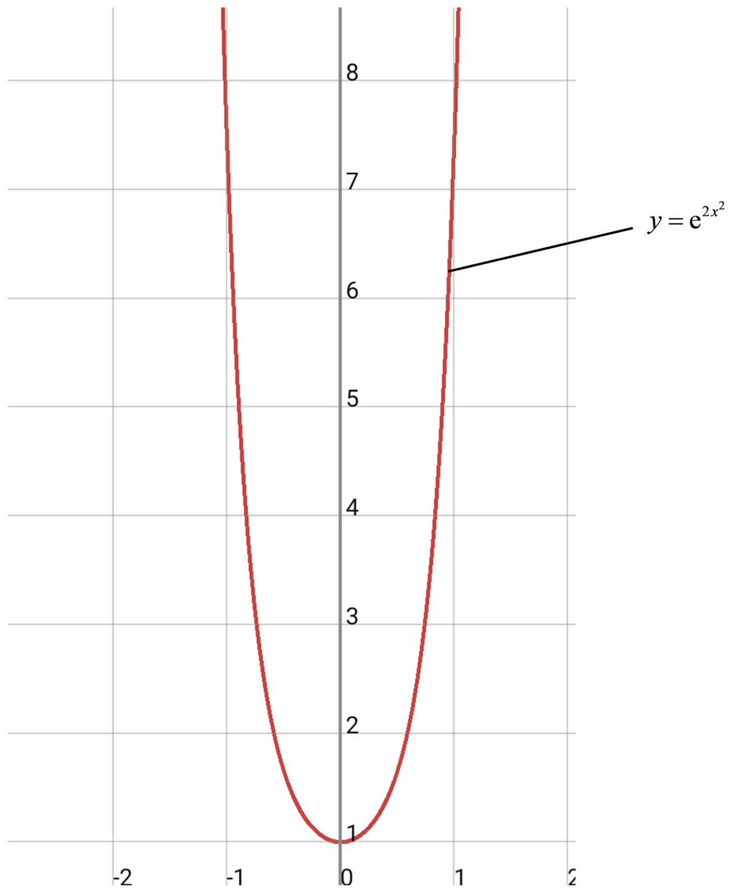

Figure 1. Graph of $y = e^{2x^2}$

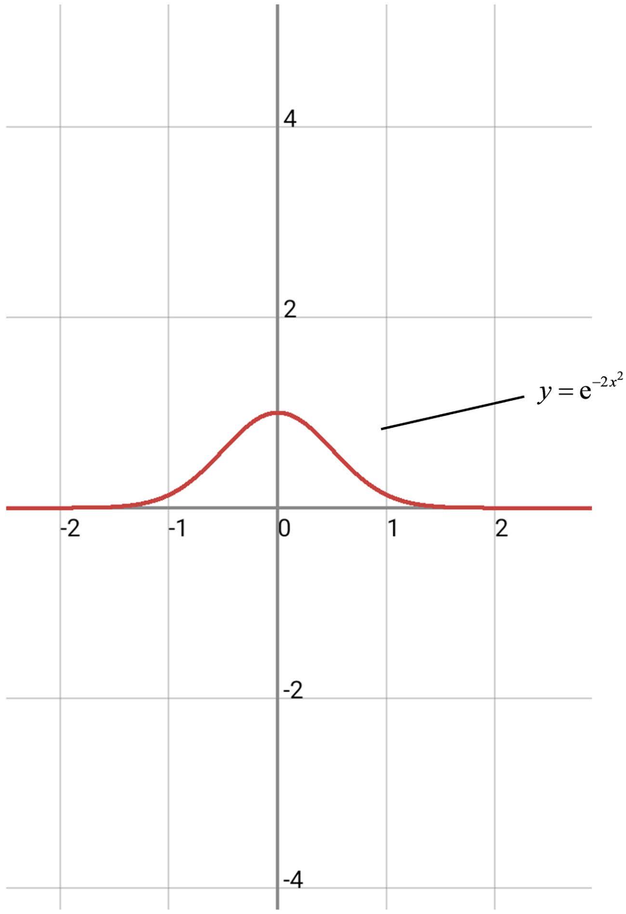

Figure 2. Graph of $y = e^{-2x^2}$

Thus for every function of the form $y = Ce^{mx^2}$, the non-differentiable region is $y = 2Cme^{mx^2}$. From the foregoing the solution to Equation (8) must therefore be of the form

$$y(x) = \left\{C_1 e^{m_1 x^2} + C_2 e^{m_2 x^2}\right\} - \left\{2C_1 m_1 e^{m_1 x^2} + 2C_2 m_2 e^{m_2 x^2}\right\}$$

Note that from the graph in **Figure 1**, $y(x)$ is differentiable within the region $-a \le x \le a$, while it is non-differentiable outside this region since $y(x)$ is either 0 constantly or it has constant values at $x = -a$ and $x = a$. At these points, the function either has a derivative of 0 or its derivative is indeterminate, hence, it is non-differentiable.

The fourth objective of this paper was to demonstrate that Equation (10) is not just an ansatz to Equation (9) *i.e.* $y = \sum_{n=0}^{\infty} C_n (x - n_0)^n$ is not just a guessed solution, the general second order linear homogenous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

From this we learn that the function in Equation (40) is actually a good solution to the Equation (9) *i.e.* $y = e^{-\frac{1}{2}\int T(x)dx}$

where $T(x) = p(x) \pm \sqrt{(p(x))^2 - 4q(x)}$ is a solution to (9) if and only if the discriminant function $L(x)=\sqrt{\left(p(x)\right)^2-4q(x)}$ is integrable. In such a case the Equation (9) will have a solution in terms of elementary functions. However, if $L(x)$ is not integrable, then $L(x)$ must be converted to a binomial power series which converges to $L(x)=p\frac{-2q}{p}$ as thus leading to the function

$$F(x)=\frac{-1}{2}\int\left\{\frac{\left(p(x)\right)^2-\left(p(x)\right)^2-2q(x)}{p(x)}\right\}\mathrm{d}x$$

So that $y=\mathrm{e}^{F(x)}$ in this case the Taylor series expansion for this function becomes

$$y=\sum_{n=0}^{\infty}\frac{Y^{(n)}(x_0)(x-x_0)^2}{n!}$$

Which reduces to $y=\sum_{n=0}^{\infty}C_n(x-x_0)^n$

where $c_n=\frac{Y^{(n)}(x)}{n!}$

5. Conclusions

From the results obtained so far, we now draw the following conclusions:

1) The Euler's exponential function $y=\mathrm{e}^{ax}$ is indeed a deductive solution to the linear homogenous differential equation $y''+py'+qy=0$.
where p and $q=0$

2) The Euler's equation $x^2y''+xpy'+qy=0$ has the function $y=x^r$ as its analytic solution and not as an ansatz.

3) The equation $y''+npy'+x^2qy=0$ has the function $y=\mathrm{e}^{mx^2}$ as its solution such that the function has a differentiable and non-differentiable part.

4) The function $y=\sum_{n=0}^{\infty}C_n(x-a)^n$ is indeed the solution to the general differential equation with variable coefficients $y''+p(x)y'+q(x)y=0$. This is not a guessed solution but an analytically derived solution to the equation above.

Conflicts of Interest

The author declares no conflicts of interest.

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