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On the Study of Binomial Theorem: Formulation of Conjecture for Odd Binomial Coefficients for Binomials with Indices of $n = 2^r - 1$, Where $r \in \mathbb{Z}^{-*}$

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Abstract

Because of the growing role of Binomial Theorem in various fields in Mathematics, such as in calculus and number theory, and even in societal advancements, such in technology and business, mathematicians continue to explore new developments in this interesting theorem. In this case, this study explored some of the unveiled concepts of Binomial Theorem for further studies, specifically on observing the relationships of the indices of the binomials to obtaining odd-valued binomial coefficients. Using the proof by exhaustion, among various mathematical proofs, a formula for n of the binomial in which the cases are separated and focus on the condition is supported. After a series of exhaustion of data, cases were determined and identified. Among the 64 values tested consecutively from 0 to 63, it was found that only the values of 0, 1, 3, 7, 15, 31, and 63 for n resulted to all odd binomial coefficients. Through trial-and-error process, the general term of the indices was expressed in the form of $n = 2^r - 1$, where $r \in \mathbb{Z}^{-*}$.

Subject Areas

Combinatorial Mathematics

Keywords

Binomial Theorem, Binomial Coefficients, Proof by Exhaustion, Conjecture

1. Introduction

Much of the development in mathematics—algebraic or analytic, pure or applied—involves the significant application of the Binomial Theorem. Its role spans even in development of calculus and number theory [1]. In this modern time, Binomial Theorem is even involved in societal fields, such as technology, economy, and engineering.

Binomial Theorem, though widely applied, starts as a mathematical body of knowledge even in the time of Nasir al-Din al-Tusi [2]. The coefficients in each term of expanded form occurred from the Binomial Theorem are known as the binomial coefficients. For any two integers $n \geq 0$ and $0 \leq k \leq n$, the number of combinations of k distinct elements of a given set composed by n different objects is conventionally denoted by C_n^k or $\binom{n}{k}$. This number occurs in many different contexts; in particular, it appears as a coefficient in binomial expansions, wherefrom it gets its name. Arranging the binomial coefficients C_n^0, …, C_n^n from left to right in a row for successive values of n, the triangular array called Pascal's triangle is obtained [3].

Because of the growing importance of Binomial Theorem and Pascal's Triangle, many researches arose which aims on its expansion in various fields of Mathematics. [4] used computer graphics, geometry, algebra and combinatorics in deriving the Binomial Theorem. [5] even used elementary methods to examine continuous binomial coefficients by deriving several representations of them including infinite product and Taylor series. Moreover, [6] showed that combinatorial methods and reflection and rotation transformations are used to prove basic binomial coefficient formula and its extended model. On the study of [1], Binomial Theorem is also used to facilitate development in the fields of Mathematics, just like in algebra and analysis. Whereas, [7] proved an identity between binomial coefficients and the high power of Fibonacci numbers.

Binomial Theorem and binomial coefficients were also used to broaden understanding in other fields. For example, employing a *p*, *q*-binomial coefficients are used to describe the magnetization distributions of the Ising model [8] [9] and *(p*, *q)*-Stirling numbers of the second kind [10]. Moreover, [11] suggested that binomial coefficient arithmetic should be applied in analyzing model cell population dynamics.

Just recently, a study on Binomial Theorem presented geometric interpretations and gathered works with binomial coefficients of binomial expressions [12]. This study, among those mentioned, presented various advancements of the theorem and widened its application in other branches in other fields, but only a few examined detailed relationships among binomial coefficients, and discovered possible connections that might exist within these coefficients. With that, this study wanted to explore some of its unveiled concepts for further studies. Specifically, it aimed to look for relationships of the indices of the binomials to obtaining odd-valued binomial coefficients.

2. Review of Related Literature

2.1. Binomial Theorem and Binomial Coefficients

As a simple yet important mathematical result, the Binomial Theorem has gained wide interest to many statistical scientists in particular. In fact, various textbooks and papers include proofs and applications in probability and mathematical statistics. The use of mathematical induction to create a standardized proof of the Binomial Theorems has involved quite a delicate argument [13].

In 1952, [14] provided a simpler proof of the Binomial Theorem, which also involved an induction argument. A very nice proof of the Binomial Theorem based on combinatorial considerations was obtained by [13]. Furthermore, [15] presented a simple proof of the Binomial Theorem using partial differential calculus. These studies indicated concerted efforts for mathematicians to strengthen claims on Binomial Theorem.

On specific aspects, however, studies are also worked out for binomial coefficients as patterns are seen through rigorous efforts of some mathematicians delving the Binomial Theorem. This allows students to connect with the theorem, and understand a deep sense of Mathematics. Along with the concept of binomial coefficient, [16] also used mathematical induction and concept of combination to analyze road network models. Moreover, [17] proved some generalized identities involving Fibonacci sequence using binomial coefficients. On the other hand, [18] introduced the concept of binomial coefficients in computing the value of "factorial of n of order K". In this process, Pascal's Triangle is expanded to an Integer Binomial Plan, which displays remarkable properties. Eventually, he discovered on the strange concepts of semi-integer derivative of what he referred to as k-derivable function, and referred for further studies.

2.2. Pascal's Triangle

Although Pascal himself used a rectangular version of this triangle based on the figurate numbers, the contemporary version of arrangement is that of an equilateral triangle of the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where each row represents the exhaustion of coefficients of a particular n, beginning with $n = 0$ and increasing by one with each subsequent row, where in any given row n, $k = 0, 1, 2, 3, \ldots, n$. Each row can also be seen as the coefficients of the expansion given by the Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, something worth noting in exploring the properties of the triangle.

Upon its discovery, many studies and applications have already been coined to the Pascal's triangle. Just recently, it has already been used to give an analysis of the game "Plinko" priced-ballgame in depth. It calculated the expectation of winnings given any starting point for the disc, as well as the probabilities of each particular prize amount. With this, Pascal's Triangle is also applied to current generalizations to the classical problem "The Gambler's Game of Points" to games of alternative point structures [19]. On the other hand, [20] was able to expand upon the fact that the image of non-zeros mod p (p prime) in Pascal's Triangle will have a fractal dimension of $D(p)=1+\log\left[p\left(\frac{p+1}{2}\right)\right]$. This can be generalized to various combinations of moduli and sets of residues.

The ramifications of the simple yet increasingly complex study of Pascal's triangle in education are enormous. In education today, educators are encouraged to differentiate instruction as to engage students at different levels of learning style and ability. This is to allow students to experience the interconnected of math topics that Pascal's triangle naturally allows [20]. Similarly, [21] broadens the mathematical pattern of the Pascal triangle by generating bilateral picture of the binomial expansion, resulting in an expansion of positive and negative powers.

2.3. Conjecture

Mathematical conjecture plays an essential role in the growth of Mathematical knowledge, as it expands mathematical theories [22]. In doing this, however, one has to designs abstraction and generalization process to concretize hypothetical ideas [23] [24]. However, the very first step involves invention, which one should pay focus religiously. These processes involve people's reasoning skills as they have to make and investigate existing theorems and conjectures, develop and evaluate arguments and proofs, and choose types and methods of proofs [25].

3. Methods

In this study, claims are tried to prove based on the observation of the concepts and relationships in the Binomial Theorem. To strengthen the claims of this study, techniques suitable for the appropriate proof are selected. Generally, this study presented kinds of proof which had been already used by the various mathematical studies. [26] presented four fundamental mathematical proving techniques used when one wishes to prove the statement $p \Rightarrow q$.

1) **Proof by counterexample.** This type of proof involves finding at least one example in which a generalization is false. The counterexample will disprove the generalization or indicate its negation.

2) **Direct/Deductive proof.** In this case, one shows that a given statement is deducible by inferring patterns from given information, previously studied definitions, postulates and theorems. Traditionally, direct proofs have been expressed using two column or paragraph formats.

3) **Indirect Proof.** With this type of proof, one assumes that the negation of a statement yet to be proven is true, then shows that this assumption leads to a contradiction. Additionally, the process of proving a proof by proving its contra-positive can be thought of as a special case of indirect proof through contradiction. Paragraph formats are often used to show indirect proofs.

4) Proof by induction. It is based on the principle of mathematical induction and can be stated as follows: "If a given property is true for 1 and if for all $n > 1$, the property being true for n implies it is true for $n + 1$". Thus, we can conclude that the property is true for all natural numbers.

In the study "Proofs and Mathematical Reasoning" by [27], proof by cases, also referred to as **proof by exhaustion**, was discussed which aimed to list all possibilities on a certain problem. This method was traced back from the time of Eudoxius and Archimedes as a general technique in proving theorem whose truths had been informally discovered by other means [28]. Moreover, it is also mentioned by Euclid on his use of the method of exhaustion to prove propositions, XII.5, XI.10, XII.11, and XII.18. But because of the difficulty of problems, new and more powerful methods are invented to further enhance mathematical knowledge, which leads to the discovery of various Mathematics, just like the integral calculus.

In this study, the proof by exhaustion was adapted. The Pascal's Triangle was explicitly expressed until the binomial coefficients cannot be expressed in simplest terms. Then, a formula for n of the binomial in which the cases are separated is identified, and focus on the condition to be supported. This was done to separate the conditions which fall under the identified cases. Since the cases to comprehensively prove the claims were not restricted, a conjecture was made to signify that the established claim is still unproven, though was exhaustibly tried.

4. Results

The study involves finding relationships of the indices of the binomials in the Pascal's triangle to obtaining odd-valued binomial coefficients. Through the proof by exhaustion, the binomial coefficients were continuously extracted in the Pascal's Triangle through the combinatorial process. In this study however, n was just limited to 63 due to resource and time constraints. Below are the binomial coefficients exhausted for the study.

- **$n = 0$**

1

- **$n = 1$**

1, 1

- $n = 2$

1, 2, 1

- **$n = 3$**

1, 3, 3, 1

- $n = 4$

1, 4, 6, 4, 1

- $n = 5$

1, 5, 10, 10, 5, 1

- $n = 6$

1, 6, 15, 20, 15, 6, 1

- ***n* = 7**

1, 7, 21, 35, 35, 21, 7, 1

- $n = 8$

1, 8, 28, 56, 70, 56, 28, 8, 1

- $n = 9$

1, 9, 36, 84, 126, 126, 84, 36, 9, 1

- $n = 10$

1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1

- $n = 11$

1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1

- $n = 12$

1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1

- $n = 13$

1, 13, 78, 286, 715, 1287, 1716, 1716, 1287, 715, 286, 78, 13, 1

- $n = 14$

1, 14, 91, 364, 1001, 2002, 3003, 3432, 3003, 2002, 1001, 364, 91, 14, 1

- ***n* = 15**

1, 15, 105, 455, 1365, 3003, 5005, 6435, 6435, 5005, 3003, 1365, 455, 105, 15, 1

- $n = 16$

1, 16, 120, 560, 1820, 4368, 8008, 11440, 12870, 11440, 8008, 4368, 1820, 560, 120, 16, 1

- $n = 17$

1, 17, 136, 680, 2380, 6188, 12376, 19448, 24310, 24310, 19448, 12376, 6188, 2380, 680, 136, 17, 1

- $n = 18$

1, 18, 153, 816, 3060, 8568, 18564, 31824, 43758, 48620, 43758, 31824, 18564, 8568, 3060, 816, 153, 18, 1

- $n = 19$

1, 19, 171, 969, 3876, 11628, 27132, 50388, 75582, 92378, 92378, 75582, 50388, 11628, 3876, 969, 171, 19, 1

- $n = 20$

1, 20, 190, 1140, 4845, 15504, 38760, 77520, 125970, 167960, 184756, 167960, 125970, 77520, 38760, 15504, 4845, 1140, 190, 20, 1

- $n = 21$

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- $n = 22$

1, 22, 231, 1540, 7315, 26334, 74613, 170544, 319770, 497420, 646646, 705432, 646646, 497420, 319770, 170544, 74613, 26334, 1540, 231, 22, 1

- $n = 23$

1, 23, 253, 1771, 8855, 33649, 100947, 245157, 490314, 817190, 1144066, 1352078, 1352078, 1144066, 817190, 490314, 245157, 100947, 33649, 8855, 1771, 253, 23, 1

- $n = 24$

1, 24, 276, 2024, 10626, 42504, 134596, 346104, 735471, 1307504, 1961256, 2496144, 2704156, 2496144, 1961256, 1307504, 735471, 346104, 134596, 42504, 10626, 2024, 276, 24, 1

- $n = 25$

1, 25, 300, 2300, 12650, 53130, 177100, 480700, 1081575, 2042975, 3268760, 4457400, 5200300, 5200300, 4457400, 3268760, 2042975, 1081575, 480700, 177100, 53130, 12650, 2300, 300, 25, 1

- $n = 26$

1, 26, 325, 2600, 14950, 65780, 230230, 657800, 1562275, 3124550, 5311735, 7726160, 9657700, 10400600, 9657700, 7726160, 5311735, 3124550, 1562275, 657800, 230230, 65780, 14950, 2600, 325, 26, 1

- $n = 27$

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- $n = 28$

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- $n = 29$

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- **$n = 31$**

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- $n = 32$

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- $n = 33$

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- $n = 34$

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- $n = 35$

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- $n = 36$

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- $n = 39$

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- $n = 42$

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- $n = 44$

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- $n = 45$

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- $n = 46$

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- $n = 49$

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- $n = 56$

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- $n = 57$

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- $n = 58$

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- $n = 59$

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- $n = 60$

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- $n = 61$

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- $n = 62$

1, 62, 1891, 37820, 557845, 6471002, 61474519, 491796152, 3381098545, 20286591270, 107518933731, 508271323092, 2160153123141, 8308281242850, 29078984349975, 93052749919920, 273342452889765, 739632519584070, 1849081298960175, 4282083008118300, 9206478467454345, 18412956934908690, 34315056105966195, 59678358445158600, 96977332473382725, 147405545359541742, 209769429934732479, 279692573246309972, 349615716557887465, 409894288378212890, 450883717216034179, 465428353255261088, 450883717216034179, 409894288378212890, 349615716557887465, 279692573246309972, 209769429934732479, 147405545359541742, 96977332473382725, 59678358445158600, 34315056105966195, 18412956934908690, 9206478467454345, 4282083008118300, 1849081298960175, 739632519584070, 273342452889765, 93052749919920, 29078984349975, 8308281242850, 2160153123141, 508271323092, 107518933731, 20286591270, 3381098545, 491796152, 61474519, 6471002, 557845, 37820, 1891, 62, 1

- **$n = 63$**

1, 63, 1953, 39711, 595665, 7028847, 67945521, 553270671, 3872894697, 23667689815, 127805525001, 615790256823, 2668424446233, 10468434365991, 37387265592825, 122131734269895, 366395202809685, 1012974972473835, 2588713818544245, 6131164307078475, 13488561475572645, 27619435402363035, 52728013040874885, 93993414551124795, 156655690918541325, 244382877832924467, 357174975294274221, 489462003181042451, 629308289804197437, 759510004936100355, 860778005594247069, 916312070471295267, 860778005594247069, 759510004936100355, 629308289804197437, 489462003181042451, 357174975294274221, 244382877832924467, 156655690918541325, 93993414551124795, 52728013040874885, 27619435402363035, 13488561475572645, 6131164307078475, 2588713818544245, 1012974972473835, 366395202809685, 122131734269895, 37387265592825, 10468434365991, 2668424446233, 615790256823, 127805525001, 23667689815, 3872894697, 553270671, 67945521, 7028847, 595665, 39711, 1953, 63, 1

5. Analysis

From the presented data with 64 consecutive indices of the binomial from 0 to 63, only the values of 0, 1, 3, 7, 15, 31, and 63 for n resulted to all odd binomial coefficients. The values are presented in Table 1.

Aside from these values of n, the binomial coefficients already led to a mix of even and odd binomial coefficients. Thus, cases are separated according to the indices of the binomials whose binomial coefficients in all odd values from those which are not.

To generalize the cases whose indices of the binomials lead to all odd values for binomial coefficients, general formula is established based on the exhausted data. In this case, the focus was on 0, 1, 3, 7, 15, 31, and 63. The general term of the indices was expressed in the form of $n = 2^r - 1$, where $r \in \mathbb{Z}^{-*}$. In this equation, it is expected that the next set of odd binomial coefficient after binomial raised by $n = 63$ will be $n = 127, 255, 511, 1023, 2047, \ldots$

Table 1. Binomial coefficients of the expanded form of binomial expressions for indices under consideration.

Indices under consideration, n	Binomial coefficients
0	1
1	1, 1
3	1, 3, 3, 1
7	1, 7, 21, 35, 35, 21, 7, 1
15	1, 15, 105, 455, 1365, 3003, 5005, 6435, 6435, 5005, 3003, 1365, 455, 105, 15, 1
31	1, 31, 465, 4495, 31465, 169911, 736281, 2629575, 7888725, 20160075, 44352165, 84672315, 141120525, 206253075, 265182525, 300540195, 300540195, 265182525, 206253075, 141120525, 84672315, 44352165, 20160075, 7888725, 2629575, 736281, 169911, 31465, 4495, 465, 31, 1
63	1, 63, 1953, 39711, 595665, 7028847, 67945521, 553270671, 3872894697, 23667689815, 127805525001, 615790256823, 2668424446233, 10468434365991, 37387265592825, 122131734269895, 366395202809685, 1012974972473835, 2588713818544245, 6131164307078475, 13488561475572645, 27619435402363035, 52728013040874885, 93993414551124795, 156655690918541325, 244382877832924467, 357174975294274221, 489462003181042451, 629308289804197437, 759510004936100355, 860778005594247069, 916312070471295267, 860778005594247069, 759510004936100355, 629308289804197437, 489462003181042451, 357174975294274221, 244382877832924467, 156655690918541325, 93993414551124795, 52728013040874885, 27619435402363035, 13488561475572645, 6131164307078475, 2588713818544245, 1012974972473835, 366395202809685, 122131734269895, 37387265592825, 10468434365991, 2668424446233, 615790256823, 127805525001, 23667689815, 3872894697, 553270671, 67945521, 7028847, 595665, 39711, 1953, 63, 1

However, the claims in this paper found through exhaustion are still open for proof as the case was not expressed in general terms. With that, a conjecture was crafted for the claim of this study, as presented below:

"If for a binomial expression $(x + y)^n$ whose index is equivalent to $n = 2^r - 1$, where $r \in \mathbb{Z}^{-*}$, then its binomial coefficients are all odd."

6. Conclusion

This paper explored relationships on the indices of the binomials to obtaining odd-valued binomial coefficients. Using the proof exhaustion, it was found out that binomials with indices of $n = 0, 1, 3, 7, 15, 31, 63$ all had odd binomial coefficients when expanded. Because indices after $n = 63$ were not explored due to the limitation of the method of proof itself, a conjecture was designed to generalize the findings. "If for a binomial expression $(x + y)^n$ whose index is equivalent to $n = 2^r - 1$, where $r \in \mathbb{Z}^{-*}$, then its binomial coefficients are all odd". With the existing limitation of proving this conjecture, further study on this is recommended.

Conflicts of Interest

The authors declare no conflicts of interest.

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