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Generalized Hyers-Ulam-Rassisa Type Stability of a Cauchy Additive (ξ_1,ξ_2)-Functional Inequalities with $3k$-Variables in Complex Banach Space

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Abstract

In this paper, we study to solve two additive (ξ_1,ξ_2)-functional inequalities with $3k$-variables and their Hyers-Ulam stability: First are investigated in complex Banach spaces with a fixed point method and last are investigated in complex Banach spaces with a direct method: These are the main results of this paper.

Subject Areas

Mathematics

Keywords

Additive (ξ_1,ξ_2)-Functional Inequality, Fixed Point Method, Direct Method, Banach Space, Hyers-Ulam Stability

Mathematics Subject Classification

46S10, 39B62, 39B52, 47H10

1. Introduction

Let $\mathbb{X}$ and $\mathbf{Y}$ be normed spaces on the same field $\mathbb{K}$, and $f:\mathbb{X}\to\mathbf{Y}$. We use the notation $\|\cdot\|$ for all the norms on both $\mathbb{X}$ and $\mathbf{Y}$. In this paper, we investisgate additive (ξ_1,ξ_2)-functional inequalitíes when $\mathbf{X}$ be a real or complex normed space and $\mathbf{Y}$ a complex Banach spaces. We solve and prove the Hyers-Ulam-Rassisa type stability of following Cauchy additive (ξ_1,ξ_2)-functional inequalities.

$$\begin{aligned}&\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\ &\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\ &\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{1}$$

and when we change the role of the function inequality (1), we continue to prove the following function inequality

$$\begin{aligned}&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\ &\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\ &\quad+\left\|\xi_2\left(2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{2}$$

So (1) and (2) are equivalent propositions, which ξ_1,ξ_2 are fixed nonzero complex numbers with $\mathbf{G}(\xi_1,\xi_2)$-functional inequality. Note that in the preliminaries, we just recap some of the essential properties for the above problem and for the specific problem, please see the document. The Hyers-Ulam stability was first investigated for the functional equation of Ulam in [1] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The Hyers [2] gave the first affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers' Theorem was generalized by Aoki [3] additive mappings and by Rassias [4] for linear mappings considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference with a general control function in the spirit of Rassias' approach.

The stability of the quadratic functional equation was proved by Skof [6] for mappings $f:\mathbf{X}\to\mathbb{Y}$, where $\mathbf{X}$ is a normed space and $\mathbf{Y}$ is a Banach space. Park [7] [8] defined additive γ-functional inequalities and proved the Hyers-Ulam stability of the additive γ-functional inequalities in Banach spaces and non-archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors on the world even term [4]-[29]. We recall a fundamental result in fixed point theory. The authors studied the Hyers-Ulam stability for the following functional inequalities

$$\left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\|\le\left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-\frac{1}{2}f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(z)\right\| \quad (3)$$

$$\left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-\frac{1}{2}f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(z)\right\|\le\left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\| \quad (4)$$

$$\|f(x+y)-f(x)-f(y)\|\le\left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \quad (5)$$

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\le\|\rho(f(x+y)-f(x)-f(y))\| \quad (6)$$

and

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f(z)\right\|\\&\le\left\|\beta\left(2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f(z)\right)\right\|\end{aligned} \quad (7)$$

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f(z)\right\|\\&\le\left\|\beta\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f(z)\right)\right\|\end{aligned} \quad (8)$$

finaly

$$\begin{aligned}&\|f(x+y)-f(x)-f(y)\|\\&\le\|\beta_1(f(x+y)+f(x-y)-2f(x))\|+\left\|\beta_2\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|\end{aligned} \quad (9)$$

next

$$\begin{aligned}&\|f(x_1+x_2+\cdots+x_n)-f(x_1)-f(x_2+\cdots+x_n)\|_{\mathbb{Y}}\\&\le\|\beta_1(f(x_1+x_2+\cdots+x_n)-f(x_1-x_2-\cdots-x_n)-2f(x_1))\|_{\mathbb{Y}}\\&\quad+\left\|\beta_2\left(2f\left(\frac{x_1+x_2+\cdots+x_n}{2}\right)-f(x_1)-f(x_2+\cdots+x_n)\right)\right\|_{\mathbb{Y}}\end{aligned} \quad (10)$$

final

$$\begin{aligned}&\left\|2f\left(\frac{x_1+x_2}{2}+\frac{x_3+x_4+\cdots+x_k}{4}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\cdots+x_k}{2}\right)\right\|_{\mathbf{Y}}\\&\le\left\|\beta_1\left(f\left(x_1+x_2+\frac{x_3+x_4+\cdots+x_k}{2}\right)+f\left(x_1-x_2-\frac{x_3+x_4+\cdots+x_k}{2}\right)-2f(x_1)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\beta_2\left(f\left(x_1+x_2+\frac{x_3+x_4+\cdots+x_k}{2}\right)-f(x_1)-f\left(x_2+\frac{x_3+x_4+\cdots+x_k}{2}\right)\right)\right\|_{\mathbf{Y}}\end{aligned} \quad (11)$$

in complex Banach spaces.

In this paper, we solve and prove the Hyers-Ulam stability for (ξ_1,ξ_2)-functional inequalities (1) and (2), *i.e.*, the (ξ_1,ξ_2)-functional inequalities with n-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfy the (ξ_1,ξ_2)-functional inequalities (1) and (2). Thus, the results in this paper are the generalization of those in [13] [14] [15] [16] [26] [27] [28] [29] for (ξ_1,ξ_2)-functional inequalities with n-variables.

The goal of the paper is to develop functional inequalities with a higher number of variables to solve problems of general nonlinear functional equations in order to develop the field of nonlinear analysis.

The paper is organized as follows: In the section preliminaries, we remind some basic notations in [13] [14] [17], such as complete generalized metric space and Solutions of the inequalities.

Section 3: In this section, I use the method of the fixed to prove the Hyers-Ulam stability of the additive (ξ_1,ξ_2)-functional inequalities (1) when $\mathbb{X}$ be a real or complete normed space and $\mathbb{Y}$ complex Banach space.

Section 4: In this section, I use the method of directly determining the solution for (1) when $\mathbf{X}$ be a real or complete normed space and $\mathbf{Y}$ complex Banach space.

Section 5: In this section, I use the method of the fixed to prove the Hyers-Ulam stability of the additive (ξ_1,ξ_2)-functional inequalities (2) when $\mathbf{X}$ be a real or complete normed space and $\mathbf{Y}$ complex Banach space.

Section 6: In this section, I use the method of directly determining the solution for (2) when $\mathbf{X}$ be a real or complete normed space and $\mathbf{Y}$ complex Banach space.

2. Preliminaries

2.1. Complete Generalized Metric Space and Solutions of the Inequalities

Theorem 1. Let $(\mathbb{X},d)$ be a complete generalized metric space and let $J:\mathbb{X}\to\mathbb{X}$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in\mathbb{X}$, either

$$d\left(J^n,J^{n+1}\right)=\infty$$

for all nonegative integers n or there exists a positive integer n_0 such that

1). $d\left(J^n,J^{n+1}\right)<\infty$, $\forall n\ge n_0$;

2). The sequence $\left\{J^n x\right\}$ converges to a fixed point y^* of J;

3). y^* is the unique fixed point of J in the set $Y=\left\{y\in\mathbb{X}\mid d\left(J^n,J^{n+1}\right)<\infty\right\}$;

4). $d\left(y,y^*\right)\le\dfrac{1}{1-l}d\left(y,Jy\right)\quad\forall y\in\mathbb{Y}$.

2.2. Solutions of the Inequalities

The functional equation

$$f\left(x+y\right)=f\left(x\right)+f\left(y\right)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

3. Establish the Solution of the Additive (ξ_1,ξ_2)-Function Inequalities Using a Fixed Point Method

Now, we first study the solutions of (1). Note that for these inequalities, when $\mathbf{X}$ be a real or complete normed space and $\mathbf{Y}$ complex Banach space.

Lemma 2. Suppose mapping $\Gamma:\mathbf{X}\to\mathbf{Y}$ satisfies $\Gamma(0)=0$ and

$$\begin{aligned}&\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{12}$$

for all $x_j,y_j,z_j\in\mathbf{X},\forall j=1\to k$, then $\Gamma:\mathbf{X}\to\mathbf{Y}$ is Cauchy additive

Proof. Assume that $\Gamma:\mathbf{X}\to\mathbf{Y}$ satisfies (12)

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,0,x,\cdots,0,0,\cdots,0)$ in (12) we have

$$\left\|2\Gamma\left(\frac{x}{2}\right)-\Gamma(x)\right\|_{\mathbf{Y}}\le 0$$

Thus

$$\Gamma\left(\frac{x}{2}\right)=\frac{1}{2}f(x)\tag{13}$$

for all $x\in\mathbf{X}$.

It follows from (12) and (13) that

$$\begin{aligned}&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&=\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{14}$$

and so

$$\begin{aligned}&(1-|\xi_2|)\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le|\xi_1|\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{15}$$

we let $u=\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j, v=\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j$, for all $j=1\to k$, we get

$$\begin{aligned}&\left(1-|\xi_2|\right)\left\|\Gamma(u)-f\left(\frac{u+v}{2}\right)-\Gamma\left(\frac{u-v}{2}\right)\right\|_{\mathbf{Y}}\\&\le|\xi_1|\left\|\Gamma(u)+\Gamma(v)-2\Gamma\left(\frac{u+v}{2}\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{16}$$

for all $u,v\in\mathbf{X}$

and so

$$\begin{aligned}&\frac{1}{2}\left(1-|\xi_2|\right)\left\|\Gamma(u+v)+\Gamma(u-v)-2\Gamma(u)\right\|_{\mathbf{Y}}\\&\le|\xi_1|\left\|\Gamma(u+v)-f(u)-\Gamma(v)\right\|_{\mathbf{Y}}\end{aligned}\tag{17}$$

for all $u,v\in\mathbf{X}$. It follows from (15) and (17) that

$$\begin{aligned}&\frac{1}{2}\left(1-|\xi_2|\right)^2\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-f\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le|\xi_1|^2\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{18}$$

Since $\sqrt{2}|\xi_1|+|\xi_2|<1$

and so

$$\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)=\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)+\Gamma\left(\sum_{j=1}^{k}z_j\right)$$

for all $x_j,y_j,z_j\in\mathbf{X},\forall j=1\to k$. Thus Γ is Cauchy additive. □

Theorem 3. Suppose $\varphi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that there exists an $L<1$ with

$$\varphi\left(\frac{x_1}{2},\cdots,\frac{x_k}{2},\frac{y_1}{2},\cdots,\frac{y_k}{2},\frac{z_1}{2},\cdots,\frac{z_k}{2}\right)\le\frac{L}{2}\varphi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)\tag{19}$$

for all $x_j,y_j,z_j\in\mathbf{X},\forall j=1\to k$. If $\Gamma:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $\Gamma(0)=0$ and

$$\begin{aligned}&\left\|2\Gamma\left(\sum_{j=1}^{n}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\varphi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)\end{aligned}\tag{20}$$

for all $x_j,y_j,z_j\in\mathbf{X},\forall j=1\to k$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|\Gamma(x)-\psi(x)\right\|_{\mathbf{Y}}\le\frac{1}{1-L}\varphi\left(x,\cdots,0,x,\cdots,0,0,\cdots,0\right)\tag{21}$$

for all $x\in\mathbf{X}$.

Proof. Replacing $(x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k,z_1,z_2,\cdots,z_k)$ by $(x,\cdots,0,x,\cdots,0,0,\cdots,0)$ in (20), we get

$$\left\|2\Gamma\left(\frac{x}{2}\right)-\Gamma(x)\right\|_{\mathbf{Y}} \le \varphi(x,0,\cdots,0,x,\cdots,0,0,\cdots,0) \tag{22}$$

for all $x\in\mathbb{X}$.

Consider the set

$$\mathbb{S}:=\{h:\mathbf{X}\to\mathbf{Y},h(0)=0\}$$

and introduce the generalized metric on $\mathbb{S}$:

$$d(g,h):=inf\left\{\lambda\in\mathbb{R}:\|g(x)-h(x)\|\le\lambda\varphi(x,0,\cdots,0,x,\cdots,0,0,\cdots,0),\forall x\in\mathbf{X}\right\},$$

where, as usual, $inf\,\phi=+\infty$. It easy to show that $(\mathbb{S},d)$ is complete [17] Now we cosider the linear mapping $J:\mathbb{S}\to\mathbb{S}$ such that

$$Jg(x):=2g\left(\frac{x}{2}\right)$$

for all $x\in\mathbf{X}$. Let $g,h\in\mathbb{S}$ be given such that $d(g,h)=\varepsilon$. Then

$$\|g(x)-h(x)\|\le\varepsilon\varphi(x,0,\cdots,0,x,\cdots,0,0,\cdots,0)$$

for all $x\in\mathbf{X}$.

Hence

$$\begin{aligned}\|Jg(x)-Jh(x)\|&=\left\|2g\left(\frac{x}{2}\right)-2h\Gamma\left(\frac{x}{2}\right)\right\|\\&\le 2\varepsilon\varphi\left(\frac{x}{2},0,\cdots,0,\frac{x}{2},0,\cdots,0,0,\cdots,0\right)\\&\le 2\varepsilon\frac{L}{2}\varphi(x,0,\cdots,0,x,\cdots,0,0,\cdots,0)\\&\le L\varepsilon\varphi(x,0,\cdots,0,x,\cdots,0,0,\cdots,0)\end{aligned}$$

for all $x\in\mathbf{X}$. So $d(g,h)=\varepsilon$ implies that $d(Jg,Jh)\le L\cdot\varepsilon$. This means that

$$d(Jg,Jh)\le Ld(g,h).$$

for all $g,h\in\mathbb{S}$ It folows from (22) that

$$d(\Gamma,J\Gamma)\le 1.$$

By Theorem 2.1, there exists a mapping $\psi:\mathbf{X}\to\mathbf{Y}$ satisfying the fllowing:

1) ψ is a fixed point of *J*, *i.e.*,

$$\psi(x)=2\psi\left(\frac{x}{2}\right) \tag{23}$$

for all $x\in\mathbf{X}$. The mapping ψ is a unique fixed point *J* in the set

$$\mathbb{M}=\{g\in\mathbb{S}:d(\Gamma,g)<\infty\}$$

This implies that ψ is a unique mapping satisfying (23) such that there exists a $\lambda\in(0,\infty)$ satisfying

$$\left\|\Gamma(x)-\psi(x)\right\|\le\lambda\varphi(x,0,\cdots,0,x,\cdots,0,0,\cdots,0)$$

for all $x\in\mathbf{X}$

2) $d\left(J^{l}f,\psi\right)\to 0$ as $l\to\infty$. This implies equality

$$\lim_{l\to\infty}2^{n}f\left(\frac{x}{2^{n}}\right)=\psi(x)$$

for all $x\in\mathbb{X}$

3) $d(\Gamma,\psi)\le\frac{1}{1-L}d(\Gamma,J\Gamma)$. which implies

$$\left\|\Gamma(x)-\psi(x)\right\|\le\frac{1}{1-L}\varphi(x,0,\cdots,0,x,\cdots,0,0,\cdots,0)$$

for all $x\in X$. It follows (19) and (20) that

$$\begin{aligned}
&\left\|2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\
&=\lim_{n\to\infty}2^{n}\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+2}}+\frac{1}{2^{n+1}}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}\right)-\Gamma\left(\frac{1}{2^{n}}\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\
&\le\lim_{n\to\infty}2^{n}\left|\xi_1\right|\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}+\frac{1}{2^{n}}\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}-\frac{1}{2^{n}}\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}\right)\right\|_{\mathbf{Y}}\\
&\quad+\lim_{n\to\infty}2^{n}\left|\xi_2\right|\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}+\frac{1}{2^{n}}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}\right)-\Gamma\left(\frac{1}{2^{n}}\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\
&\quad+\lim_{n\to\infty}2^{n}\varphi\left(\frac{x_1}{2^{n}},\cdots,\frac{x_n}{2^{n}},\frac{y_1}{2^{n}},\cdots,\frac{y_n}{2^{n}},\frac{z_1}{2^{n}},\cdots,\frac{z_n}{2^{n}}\right)\\
&\quad=\left\|\xi_1\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\
&\quad+\left\|\xi_2\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}
\end{aligned}\tag{24}$$

for all $x_j,y_j,z_j\in\mathbf{X},\ j=1\to k$. So

$$\begin{aligned}
&\left\|2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\
&\le\left\|\beta_1\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\
&\quad+\left\|\beta_2\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\
&\quad+\varphi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)
\end{aligned}$$

for all $x_j,y_j,z_j\in\mathbf{X},\ j=1\to k$. By *Lemma* 3.1, the mapping $\psi:\mathbf{X}\to\mathbf{Y}$ is additive. Ei

$$\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)=0$$

□

Theorem 4. Let $\varphi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that there exists an $L<1$ with

$$\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\le 2L\varphi\left(\frac{x_1}{2},\cdots,\frac{x_k}{2},\frac{y_1}{2},\cdots,\frac{y_k}{2},\frac{z_1}{2},\cdots,\frac{z_k}{2}\right)\tag{25}$$

for all $x,y,z\in\mathbb{X}$. Let $\Gamma:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $\Gamma(0)=0$ and

$$\begin{aligned}&\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned}\tag{26}$$

for all $x_j,y_j,z_j\in\mathbf{X},\ j=1\to k$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\|\Gamma(x)-\psi(x)\|_{\mathbf{Y}}\le\frac{L}{1-L}\varphi(x,\cdots,0,x,\cdots,0,0,\cdots,0)\tag{27}$$

for all $x\in\mathbf{X}$.

The rest of the proof is similar to the proof of Theorem 3.

From proving the theorems we have consequences:

Corollary 1. Let $r>1$ and θ be nonnegative real numbers and let $\Gamma:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $\Gamma(0)=0$ and

$$\begin{aligned}&\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)\end{aligned}\tag{28}$$

for all $x_j,y_j,z_j\in\mathbf{X},\forall j=1\to k$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\|f(x)-\psi(x)\|_{\mathbb{Y}}\le\frac{2.2^r\theta}{2^r-2}\|x\|_{\mathbf{X}}^r\tag{29}$$

for all $x\in\mathbf{X}$.

Corollary 2. Let $r<1$ and θ be nonnegative real numbers and let $\Gamma:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $\Gamma(0)=0$ and

$$\begin{aligned}&\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)\end{aligned}\tag{30}$$

for all $x_j, y_j, z_j \in \mathbf{X}, \forall j \to k$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\|\Gamma(x)-\psi(x)\|_{\mathbf{Y}}\le\frac{2.2^r\theta}{2-2^r}\|x\|_{\mathbf{X}}^r\tag{31}$$

for all $x\in\mathbf{X}$.

4. Establish the Solution of the Additive (ξ_1,ξ_2)-Function Inequalities Using a Direect Method

Next, we study the solutions of (1). Note that for these inequalities, when $\mathbf{X}$ be a real or complete normed space and $\mathbf{Y}$ complex Banach space.

Theorem 5. Suppose $\varphi:\mathbf{X}^{3k}\to[0,\infty)$ be a function such that

$$\begin{aligned}&\phi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\\&:=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x_1}{2^j},\cdots,\frac{x_k}{2^j},\frac{y_1}{2^j},\cdots,\frac{y_k}{2^j},\frac{z_1}{2^j},\cdots,\frac{z_k}{2^j}\right)<\infty\end{aligned}\tag{32}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j=1\to k$ and let $\Gamma:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfies $\Gamma(0)=0$ and

$$\begin{aligned}&\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned}\tag{33}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j=1\to k$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\|\Gamma(x)-\psi(x)\|_{\mathbb{Y}}\le\phi(x,\cdots,0,x,\cdots,0,0,\cdots,0)\tag{34}$$

for all $x\in\mathbf{X}$

Proof. Replacing $(x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k,z_1,z_2,\cdots,z_k)$ by $(x,\cdots,0,x,\cdots,0,0,\cdots,0)$ in (33), we get

$$\left\|2\Gamma\left(\frac{x}{2}\right)-\Gamma(x)\right\|_{\mathbf{Y}}\le\varphi(x,0,\cdots,0,x,\cdots,0,0,\cdots,0)\tag{35}$$

for all $x \in \mathbf{X}$.

Hence

$$\begin{aligned}&\left\|2^l\Gamma\left(\frac{x}{2^l}\right)-2^m\Gamma\left(\frac{x}{2^m}\right)\right\|_{\mathbf{Y}}\\ &\le \sum_{j=l}^{m-1}\left\|2^j\Gamma\left(\frac{x}{2^j}\right)-2^{j+1}\Gamma\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathbf{Y}}\\ &\le \sum_{j=l}^{m-1}2^j\varphi\left(\frac{x}{2^{j+1}},0,\cdots,0,\frac{x}{2^{j+1}},0,\cdots,0,0,\cdots,0\right)\end{aligned} \tag{36}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{X}$. It follows from (36) that the sequence $\left\{2^n\Gamma\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete, the sequence $\left\{2^n\Gamma\left(\frac{x}{2^n}\right)\right\}$ coverges. So one can define the mapping $\psi : \mathbf{X} \to \mathbf{Y}$ by

$$\psi(x) := \lim_{n\to\infty} 2^n\Gamma\left(\frac{x}{2^n}x\right) \tag{37}$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (37), we get (34) It follows from (32) and (33) that

$$\begin{aligned}&\left\|2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\ &=\lim_{n\to\infty}2^n\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+2}}+\frac{1}{2^{n+1}}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}\right)-f\left(\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\ &\le\lim_{n\to\infty}2^n\left|\xi_1\right|\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}+\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}-\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}\right)\right\|_{\mathbf{Y}}\\ &\quad+\lim_{n\to\infty}2^n\left|\xi_2\right|\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}+\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}\right)-\Gamma\left(\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\ &\quad+\lim_{n\to\infty}2^n\varphi\left(\frac{x_1}{2^n},\cdots,\frac{x_k}{2^n},\frac{y_1}{2^n},\cdots,\frac{y_k}{2^n},\frac{z_1}{2^n},\cdots,\frac{z_k}{2^n}\right)\\ &=\left\|\beta_1\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\ &\quad+\left\|\beta_2\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{38}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j = 1 \to k$. So

$$\begin{aligned}&\left\|2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\ &\le\left\|\xi_1\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\ &\quad+\left\|\xi_2\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\ &\quad+\varphi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)\end{aligned}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j = 1 \to n$. By *Lemma* 3.1, the mapping $\psi : \mathbf{X} \to \mathbf{Y}$ is additive. Ei

$$\psi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) - \psi\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right) - \psi\left(\sum_{j=1}^{k} z_j\right) = 0$$

Now, let $\psi' : \mathbb{X} \to \mathbb{Y}$ be another additive mapping satisfying (34). Then we have

$$\begin{aligned}\left\|\psi(x) - \psi'(x)\right\| &= \left\|2^q \psi\left(\frac{x}{2^q}\right) - 2^q \psi'\left(\frac{x}{2^q}\right)\right\|_{\mathbf{Y}} \\ &\le \left\|2^q \psi\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q \psi'\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|_{\mathbb{Y}} \\ &\le 2^q \phi\left(\frac{x}{2^q}, 0, \cdots, 0, \frac{x}{2^q}, 0, \cdots, 0, 0, \cdots, 0\right)\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $\psi(x) = \psi'(x)$ for all $x \in \mathbf{X}$. This proves the uniqueness of ψ.

Theorem 6. Suppose $\varphi : \mathbf{X}^{3k} \to [0, \infty)$ be a function such that

$$\begin{aligned}&\psi(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k) \\ &:= \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left(2^j x_1, \cdots, 2^j x_k, 2^j y_1, \cdots, 2^j y_k, 2^j z_1, \cdots, 2^j z_k\right) < \infty\end{aligned} \tag{39}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j = 1 \to k$ and let $\Gamma : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfies $\Gamma(0) = 0$ and

$$\begin{aligned}&\left\|2\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{4} + \frac{1}{2}\sum_{j=1}^{k} z_j\right) - \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right) - \Gamma\left(\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} \\ &\le \left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) + \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} - \sum_{j=1}^{k} z_j\right) - 2\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right)\right)\right\|_{\mathbf{Y}} \\ &\quad + \left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) - \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right) - \Gamma\left(\sum_{j=1}^{k} z_j\right)\right)\right\|_{\mathbf{Y}} \\ &\quad + \varphi(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)\end{aligned} \tag{40}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j = 1 \to k$.

Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\begin{aligned}&\left\|\Gamma(x) - \psi(x)\right\|_{\mathbf{Y}} \\ &\le \phi(x, 0, \cdots, 0, x, \cdots, 0, 0, \cdots, 0)\end{aligned} \tag{41}$$

for all $x \in \mathbb{X}$

The rest of the proof is similar to the proof of theorem 5.

From proving the theorems we have consequences:

Corollary 3. Let $r > 1$ and θ be nonnegative real numbers and let $\Gamma : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfy $\Gamma(0) = 0$ and

$$\begin{aligned}&\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-f\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|y_j\right\|^r+\sum_{j=1}^{k}\left\|z_j\right\|^r\right)\end{aligned}\tag{42}$$

for all $x_j\in\mathbf{X}$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbb{Y}}\le\frac{2.2^r\theta}{2^r-2}\left\|x\right\|_{\mathbf{X}}^r\tag{43}$$

for all $x\in\mathbf{X}$

Corollary 4. Let $r<1$ and θ be non-negative real numbers and let $\Gamma:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $f(0)=0$ and

$$\begin{aligned}&\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|y_j\right\|^r+\sum_{j=1}^{k}\left\|z_j\right\|^r\right)\end{aligned}\tag{44}$$

for all $x_j,y_j,z-j\in\mathbf{X}$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|\Gamma(x)-\psi(x)\right\|_{\mathbf{Y}}\le\frac{2.2^r\theta}{2-2^r}\left\|x\right\|_{\mathbf{X}}^r\tag{45}$$

for all $x\in\mathbf{X}$

5. Establish the Solution of the Cauchy Additive (ξ_1,ξ_2)-Function Inequalities Using a Fixed Point Method

Now, we first study the solutions of (2). Note that for these inequalities, when $\mathbf{X}$ be a real or complete normed space and $\mathbf{Y}$ complex Banach space.

Lemma 7. Suppose mapping $\Gamma:\mathbb{X}\to\mathbf{Y}$ satisfies $\Gamma(0)=0$ and

$$\begin{aligned}&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{46}$$

for all $x_j, y_j, z_j \in \mathbb{X}, \forall j = 1 \to k$ if and only if $\Gamma : \mathbf{X} \to \mathbf{Y}$ is Cauchy additive

Proof. Assume that $\Gamma : \mathbf{X} \to \mathbf{Y}$ satisfies (46)

$(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(x, \cdots, 0, x, \cdots, 0, x, \cdots, 0)$ in (46) we have

$$\left\|\Gamma(2x) - 2\Gamma(x)\right\|_{\mathbf{Y}} \le |\beta_1| \left\|\Gamma(2x) - 2\Gamma(x)\right\|_{\mathbf{Y}}$$

and so $\Gamma(2x) = 2\Gamma(x)$ for all $x \in \mathbf{X}$.

Thus

$$\Gamma\left(\frac{x}{2}\right) = \frac{1}{2}\Gamma(x) \tag{47}$$

for all $x \in \mathbf{X}$

It follows from (46) and (47) that

$$\begin{aligned} &\left\|\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) - \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right) - \Gamma\left(\sum_{j=1}^{k} z_j\right)\right\|_{\mathbb{Y}} \\ &\le \left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) + \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} - \sum_{j=1}^{k} z_j\right) - 2\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right)\right)\right\|_{\mathbb{Y}} \\ &\quad + \left\|\xi_2\left(2\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{4} + \frac{1}{2}\sum_{j=1}^{k} z_j\right) - \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right) - f\left(\sum_{j=1}^{k} z_j\right)\right)\right\|_{\mathbb{Y}} \\ &= \left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) + \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} - \sum_{j=1}^{k} z_j\right) - 2\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right)\right)\right\|_{\mathbb{Y}} \\ &\quad + \left\|\xi_2\left(\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) - \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right) - \Gamma\left(\sum_{j=1}^{k} z_j\right)\right)\right\|_{\mathbb{Y}} \end{aligned} \tag{48}$$

and so

$$\begin{aligned} &(1 - |\xi_2|)\left\|\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) - \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right) - \Gamma\left(\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} \\ &\le |\xi_1|\left\|\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) + \Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2} - \sum_{j=1}^{k} z_j\right) - 2\Gamma\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2}\right)\right\|_{\mathbf{Y}} \end{aligned} \tag{49}$$

we let $u = \sum_{j=1}^{k} \frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j, v = \sum_{j=1}^{k} \frac{x_j + y_j}{2} - \sum_{j=1}^{k} z_j$, for all $j = 1 \to k$, we get

$$\begin{aligned} &(1 - |\xi_2|)\left\|\Gamma(u) - \Gamma\left(\frac{u + v}{2}\right) - \Gamma\left(\frac{u - v}{2}\right)\right\| \\ &\le |\xi_1|\left\|\Gamma(u) + \Gamma(v) - 2\Gamma\left(\frac{u + v}{2}\right)\right\|_{\mathbf{Y}} \end{aligned} \tag{50}$$

for all $u, v \in \mathbf{X}$

and so

$$\begin{aligned} &\frac{1}{2}(1 - |\xi_2|)\left\|\Gamma(u + v) + \Gamma(u - v) - 2\Gamma(u)\right\|_{\mathbb{Y}} \\ &\le |\xi_1|\left\|\Gamma(u + v) - \Gamma(u) - f(v)\right\|_{\mathbf{Y}} \end{aligned} \tag{51}$$

for all $u, v \in \mathbf{X}$ It follows from (49) and (51) that

$$\begin{aligned}&\frac{1}{2}\left(1-|\xi_2|\right)^2\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le|\xi_1|^2\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{52}$$

Since $\sqrt{2}|\xi_1|+|\xi_2|<1$

and so

$$\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)=\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)+\Gamma\left(\sum_{j=1}^{k}z_j\right)$$

for all $x_j, y_j, z_j \in \mathbf{X}, \forall j=1\to k$. Thus *f* is Cauchy additive. □

The rest of the proof is similar to the proof of Lemma 2.

Theorem 8. Suppose $\varphi:\mathbf{X}^{3n}\to[0,\infty)$ be a function such that there exists an $L<1$ with

$$\varphi\left(\frac{x_1}{2},\cdots,\frac{x_n}{2},\frac{y_1}{2},\cdots,\frac{y_n}{2},\frac{z_1}{2},\cdots,\frac{z_n}{2}\right)\le\frac{L}{2}\varphi\left(x_1,\cdots,x_n,y_1,\cdots,y_n,z_1,\cdots,z_n\right) \tag{53}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j=1\to k$. If $f:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $\Gamma(0)=0$ and

$$\begin{aligned}&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&+\left\|\xi_2\left(2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&+\varphi\left(x_1,\cdots,x_n,y_1,\cdots,y_n,z_1,\cdots,z_n\right)\end{aligned} \tag{54}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j=1\to k$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}}\le\frac{L}{2(1-L)\left(1-|\xi_1|\right)}\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0) \tag{55}$$

for all $x \in \mathbf{X}$

Proof. Replacing $(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_n,z_1,z_2,\cdots,z_n)$ by $(x,0,\cdots,0,x,\cdots,0,0,\cdots,0)$ in (54), we get

$$\left(1-|\xi_1|\right)\left\|\Gamma(2x)-2\Gamma(x)\right\|_{\mathbf{Y}}\le\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0) \tag{56}$$

for all $x \in \mathbf{X}$.

Consider the set

$$\mathbb{S}:=\left\{h:\mathbf{X}\to\mathbf{Y},h(0)=0\right\}$$

and introduce the generalized metric on $\mathbb{S}$:

$$d(g,h) := inf\left\{\lambda \in \mathbb{R} : \|g(x)-h(x)\| \le \lambda\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0), \forall x \in \mathbf{X}\right\},$$

where, as usual, $inf\,\phi = +\infty$. It easy to show that $(\mathbb{S},d)$ is complete [17] Now we cosider the linear mapping $J : \mathbb{S} \to \mathbb{S}$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in \mathbf{X}$. Let $g,h \in \mathbb{S}$ be given such that $d(g,h) = \varepsilon$ then

$$\|g(x)-h(x)\| \le \varepsilon\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0)$$

for all $x \in \mathbf{X}$.

Hence

$$\|Jg(x)-Jh(x)\| = \left\|2g\left(\frac{x}{2}\right)-2h\Gamma\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\varphi\left(\frac{x}{2},0,\cdots,0,\frac{x}{2},\cdots,0,\frac{x}{2},\cdots,0\right)$$

$$\le 2\varepsilon\frac{L}{2}\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0) \le L\varepsilon\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0)$$

for all $x \in \mathbf{X}$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \le L\cdot\varepsilon$. This means that

$$d(Jg,Jh) \le Ld(g,h)$$

for all $g,h \in \mathbf{X}$ It follows from (56) that

$$\left\|\Gamma(x)-2\Gamma\left(\frac{x}{2}\right)\right\| \le \frac{1}{1-|\xi_1|}\varphi\left(\frac{x}{2},0,\cdots,0,\frac{x}{2},\cdots,0,\frac{x}{2},\cdots,0\right)$$

$$\le \frac{L}{2(1-|\xi_1|)}\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0)$$

for all $x \in \mathbf{X}$. So $d(\Gamma,J\Gamma) \le \frac{L}{2(1-|\xi_1|)}$ for all $x \in \mathbf{X}$ By Theorem 2.3, there exists a mapping $\psi : \mathbf{X} \to \mathbb{Y}$ satisfying the following:

1) ψ is a fixed point of *J*, *i.e.*,

$$\psi(x) = 2\psi\left(\frac{x}{2}\right) \tag{57}$$

for all $x \in X$. The mapping ψ is a unique fixed point *J* in the set

$$\mathbb{M} = \{g \in \mathbb{S} : d(f,g) < \infty\}$$

This implies that ψ is a unique mapping satisfying (57) such that there exists a $\lambda \in (0,\infty)$ satisfying

$$\|\Gamma(x)-\psi(x)\| \le \lambda\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0)$$

for all $x \in \mathbf{X}$

2) $d(J^l\Gamma,\psi) \to 0$ as $l \to \infty$. This implies equality

$$\lim_{l\to\infty} 2^n\Gamma\left(\frac{x}{2^n}\right) = \psi(x)$$

for all $x \in \mathbf{X}$

3) $d(\Gamma,\psi) \le \frac{1}{1-L}d(\Gamma,J\Gamma)$. which implies

$$\left\|\Gamma(x)-\psi(x)\right\| \le \frac{L}{2(1-L)\left(1-|\xi_1|\right)}\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0)$$

for all $x \in \mathbf{X}$. □

The rest of the proof is similar to the proof of Theorem 3.

Theorem 9. Let $\varphi:\mathbf{X}^{3n} \to [0,\infty)$ be a function such that there exists an $L<1$ with

$$\varphi(x_1,\cdots,x_n,y_1,\cdots,y_n,z_1,\cdots,z_n) \le 2L\varphi\left(\frac{x_1}{2},\cdots,\frac{x_n}{2},\frac{y_1}{2},\cdots,\frac{y_n}{2},\frac{z_1}{2},\cdots,\frac{z_n}{2}\right) \tag{58}$$

for all $x,y,z \in \mathbf{X}$. Let $\Gamma:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $\Gamma(0)=0$ and

$$\begin{aligned}
&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\
&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\
&\quad+\left\|\xi_2\left(2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\
&\quad+\varphi(x_1,\cdots,x_n,y_1,\cdots,y_n,z_1,\cdots,z_n)
\end{aligned} \tag{59}$$

for all $x_j,y_j,z_j \in \mathbf{X}, j=1\to k$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|\Gamma(x)-\psi(x)\right\|_{\mathbf{Y}} \le \frac{L}{2(1-L)\left(1-|\xi_1|\right)}\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0) \tag{60}$$

for all $x \in \mathbf{X}$

The rest of the proof is similar to the proof of Theorem 8.

From proving the theorems we have consequences:

Corollary 5. Let $r>1$ and θ be nonnegative real numbers and let $\Gamma:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $\Gamma(0)=0$ and

$$\begin{aligned}
&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\
&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\
&\quad+\left\|\xi_2\left(2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\
&\quad xi+\theta\left(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^{r}+\sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^{r}+\sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^{r}\right)
\end{aligned} \tag{61}$$

for all $x_j,y_j,z_j \in \mathbf{X}, j=1\to k$.

Then there exists a unique mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|\Gamma(x)-\psi(x)\right\|_{\mathbf{Y}} \le \frac{3\theta}{(2^r-2)\left(1-|\xi_1|\right)}\|x\|_{\mathbf{X}}^{r} \tag{62}$$

for all $x \in \mathbf{X}$

Corollary 6. Let $r < 1$ and θ be nonnegative real numbers and let $\Gamma : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfy $\Gamma(0) = 0$ and

$$\begin{aligned}&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{r}+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbb{X}}^{r}+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbb{X}}^{r}\right)\end{aligned} \tag{63}$$

for all $x_j, y_j, z_j \in \mathbb{X}, j = 1 \to k$.

Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x)-\psi(x)\right\|_{\mathbf{Y}} \leq \frac{3\theta}{\left(2-2^r\right)\left(1-\left|\xi_1\right|\right)}\left\|x\right\|_{\mathbf{X}}^{r} \tag{64}$$

for all $x \in \mathbf{X}$.

6. Establish the Solution of the Additive (ξ_1,ξ_2)-Function Inequalities Using a Direect Method

Next, we study the solutions of (2). Note that for these inequalities, when $\mathbf{X}$ be a real or complete normed space and $\mathbf{Y}$ complex Banach space.

Theorem 10. Suppose $\varphi : \mathbf{X}^{3k} \to [0, \infty)$ be a function such that

$$\begin{aligned}&\phi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)\\&:=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x_1}{2^j},\cdots,\frac{x_k}{2^j},\frac{y_1}{2^j},\cdots,\frac{y_k}{2^j},\frac{z_1}{2^j},\cdots,\frac{z_k}{2^j}\right)<\infty\end{aligned} \tag{65}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j = 1 \to k$ and let $\Gamma : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfies $\Gamma(0) = 0$ and

$$\begin{aligned}&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\varphi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)\end{aligned} \tag{66}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j = 1 \to k$.

Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|\Gamma(x)-\psi(x)\right\|_{\mathbb{Y}} \leq \phi(x,\cdots,0,x,\cdots,0,x,\cdots,0) \tag{67}$$

for all $x \in \mathbf{X}$

Proof. Replacing $\left(x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k, z_1, z_2, \cdots, z_k\right)$ by $\left(x, \cdots, 0, x, \cdots, 0, x, 0, \cdots, 0\right)$ in (66), we get

$$\left(1-\left|\xi_1\right|\right)\left\|\Gamma\left(2x\right)-2\Gamma\left(x\right)\right\|_{\mathbf{Y}} \le \varphi\left(x, 0, \cdots, 0, x, \cdots, 0, x, 0, \cdots, 0\right) \tag{68}$$

for all $x \in \mathbf{X}$.

So

$$\left\|\Gamma\left(x\right)-\frac{1}{2}\Gamma\left(2x\right)\right\|_{\mathbf{Y}} \le \frac{1}{2\left(1-\left|\xi_1\right|\right)}\varphi\left(x, 0, \cdots, 0, x, \cdots, 0, x, \cdots, 0\right) \tag{69}$$

for all $x \in \mathbf{X}$.

Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}\Gamma\left(2^l x\right)-\frac{1}{2^m}\Gamma\left(2^m x\right)\right\|_{\mathbf{Y}} \\
&\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1}\Gamma\left(\frac{x}{2^{j+1}}\right)\right\|_{\mathbf{Y}} \\
&\le \sum_{j=l}^{m-1}\frac{2^{j+1}}{2\left(1-\left|\xi_1\right|\right)}\varphi\left(\frac{x}{2^{j+1}}, 0, \cdots, 0, \frac{x}{2^{j+1}}, 0, \cdots, 0, \frac{x}{2^{j+1}}, \cdots, 0\right)
\end{aligned} \tag{70}$$

for all nonnegative integers *m* and *l* with $m > l$ and all $x \in \mathbf{X}$. It follows from (70) that the sequence $\left\{2^n\Gamma\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{2^n\Gamma\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $\psi : \mathbf{X} \to \mathbf{Y}$ by

$$\psi\left(x\right) := \lim_{n\to\infty} 2^n\Gamma\left(\frac{x}{2^n}x\right) \tag{71}$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (37), we get (67) It follows from (65) and (66) that

$$\begin{aligned}
&\left\|\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}} \\
&= \lim_{n\to\infty} 2^n\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}+\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}\right)-\Gamma\left(\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}} \\
&\le \lim_{n\to\infty} 2^n\left|\xi_1\right|\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}+\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}-\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}\right)\right\|_{\mathbf{Y}} \\
&\quad+\lim_{n\to\infty} 2^n\left|\xi_2\right|\left\|2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+2}}+\frac{1}{2^{n+1}}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2^{n+1}}\right)-\Gamma\left(\frac{1}{2^n}\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}} \\
&\quad+\lim_{n\to\infty} 2^n\varphi\left(\frac{x_1}{2^n}, \cdots, \frac{x_k}{2^n}, \frac{y_1}{2^n}, \cdots, \frac{y_k}{2^n}, \frac{z_1}{2^n}, \cdots, \frac{z_k}{2^n}\right) \\
&= \left\|\xi_1\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}} \\
&\quad+\left\|\xi_2\left(2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{72}$$

for all $x_j, y_j, z_j \in \mathbb{X}, j = 1 \to k$. So

$$\left\|\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}$$
$$\leq\left\|\xi_1\left(\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}$$
$$+\left\|\xi_2\left(2\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\psi\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j = 1 \to n$. By *Lemma* 5.1, the mapping $\psi : \mathbf{X} \to \mathbf{Y}$ is additive. Ei

$$\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)=\psi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)+\psi\left(\sum_{j=1}^{k}z_j\right)$$

Now, let $\psi' : \mathbf{X} \to \mathbf{Y}$ be another additive mapping satisfying (67). Then we have

$$\begin{aligned}\left\|\psi(x)-\psi'(x)\right\| &= \left\|2^q\psi\left(\frac{x}{2^q}\right)-2^q\psi'\left(\frac{x}{2^q}\right)\right\|_{\mathbf{Y}}\\ &\leq \left\|2^q\psi\left(\frac{x}{2^q}\right)-2^q\Gamma\left(\frac{x}{2^q}\right)\right\|+\left\|2^q\psi'\left(\frac{x}{2^q}\right)-2^q\Gamma\left(\frac{x}{2^q}\right)\right\|_{\mathbf{Y}}\\ &\leq 2^q\phi\left(\frac{x}{2^q},0,\cdots,0,\frac{x}{2^q},0,\cdots,0,\frac{x}{2^q}0,\cdots,0\right)\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $\psi(x) = \psi'(x)$ for all $x \in \mathbf{X}$. This proves the uniqueness of ψ. □

Theorem 11. Suppose $\varphi : \mathbf{X}^{3k} \to [0,\infty)$ be a function such that

$$\begin{aligned}&\psi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\\ &:=\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi\left(2^jx_1,\cdots,2^jx_k,2^jy_1,\cdots,2^jy_k,2^jz_1,\cdots,2^jz_k\right)<\infty\end{aligned}\tag{73}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j = 1 \to k$ and let $\Gamma : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfies $\Gamma(0) = 0$ and

$$\begin{aligned}&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\ &\leq\left\|\beta_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\ &\quad+\left\|\beta_2\left(2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\ &\quad+\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned}\tag{74}$$

for all $x_j, y_j, z_j \in \mathbf{X}, j = 1 \to k$.

Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|\Gamma(x)-\psi(x)\right\|_{\mathbf{Y}}\leq\phi(x,0,\cdots,0,x,\cdots,0,x,0,\cdots,0)\tag{75}$$

for all $x \in \mathbb{X}$

Proof. Replacing $(x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k,z_1,z_2,\cdots,z_k)$ by $(x,\cdots,0,x,\cdots,0,x,\cdots,0)$ in (74), we get

$$\left(1-|\xi_1|\right)\left\|\Gamma(2x)-2\Gamma(x)\right\|_{\mathbb{Y}}\le\varphi(x,0,\cdots,0,x,\cdots,0,x,\cdots,0)\tag{76}$$

for all $x\in\mathbf{X}$.

So Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,0,\cdots,0,x,0,\cdots,0,x,0,\cdots,0)$ in (74), we get

$$\left\|2\Gamma\left(\frac{x}{2}\right)-\Gamma(x)\right\|_{\mathbf{Y}}\le\varphi(x,0,\cdots,0,x,0,\cdots,0,x,0,\cdots,0)\tag{77}$$

for all $x\in\mathbf{X}$. So

$$\left\|\Gamma(x)-\frac{1}{2}\Gamma(2x)\right\|_{\mathbf{Y}}\le\frac{1}{2}\varphi(2x,0,\cdots,0,2x,0,\cdots,0,2x,0,\cdots,0)\tag{78}$$

for all $x\in\mathbf{X}$. Hence

$$\begin{aligned}&\left\|\frac{1}{2^l}\Gamma\left(2^lx\right)-\frac{1}{2^m}\Gamma\left(2^mx\right)\right\|_{\mathbf{Y}}\\&\le\sum_{j=l}^{m-1}\left\|\frac{1}{2^j}\Gamma\left(2^jx\right)-\frac{1}{2^{j+1}}\Gamma\left(2^{j+1}x\right)\right\|_{\mathbf{Y}}\\&\le\sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\varphi\left(2^{j+1}x,0,\cdots,0,2^{j+1}x,0,\cdots,0,2^{j+1}x,0,\cdots,0\right)\end{aligned}\tag{79}$$

for all nonnegative integers m and l with $m>l$ and all $x\in\mathbf{X}$. It follows from (79) that the sequence $\left\{\frac{1}{2^n}\Gamma\left(2^nx\right)\right\}$ is a Cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{\frac{1}{2^n}\Gamma\left(2^nx\right)\right\}$ converges. So one can define the mapping $\psi:\mathbf{X}\to\mathbf{Y}$ by

$$\psi(x):=\lim_{n\to\infty}\frac{1}{2^n}\Gamma\left(2^nx\right)\tag{80}$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (79), we get (75).

The rest of the proof is similar to the proof of theorem 10. □

From proving the theorems we have consequences:

Corollary 7. Let $r>1$ and θ be nonnegative real numbers and let $\Gamma:\mathbf{X}\to\mathbf{Y}$ be a mapping satisfy $\Gamma(0)=0$ and

$$\begin{aligned}&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}+\sum_{j=1}^{k}z_j\right)+\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)-2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\left\|\xi_2\left(2\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{4}+\frac{1}{2}\sum_{j=1}^{k}z_j\right)-\Gamma\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}\right)-\Gamma\left(\sum_{j=1}^{k}z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|y_j\right\|^r+\sum_{j=1}^{k}\left\|z_j\right\|^r\right)\end{aligned}\tag{81}$$

for all $x_j \in \mathbf{X}$.

Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|\Gamma(x) - \psi(x)\right\|_{\mathbf{Y}} \le \frac{3\theta}{\left(2^r - 2\right)\left(1 - \left|\xi_1\right|\right)}\left\|x\right\|_{\mathbf{X}}^r \tag{82}$$

for all $x \in \mathbf{X}$

Corollary 8. Let $r < 1$ and θ be nonnegative real numbers and let $\Gamma : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfy $\Gamma(0) = 0$ and

$$\begin{aligned}
&\left\|\Gamma\left(\sum_{j=1}^{k}\frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) - \Gamma\left(\sum_{j=1}^{k}\frac{x_j + y_j}{2}\right) - \Gamma\left(\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} \\
&\le \left\|\xi_1\left(\Gamma\left(\sum_{j=1}^{k}\frac{x_j + y_j}{2} + \sum_{j=1}^{k} z_j\right) + \Gamma\left(\sum_{j=1}^{k}\frac{x_j + y_j}{2} - \sum_{j=1}^{k} z_j\right) - 2\Gamma\left(\sum_{j=1}^{k}\frac{x_j + y_j}{2}\right)\right)\right\|_{\mathbf{Y}} \\
&\quad + \left\|\xi_2\left(2\Gamma\left(\sum_{j=1}^{k}\frac{x_j + y_j}{4} + \frac{1}{2}\sum_{j=1}^{k} z_j\right) - \Gamma\left(\sum_{j=1}^{k}\frac{x_j + y_j}{2}\right) - \Gamma\left(\sum_{j=1}^{k} z_j\right)\right)\right\|_{\mathbf{Y}} \\
&\quad + \theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r + \sum_{j=1}^{k}\left\|y_j\right\|^r + \sum_{j=1}^{k}\left\|z_j\right\|^r\right)
\end{aligned} \tag{83}$$

for all $x_j, y_j, z - j \in \mathbf{X}$.

Then there exists a unique mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|\Gamma(x) - \psi(x)\right\|_{\mathbf{Y}} \le \frac{3\theta}{\left(2 - 2^r\right)\left(1 - \left|\xi_1\right|\right)}\left\|x\right\|_{\mathbf{X}}^r \tag{84}$$

for all $x \in \mathbf{X}$

7. Conclusion

In this paper, I have given two functional inequalities with 3k variables and fully solved complex Banach space by fixed point methods and direct methods. This result is based on special results such as [25] [26].

Conflicts of Interest

The author declares no conflicts of interest.

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