Open Access Library Journal
2020, Volume 7, e6719
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Subdivide the Shape of Numbers and a Theorem of Ring

Ji Peng

Department of Electronic Information, Nanjing University, Nanjing, China
Email: mcfroo@sina.com

How to cite this paper: Peng, J. (2020) Subdivide the Shape of Numbers and a Theorem of Ring. *Open Access Library Journal*, **7**: e6719.
https://doi.org/10.4236/oalib.1106719

Received: August 17, 2020
Accepted: September 19, 2020
Published: September 22, 2020

Abstract

This article is based on the concept of Shape of numbers, introduce more shapes, obtain the calculation formulas and find an explanation of the formulas. By observing and associating, show a law about the symmetry of Ring.

Subject Areas

Discrete Mathematics

Keywords

Shape of Numbers, Calculation Formula, Symmetry, Combinatorics, Ring

1. Introduction

Peng, J. introduced the definition of Shape of numbers in [1]:

$$(K_1, K_2, \cdots, K_M), K_i \in N, K_1 < K_2 < \cdots < K_M$$

there are *M*-1 intervals between adjacent numbers. Use *A* for continuity and *B* for discontinuity, record as a string of *M*-1 characters (e.g. *AABB*...) to represents a catalog.

Define collection of a catalog as Shape of numbers. Use the symbol *PX* to represent a catelog (if *M* = 1 then *PX* = 1).

The single $(K_1, K_2, \cdots, K_M)$ is an Item, $K_1 K_2 \cdots K_M$ is the product of an item.

For example:

$$(1,2,4),(1,2,8),(2,3,6) \in PX = AB, \ (1,3,5),(1,3,6),(2,4,6) \in PX = BB,$$

$$(1,2,5,6),(2,3,6,7),(2,3,7,8),(1002,1003,6789,6790) \in PX = ABA$$

$PM(PX)$ **= Count of numbers of *PX*,** $PA(PX)$ **= Count of *A*,** $PB(PX)$

= Count of B

$$\rightarrow PM(PX) = PA(PX) + PB(PX) + 1$$

$|PX|$ = Count of items belonging to PX

$MIN(PX)$ = Minimum product of PX: $MIN(AA) = 1\times2\times3$, $MIN(AB) = 1\times2\times4$

$IDX(PX) = 2 + PA(PX) + 2\times PB(PX) = PM(PX) + PB(PX) + 1$: $IDX(AA) = 4$, $IDX(AB) = 5$

$SUM(N, PX)$ = Sum of all product of items belonging to PX in $[1, N-1]$

For example: $SUM(6, AB) = 1\times2\times4 + 1\times2\times5 + 2\times3\times5$

$END(N, PX)$ = Set of items belonging to PX with the maximum factor = N-1

For example: $END(6, B) = \{(1,5),(2,5),(3,5)\}$

[1] obtain the conclusion:

$$|\text{Items in } SUM(N, PX)| = \binom{N - PM(PX)}{PB(PX)+1} = \binom{N - IDX(PX) + PB(PX) + 1}{PB(PX)+1} \quad (1)$$

$$SUM(N, PX) = MIN(PX)\binom{N}{IDX(PX)} \quad (2)$$

Definition: Subdividethe $(K_1, K_2, \cdots, K_M)$ by interval of adjacent numbers. If the discontinuity interval is $D > 1$, the interval of adjacent numbers $\geq D$ is classified into a same catelog. Use the symbol PY represent a catelog and represented by [min item]

For example:

$$PY = [1,3],\ (1,3),(1,4),(2,4),(1,5),(2,5),(3,5) \in PY.\ \text{Same as } PX = B$$

$$PY = [1,4],\ (1,4),(1,5),(2,5),(1,6),(2,6),(3,6) \in PY,\ (3,5),(4,6) \notin PY$$

$$PY = [1,4,6],\ (1,4,7),(1,5,7),(2,5,7) \in PY,\ (3,5,7) \notin PY$$

Redefinition: $PB(PY)$ = Count of discontinuity intervals in PY, compatible with PX

Redefinition: $IDX(PY)$ = The maximum factor of $MIN(PY)$ + 1, compatible with PX

Definition: $BASE(PY) = PX$, If $PB(PX) = PB(PY)$, $PM(PX) = PM(PY)$, PX has discontinuity intervals at the same positions of PY.

For example:

$$PY = [1,3],[1,4],[1,K>2];\ BASE(PY) = [1,3] = B$$

$$PY = [1,3,4],[1,4,5],[1,K>2,X=K+1];\ BASE(PY) = [1,3,4] = BA$$

$$PY = [1,3,5],[1,4,9],[1,K>2,X>K+1];\ BASE(PY) = [1,3,5] = BB$$

Definition: if $f(n) = \sum K_i \binom{N-n_i}{m_i}$, then $D^1 f(n) = \sum K_i \binom{N-n_i-1}{m_i-1}$

Definition: $PH(PY) = IDX(PY) - PB(PY) - 2$ = Maximum factor of $MIN(PY) - PB(PY) - 1$

$$PY = [1, K_1 \cdots K_M], \quad BS = BASE(PY) = [1, G_1 \cdots G_M] \rightarrow$$
$$PH(PY) = K_M - IDX(BS) + PM(BS) \tag{1.0}$$

[Proof]

$$\begin{aligned} PH(PY) &= K_M - PB(PY) - 1 = K_M - PB(BS) - 1 = K_M - G_M + M \\ &= K_M - (G_M + 1) + (M + 1) = \text{Right} \end{aligned}$$

$$\sum_{n=K}^{N-1} n \binom{n-K}{M} = (M+1)\binom{N-K}{M+2} + (M+K)\binom{N-K}{M+1} \tag{1.1}$$

[Proof]

$$\begin{aligned} \text{Left} &= \sum_{n=K}^{N-1} (n-K+1)\binom{n-K}{M} + (K-1)\sum_{n=K}^{N-1}\binom{n-K}{M} \\ &= \sum_{n=K}^{N-1} (M+1)\binom{n-K+1}{M+1} + (K-1)\sum_{n=K}^{N-1}\binom{n-K}{M} \\ &= (M+1)\binom{N-K+1}{M+2} + (K-1)\binom{N-K}{M+1} \\ &= (M+1)\binom{N-K}{M+2} + (M+1)\binom{N-K}{M+1} + (K-1)\binom{N-K}{M+1} = \text{Right} \end{aligned}$$

By definition:

$$SUM(N, PY) = \sum_{n=0}^{N} \sum END(n, PY) \tag{1.2}$$

Derived from (1.2)

$$\sum END(N, PY) = D^1 SUM(N, PY) \tag{1.3}$$

By definition:

$$SUM(N, [1, \cdots, K, K+1]) = \sum_{n=0}^{N-1} n \times \sum END(n, [1, \cdots, K]) \tag{1.4}$$

$$SUM(N, [1, \cdots, K, X > K+1]) = \sum_{n=0}^{N-1} n \times SUM(n - X + K + 1, [1, \cdots, K]) \tag{1.5}$$

According to the method in [1]:

$$\begin{aligned} |\text{Items in } SUM(N, PY)| &= \binom{N - IDX(PY) + PB(PY) + 1}{PB(PY) + 1} \\ &= \binom{N - PH(PY) - 1}{PB(PY) + 1}, \end{aligned} \tag{1.6}$$

compatible with (1).

2. Calculation Formula

If $PY = BASE(PY)$, the calculation formula has been given by (2).

Otherwise, it can be deduced from (1.1)-(1.5).

$$SUM(N, [1, K \geq 3]) = 3\binom{N-K+2}{4} + K\binom{N-K+2}{3} \tag{2.1}$$

[Proof]

$$\text{Left} \xrightarrow{1.2)} \sum_{n=0}^{N} \sum END\left(n,[1,K]\right) = \sum_{n=K+1}^{N} (n-1)\left(1+2+\cdots+(n-K)\right)$$

$$= \sum_{n=K+1}^{N} (n-1)\binom{n-K+1}{2} = \sum_{n=K}^{N-1} n\binom{n-K+2}{2} \xrightarrow{1.1)} \text{Right}$$

$$SUM\left(N,[1,2,K>3]\right) = 2\times 4\binom{N-K+3}{5} + 2\times K\binom{N-K+3}{4} \tag{2.2}$$

[Proof]

$$\xrightarrow{1.5)} \sum_{n=0}^{N-1} n\times SUM\left(n-K+3,[1,2]\right)$$

$$\xrightarrow{(2)} \sum_{n=0}^{N-1} n\times 2\binom{n-K+3}{3} \xrightarrow{1.1)} \text{Right}$$

$$\begin{aligned} &SUM\left(N,[1,K\geq 3, X=K+1]\right) \\ &=12\binom{N-X+2}{5} + 3(X+1)\binom{N-X+2}{4} \\ &\quad +3K\binom{N-X+2}{4} + KX\binom{N-X+2}{3} \end{aligned} \tag{2.3}$$

[Proof]

$$\xrightarrow{1.2)\,1.4)\,2.1)} \sum_{n=0}^{N-1} n\times D^1\left[3\binom{n-K+2}{4} + K\binom{n-K+2}{3}\right]$$

$$= \sum_{n=0}^{N-1} n\times\left[3\binom{n-K+1}{3} + K\binom{n-K+1}{2}\right] \xrightarrow{1.1)}$$

$$=12\binom{N-K+1}{5} + 3(K+2)\binom{N-K+1}{4} + 3K\binom{N-K+1}{4}$$

$$+K(K+1)\binom{N-K+1}{3}$$

$$= \text{Right}$$

$$\begin{aligned} &SUM\left(N,[1,K\geq 3,\ X>K+1]\right) \\ &=15\binom{N-X+3}{6} + 3(X+1)\binom{N-X+3}{5} \\ &\quad +4K\binom{N-X+3}{5} + KX\binom{N-X+3}{4} \end{aligned} \tag{2.4}$$

[Proof]

$$\xrightarrow{1.5)\,2.1)} \sum_{n=0}^{N-1} n\times\left[3\binom{n-X+3}{4} + K\binom{n-X+3}{3}\right] \xrightarrow{1.1)} \text{Right}$$

2.1. $SUM\left(N,[1,K_1\geq 3,\cdots,K_M]\right)$

$$PY=[1,K_1\geq 3,\cdots,K_M],\quad BASE(PY)=BS=[1,G_1,\cdots,G_M]$$

1*) $SUM(N,PY)=\sum A_i\binom{P}{M_i}$, 2^M items in total.

2*) $M_i = IDX(BS), IDX(BS)-1, \cdots, IDX(BS)-M$

3*) $P = N - PH(PY)$

Use the form $(G_1+K_1)(G_2+K_2)\cdots(G_M+K_M) = \sum X_1 X_2 \cdots X_M$. The expansion function has 2^M items in total.

4*) $M_i = IDX(BS) - (\text{Count of } X \in K)$

5*) $A_i = \prod_{i=1}^{M}(X_i + D_i)$,

$$D_i = \begin{cases} -m, X_i \in G, m = \text{count of } \{X_i \cdots X_{i-1}\} \in K & 5.1^*) \\ +m, X_i \in K, m = \text{count of } \{X_i \cdots X_{i-1}\} \in G & 5.2^*) \end{cases}$$

[Proof]

$$PY = [1, K_1 \geq 3, \cdots, K_M],\quad BS = BASE(PY)$$

Suppose $SUM(N, PY) = \sum X_1 \cdots X_M \binom{P}{M_i}$, $P = N - PH(PY)$.

According to inductive hypothesis:

$$\begin{aligned} &(G_1+K_1)(G_2+K_2)\cdots(G_M+K_M)\{(G_M+1)+(K_M+1)\} \\ &= \sum X_1 X_2 \cdots X_M \{(G_M+1)+(K_M+1)\} \end{aligned}$$

$$C = \text{Count of } X \in K,\quad M - C = \text{Count of } X \in G,\quad M_i = IDX(BS) - C$$

$$PY1 = [PY, K_M + 1],\quad BS1 = BASE(PY1) \tag{2.1.1}$$

$$\begin{aligned} &SUM(N, PY1) \xrightarrow{1,2)\,1.3)} \sum_{n=0}^{N-1} n \times D^1 SUM(n, PY) \\ &= \sum_{n=0}^{N-1} n \times \sum X_1 \cdots X_M \binom{P-1}{M_i - 1} \xrightarrow{1.1)} \\ &= \sum \left(X_1 \cdots X_M M_i \binom{P-1}{M_i+1} + X_1 \cdots X_M (PH(PY) + M_i) \binom{P-1}{M_i} \right) \\ &= \sum X_1 \cdots X_M (IDX(BS) - C) \binom{P-1}{M_i+1} \\ &\quad + \sum X_1 \cdots X_M (PH(PY) + IDX(BS) - C) \binom{P-1}{M_i} \\ &= \sum X_1 \cdots X_M ((G_M+1) - C) \binom{P-1}{M_i+1} \\ &\quad + \sum X_1 \cdots X_M ((K_M+1) + (PM(BS) - 1) - C) \binom{P-1}{M_i} \\ &= \sum X_1 \cdots X_M (G_{M+1} - C) \binom{P-1}{IDX(BS) - C + 1} \\ &\quad + \sum X_1 \cdots X_M (K_{M+1} + M - C) \binom{P-1}{IDX(BS) - C} \\ &= \sum X_1 \cdots X_M (G_{M+1} - C) \binom{P-1}{IDX(BS1) - C} \\ &\quad + \sum X_1 \cdots X_M (K_{M+1} + M - C) \binom{P-1}{IDX(BS1) - (C+1)} \end{aligned}$$

1*) is obvious.

M_i change form $IDX(BS),\cdots,IDX(BS)-M$ to

$IDX(BS)+1,\cdots,IDX(BS)-M = IDX(BS1),\cdots,IDX(BS1)-(M+1)$

2*) proved

$$\begin{aligned} P-1 &= N-PH(PY)-1 = N-\left(K_M - IDX(BS)+PM(BS)\right)-1 \\ &= N-\left\{\left(K_M+1\right)-\left(IDX(BS)+1\right)+\left(PM(BS)+1\right)\right\} \\ &= N-\left(K_{M+1}-IDX(BS1)+PM(BS1)\right) = N-PH(PY1) \end{aligned}$$

3*) proved

$$X_1\cdots X_M\left(G_{M+1}-C\right)\binom{P-1}{IDX(BS1)-C} \to \text{ 4*) 5.1*)}$$

$$X_1\cdots X_M\left(G_{M+1}-C\right)\binom{P-1}{IDX(BS1)-(C+1)} \to \text{ 4*) 5.2*)}$$

4*) 5*) proved

$$PY1=\left[PY,K_{M+1}>K_M+1\right],\quad BS1=BASE(PY1) \tag{2.1.2}$$

$$\begin{aligned} &SUM(N,PY1) \xrightarrow{1.5)} \sum_{n=0}^{N-1} n\times SUM\left(n-K_{M+1}+K_M+1,PY\right) \\ &= \sum_{n=0}^{N-1} n\times\sum X_1\cdots X_M\binom{n-K_{M+1}+K_M+1-PH(PY)}{M_i} \\ &= \sum_{n=0}^{N-1} n\times\sum X_1\cdots X_M\binom{n-K_{M+1}+K_M+1-\left(K_M-IDX(BS)+PM(BS)\right)}{M_i} \\ &= \sum_{n=0}^{N-1} n\times\sum X_1\cdots X_M\binom{n-\left(K_{M+1}-IDX(BS)+PM(BS)-1\right)}{M_i} \\ &= \sum X_1\cdots X_M\left(M_i+1\right)\binom{P1}{M_i+2} \\ &\quad+\sum X_1\cdots X_M\left(K_{M+1}-IDX(BS)+M+M_i\right)\binom{P1}{M_i+1} \\ &= \sum X_1\cdots X_M\left(IDX(BS)-C+1\right)\binom{P1}{M_i+2} \\ &\quad+\sum X_1\cdots X_M\left(K_{M+1}+M-C\right)\binom{P1}{M_i+1} \\ &= \sum X_1\cdots X_M\left(G_{M+1}-C\right)\binom{P1}{IDX(BS)-C+2} \\ &\quad+\sum X_1\cdots X_M\left(K_{M+1}+M-C\right)\binom{P1}{IDX(BS)-C+1} \\ &= \sum X_1\cdots X_M\left(G_{M+1}-C\right)\binom{P1}{IDX(BS1)-C} \\ &\quad+\sum X_1\cdots X_M\left(K_{M+1}+M-C\right)\binom{P1}{IDX(BS1)-(C+1)} \end{aligned}$$

1*) is obvious.

M_i change form $IDX(BS),\cdots,IDX(BS)-M$ to
$IDX(BS)+2,\cdots,IDX(BS)-M+1=IDX(BS1),\cdots,IDX(BS1)-(M+1)$

2*) proved

$$P1=N-\left\{K_{M+1}-\left(IDX(BS)+2\right)+\left(PM(BS)+1\right)\right\}=N-PH(PY1)$$

3*) proved

$$X_1\cdots X_M\left(G_{M+1}-C\right)\binom{P1}{IDX(BS1)-C}\to\ 4^*)\ 5.1^*)$$

$$X_1\cdots X_M\left(K_{M+1}+M-C\right)\binom{P1}{IDX(BS1)-(C+1)}\to\ 4^*)\ 5.2^*)$$

4*) 5*) proved

q.e.d.

Example 2.1:

$$\begin{aligned}&(3+K_1)(5+K_2)(7+K_3)\\&=3\times5\times7+3\times5\times K_3+3\times K_2\times7+3\times K_2\times K_3+K_1\times5\times7\\&\quad+K_1\times5\times K_3+K_1\times K_2\times7+K_1\times K_2\times K_3\end{aligned}$$

$$P=N-K_3+IDX([1,3,5,7])-PM([1,3,5,7])=N-K_3+8-4=N-K_3+4$$

$$\begin{aligned}&SUM\left(N,[1,K_1\ge3,K_2\ge K_1+2,K_3\ge K_2+2]\right)\\&=3\times5\times7\binom{P}{8}+3\times5\times(K_3+2)\binom{P}{7}+3\times(K_2+1)\times(7-1)\binom{P}{7}\\&\quad+3\times(K_2+1)\times(K_3+1)\binom{P}{7}+K_1\times(5-1)\times(7-1)\binom{P}{7}\\&\quad+K_1\times(5-1)\times(K_3+1)\binom{P}{6}+K_1\times K_2\times(7-2)\binom{P}{6}\\&\quad+K_1\times K_2\times K_3\binom{P}{5}\end{aligned}$$

2.2. $SUM\left(N,[1,2,\cdots,n,K_1\ge n+2,\cdots,K_M]\right)$

$$PY=[1,2,3,\cdots,n,K_1\ge n+2,\cdots,K_M],$$

$$BASE(PY)=BS=[1,2,3,\cdots,n,G_1,\cdots,G_M]$$

1*) $SUM(N,PY)=\sum A_i\binom{P}{M_i}$, 2^M items in total.

2*) $M_i=IDX(BS),IDX(BS)-1,\cdots,IDX(BS)-M$

3*) $P=N-K_M+IDX(BS)-(M+1)=N-\left(K_M-PB(PY)-1\right)=N-PH(PY)$

Use the form $(G_1+K_1)(G_2+K_2)\cdots(G_M+K_M)=\sum X_1X_2\cdots X_M$.

4*) $M_i=IDX(BS)-(\text{Count of } X\in K)$

5*) $A_i=n!\times$ (Same as 2.1)

Example 2.2:

$$(5+K_1)(7+K_2)=5\times7+5\times K_2+K_1\times7+K_1\times K_2,$$

$$P = N - K_2 + IDX([1,2,3,5,7]) - 3 = N - K_2 + 5$$

$$SUM(N,[1,2,3,K_1 \ge 5, K_2 \ge K_1 + 2])$$
$$= 3!\left\{5\times 7\binom{P}{8} + 5\times(K_2+1)\binom{P}{7} + K_1\times(7-1)\binom{P}{7} + K_1K_2\binom{P}{6}\right\}$$

2.3. The Meaning of the Expansion of SUM (N, PY)

$$PY = [1, K_1, \cdots, K_M], \quad \text{an Item} = \{\text{begin}, K_1 + E_1, \cdots, K_M + E_M\}$$

$$(\text{product of an item}) = \text{begin}\times(K_1+E_1)\cdots(K_M+E_M) = \text{begin}\times\sum F_1\cdots F_M$$

$$F_i = E \quad (\text{means} \quad F_i = E_i) \text{ or } \quad F_i = K_i,$$

$$SUM(N,PY) = \sum \text{product} = \sum\sum \text{begin}\times F_1\cdots F_M \qquad (2.3.1^*)$$

Define 2.3. $SUM\_K(N, PY, PF = F_1F_2\cdots F_M) = \sum \text{Items}$ **in (2.3.1*) with the same** ***PF***

Example 2.3:

$$SUM(N,[1,K_1 \ge 3, K_2 \ge K_1 + 2])$$
$$= 15\binom{N-K_2+3}{6} + 3(K_2+1)\binom{N-K_2+3}{5}$$
$$+ 4K_1\binom{N-K_2+3}{5} + K_1K_2\binom{N-K_2+3}{4}$$
$$= 15\binom{N-K_2+3}{6} + 3\binom{N-K_2+3}{5} + 3K_2\binom{N-K_2+3}{5}$$
$$+ 4K_1\binom{N-K_2+3}{5} + K_1K_2\binom{N-K_2+3}{4}$$
$$= \sum_{n=0}^{N-K_2}\sum n\times(K_1+E_{1,i})(K_2+E_{2,i})$$

then can prove

$$SUM\_K(N,PY,EE)$$
$$= \sum_{\text{all items}} \text{begin} * E_{1,i}E_{2,i} = 15\binom{N-K_2+3}{6} + 3\binom{N-K_2+3}{5}$$

$$SUM\_K(N,PY,K_1E) = \sum_{\text{all items}} \text{begin} * K_1E_{2,i} = 4K_1\binom{N-K_2+3}{5}$$

$$SUM\_K(N,PY,EK_2) = \sum_{\text{all items}} \text{begin} * E_{1,i}K_2 = 3K_2\binom{N-K_2+3}{5}$$

$$SUM\_K(N,PY,K_1K_2) = \sum_{\text{all items}} \text{begin} * K_1K_2 = K_1K_2\binom{N-K_2+3}{4}$$

$$SUM(N,PY) = \sum\prod_{i=1}^{M}(X_i + D_i)\binom{P}{M_i},$$

$$X_i + D_i = \begin{cases} \{G_i - D_i\}, X_i \in G, D_i = \text{count of } \{X_1 \cdots X_{i-1}\} \in K \\ \{K_i\} + \{D_i\}, X_i \in K, D_i = \text{count of } \{X_1 \cdots X_{i-1}\} \in G \end{cases} \tag{2.3.2*}$$

Theorem 2.3 $SUM\_K(N, PY, PF) = \sum \text{Items}$ **in (2.3.2*) expand by {}, factors has the same** $\{K_i\} \in F$

$$= \sum \prod_{i=1}^{M} Y_i \binom{P}{M_i},$$

$$Y_i = \begin{cases} 0, F_i \in K, X_i \in G \\ K_i, F_i \in K, X_i \in K \\ G_i - D_i, F_i = E, X_i \in G, D_i = \text{count of } \{\cdots X_{i-1}\} \in K \\ D_i, F_i = E, X_i \in K, D_i = \text{count of } \{\cdots X_{i-1}\} \in G \end{cases} \tag{2.3.3*}$$

$$P = N - PH(PY), \quad M_i = IDX(BS) - (\text{Count of } X \in K)$$

[Proof]

Suppose $SUM\_K(N, PY, PZ) = \sum \prod_{i=1}^{M} Y_i \binom{P}{M_i}$, $P = N - PH(PY)$.

$$PY1 = [PY, K_{M+1}], \quad C = \text{Count of } X \in K,$$

$$M - C = \text{Count of } X \in G, \quad M_i = IDX(BS) - C$$

When $K_{M+1} = K_M + 1$

$$SUM(N, PY1) \xrightarrow{1,2)\,1.3)} \sum_{n=0}^{N-1} n \times D^1 SUM(n, PY)$$
$$\to \sum_{n=0}^{N-1} n \times D^1 SUM\_K(n, PY, PF)$$
$$= \sum_{n=0}^{N-1} n \times \sum \prod_{i=1}^{M} Y_i \binom{P-1}{M_i - 1} \xrightarrow{\text{reference (2.1.1)}}$$
$$= \sum \prod_{i=1}^{M} Y_i (G_{M+1} - C) \binom{P-1}{IDX(BS1) - C}$$
$$+ \sum \prod_{i=1}^{M} Y_i (M - C) \binom{P-1}{IDX(BS1) - (C+1)}$$
$$+ \sum \prod_{i=1}^{M} Y_i K_{M+1} \binom{P-1}{IDX(BS1) - (C+1)}$$
$$= SUM\_K(N, PY1, [PF, E]) + SUM\_K(N, PY1, [PF, K_{M+1}])$$

$$P - 1 = N - PH(PY1) \to \tag{2.3.3*}$$

$$SUM\_K(N, PY1, [PF, K_{M+1}]) = \sum \prod_{i=1}^{M} Y_i K_{M+1} \binom{P-1}{IDX(BS1) - (C+1)} \to \text{(2.3.3*)}$$

$$SUM\_K(N, PY1, [PF, E])$$
$$= \sum \prod_{i=1}^{M} Y_i (G_{M+1} - C) \binom{P-1}{IDX(BS1) - C}$$
$$+ \sum \prod_{i=1}^{M} Y_i (M - C) \binom{P-1}{IDX(BS1) - (C+1)} \to \tag{2.3.3*}$$

When $K_{M+1} > K_M + 1$

$$SUM(N, PY1) \xrightarrow{1,5)} \sum_{n=0}^{N-1} n \times SUM(n - K_{M+1} + K_M + 1, PY)$$
$$\to \sum_{n=0}^{N-1} n \times SUM\_K(n - K_{M+1} + K_M + 1, PY, PF)$$
$$= \sum_{n=0}^{N-1} n \times \sum \prod_{i=1}^{M} Y_i \binom{P1}{M_i} \xrightarrow{\text{reference (2.1.2)}}$$
$$= \sum \prod_{i=1}^{M} Y_i (G_{M+1} - C) \binom{P1}{IDX(BS1) - C}$$
$$+ \sum \prod_{i=1}^{M} Y_i (M - C) \binom{P1}{IDX(BS1) - (C+1)}$$
$$+ \sum \prod_{i=1}^{M} Y_i K_{M+1} \binom{P1}{IDX(BS1) - (C+1)}$$
$$= SUM\_K(N, PY1, [PF, E]) + SUM\_K(N, PY1, [PF, K_{M+1}])$$

$$P1 = N - PH(PY1) \to \tag{2.3.3*}$$

$$SUM\_K(N, PY1, [PF, K_{M+1}]) = \sum \prod_{i=1}^{M} Y_i K_{M+1} \binom{P1}{IDX(BS1) - (C+1)} \to \tag{2.3.3*}$$

$$SUM\_K(N, PY1, [PF, E])$$
$$= \sum \prod_{i=1}^{M} Y_i (G_{M+1} - C) \binom{P1}{IDX(BS1) - C}$$
$$+ \sum \prod_{i=1}^{M} Y_i (M - C) \binom{P1}{IDX(BS1) - (C+1)} \to \tag{2.3.3*}$$

q.e.d.

2.3.1) $SUM\_K(N, PY = [1, 2, \cdots, n, K_1 \ge n+2, \cdots, K_M], PF)$ has the similar conclusions.

2.3.2) $SUM\_K(N, PY = [1, 2, \cdots, n, K_1 \ge n+2, \cdots, K_M], K_1 K_2 \cdots K_M)$,

$$BS = BASE(PY) = MIN(PY) \binom{N - PH(PY)}{IDX(BS) - M}$$
$$= MIN(PY) \sum_{n=0}^{N-1} \binom{n - PH(PY) - 1}{IDX(BS) - M - 1}$$
$$= MIN(PY) \sum_{n=0}^{N-1} \binom{n - PH(PY) - 1}{PB(BS) + 1}$$
$$= MIN(PY) \sum_{n=0}^{N-1} \binom{n - PH(PY) - 1}{PB(PY) + 1}$$
$$= MIN(PY) \sum_{n=0}^{N-1} \left| \text{Items in } SUM(n+1, PY) \right|$$

3. A Theorem of Ring

$$PY = [1, K_1 \ge 3, \cdots, K_M], \quad BASE(PY) = BS = [1, G_1, \cdots, G_M]$$

$$SUM(N, PY) = \sum A_i \binom{P}{M_i},$$

When $PY = BS$, $SUM(N, PY) = MIN(BS)\binom{N}{IDX(BS)}$

This inspired us:

In the form $(G_1 + K_1)(G_2 + K_2)\cdots(G_M + K_M)$, $K_i = G_i \to MIN(BS) \mid \sum A_i$ (M_i is same).

Definition 3.1:

$S, \{K_i\} \in \text{Ring}$; $K_i = K_j$, $K_i < K_j$, $K_i > K_j$ **are allowed.**

Choice N **from** $(K_1, K_2, \cdots, K_M)$, $R = R_1 \cdots R_M$

$R_i = +1$ **Indicates that** K_i **was selected,** $R_i = -1$ **Indicates that** K_i **was unselected,**

$$F(N, M, R, K, S) = \prod_{i=1}^{M}(K_i + D_i),$$

$$D_i = \begin{cases} -mS, R_i = -1, m = \text{count of } \{K_1, \cdots, K_{i-1}\} \text{ selected} \\ +mS, R_i = +1, m = \text{count of } \{K_1, \cdots, K_{i-1}\} \text{ unselected} \end{cases} \qquad (*)$$

When $S = 1$**, abbreviated as** $F(N, M, R, K)$

Definition 3.2:

$H(N_1, N_2, K, S) = \sum F(N_1, N_1 + N_2, R, K, S)$, **Sum traverses all** N_1**-Choice of** K

Theorem 3.1: $H(N_1, N_2, K, S) = \binom{M}{N_1, N_2} K_1 \cdots K_M$, $M = N_1 + N_2$

[Proof]

When $N_1 = 0$ or $N_2 = 0$, it is obvious.

$$\begin{aligned} H(1,1,K,S) &= \text{select } K_1 + \text{select } K_2 \\ &= \text{unselect } K_2 + \text{select } K_2 \\ &= K_1(K_2 - S) + K_1(K_2 + S) \\ &= 2K_1K_2 \end{aligned}$$

Suppose $H(1, M-1, K, S) = \binom{M}{1} K_1 \cdots K_M$

$$\begin{aligned} & H(1, M, K, S) \\ &= \text{select } K_{M+1} + \text{unselect } K_{M+1} \\ &= H(0, M, K, S)(K_{M+1} + MS) + H(1, M-1, K, S)(K_{M+1} - S) \\ &= \binom{M}{0} K_1 \cdots K_M (K_{M+1} + MS) + \binom{M}{1} K_1 \cdots K_M (K_{M+1} - S) \\ &= \left\{\binom{M}{0} + \binom{M}{1}\right\} K_1 \cdots K_M K_{M+1} + \left\{M\binom{M}{0} - \binom{M}{1}\right\} S \times K_1 \cdots K_M K_{M+1} \\ &= \binom{M+1}{1} K_1 \cdots K_M K_{M+1} \end{aligned}$$

$\to H(1, N_2, K, S)$ holds $\to$ Symmetry $\to H(N_2, 1, K, S)$ holds

Suppose $H(N_1, M - N_1, K, S) = \binom{M}{N_1} K_1 \cdots K_M$

$$
\begin{aligned}
&H\left(N_1+1, M-N_1, K, S\right)=\text{select } K_{M+1}+\text{unselect } K_{M+1} \\
&=H\left(N_1, M-N_1, K, S\right)\left(K_{M+1}+\left(M-N_1\right) S\right) \\
&\quad+H\left(N_1+1, M-N_1-1, K, S\right)\left(K_{M+1}-\left(N_1+1\right) S\right) \\
&=\binom{M}{N_1} K_1 \cdots K_M\left(K_{M+1}+\left(M-N_1\right) S\right) \\
&\quad+\binom{M}{N_1+1} K_1 \cdots K_M\left(K_{M+1}-\left(N_1+1\right) S\right) \\
&=\left\{\binom{M}{N_1}+\binom{M}{N_1+1}\right\} K_1 \cdots K_{M+1} \\
&\quad+\left\{\left(M-N_1\right)\binom{M}{N_1}-\left(N_1+1\right)\binom{M}{N_1+1}\right\} S \times K_1 \cdots K_{M+1} \\
&=\binom{M+1}{N_1+1} K_1 \cdots K_M K_{M+1}
\end{aligned}
$$

$\rightarrow H\left(N_1, N_2, K, S\right)$ holds

q.e.d.

Example 3.1: $\left\{K_1, K_2, K_3\right\}=\{A, B, C\}$

$$
\begin{aligned}
H(1,2, K) &=\text{select } A+\text{select } B+\text{select } C \\
&=A(B-1)(C-1)+A(B+1)(C-1)+AB(C+2) \\
&=3ABC
\end{aligned}
$$

$$
\begin{aligned}
H(2,1, K) &=\text{select } AB+\text{select } BC+\text{select } AC \\
&=AB(C-2)+A(B+1)(C+1)+A(B-1)(C+1) \\
&=3ABC
\end{aligned}
$$

Definition 3.3:

K_i come from q sources: $S_1, S_2, \cdots, S_q$**,** $K_i \in S_j$ **indicates K_i come from S_j** $diff\left(K_i, K_j\right)=diff\left(K_i \in S_a, K_j \in S_b\right)=diff(a, b)$ **,** $diff(a, b)=S$ **, (** $a \neq b$ **);** $diff(a, b)=-diff(b, a)$**;** $diff(a, a)=0$ **then can change related parts of definition 3.1 to**

$$
\prod_{i=1}^M\left(K_i+D_i\right), D_i=\sum_{j<i} diff\left(K_i, K_j\right)
$$

and define $H\left(N_1, N_2, \cdots, N_q, K, S\right)$

Theorem 3.2 $H\left(N_1, N_2, \cdots, N_q, K, S\right)=\binom{M}{N_1, N_2, \cdots, N_q} K_1 \cdots K_M$,

$M=N_1+N_2+\cdots+N_q$

[Proof]

Only need to prove S=1, and specify $diff(a, b)=1$, ($a<b$)

$H\left(N_1, N_2, K\right)=\sum \text{Item}$, $K_i \in\left\{S_1, S_2\right\}$. An item has M factors,

Choice N_1 factors, K_i in these factors, $K_i \in S_1$, It is called invariant factor{F}.

Others are called variable factors {V}

$L_2+L_3=N_2$, choice L_2 factors in {V} to S_2, L_3 to S_3

By definition, $H\left(L_2, L_3, V\right)=\sum\left\{\left(L_2, L_3\right)-\text{Choice of } V\right\}$

$$\sum\left(\prod\{F\}\times\sum\{(L_2,L_3)-\text{Choice of } V\}\right)$$
$$=\sum\left(\prod\{F\}\times H(L_2,L_3,V)\right)=\sum\left(\prod\{F\}\times\binom{N_2}{L_1,L_2}\times\prod\{V\}\right)$$
$$=\binom{N_2}{L_2,L_3}\sum\text{Item}=\binom{N_2}{L_2,L_3}\binom{M}{N_1,N_2}K_1\cdots K_M$$
$$=\binom{M}{N_1,L_2,L_3}K_1\cdots K_M=H(N_1,L_2,L_3,K)$$

and items gives all (N_1, L_2, L_3)-Choice of K.

Prove $\sum\left(\prod\{F\}\times\sum\{(L_2,L_3)-\text{Choice of } V\}\right)$ satisfies the definition of $H(N_1,\cdots,K)$

Let $\text{Item}=\prod_{i=1}^{M}(K_i+D_i)$, $(\text{an item changed})=\prod_{i=1}^{M}(H_i+E_i)$, $K_i=H_i$

$$A1=\text{Count of }\{K_1,\cdots,K_{i-1}\}\in S_1,\quad A2=\text{Count of }\{K_1,\cdots,K_{i-1}\}\in S_2$$
$$B1=\text{Count of }\{\cdots,H_{i-1}\}\in S_1,\quad B2=\text{Count of }\{\cdots,H_{i-1}\}\in S_2,$$
$$B3=\text{Count of }\{\cdots,H_{i-1}\}\in S_3$$
$$\to A1=B1,\quad A2=B2+B3$$
$$K_{i+1}\in S_1\to H_{i+1}\in S_1\to D_{i+1}=A2=i-A1=i-B1=B2+B3=E_{i+1}$$

$\to$ The invariant factors match the definition

$$K_{i+1}\in S_2,H_{i+1}\in S_2\xrightarrow{\text{definition of } H(L_2,L_3,V)}E_i=D_i+B3=-A1+B3=-B1+B3$$
$$\xrightarrow{\text{definition of } H(N,L_2,L_3,V)}\text{Match the definition}$$
$$K_{i+1}\in S_2,H_{i+1}\in S_3\xrightarrow{\text{definition of } H(L_2,L_3,V)}E_i=D_i-B2=-A1-B2=-B1-B2$$
$$\xrightarrow{\text{definition of } H(N,L_2,L_3,V)}\text{Match the definition}$$
$$\to\ \text{Match the definition of } H(N_1,N_2,N_3,K)\xrightarrow{\text{recursion}}\cdots$$

q.e.d.

Example 3.2 $\{K_1,K_2,K_3\}=\{A,B,C\}$

$$H(1,2,K)=A(B-1)(C-1)+A(B+1)(C-1)+AB(C+2)=3ABC$$
$$A(B-1)(C-1)\in S_1S_2S_2\to\{F\}=\{A\}\to A\{(B-1)(C-2)+(B-1)C\}$$
$$\to A(B-1)(C-2)+A(B-1)C\in S_1S_2S_3+S_1S_3S_2$$
$$A(B+1)(C-1)\in S_2S_1S_2\to\{F\}=\{B+1\}\to(B+1)\{A(C-2)+AC\}$$
$$\to A(B+1)(C-2)+A(B+1)C\in S_2S_1S_3+S_3S_1S_2$$
$$AB(C+2)\in S_2S_2S_1\to\{F\}=\{C+2\}\to(C+2)\{A(B-1)+A(B+1)\}$$
$$\to A(B-1)(C+2)+A(B+1)(C+2)\in S_2S_3S_1+S_3S_2S_1$$
$$H(2,1,K)=AB(C-2)+A(B+1)(C+1)+A(B-1)(C+1)=3ABC$$
$$AB(C-2)\in S_1S_1S_2\to\{F\}=\{C-2\}\to(C-2)\{A(B-1)+A(B+1)\}$$
$$\to A(B-1)(C-2)+A(B+1)(C-2)\in S_1S_2S_3+S_2S_1S_3$$

$$A(B+1)(C+1) \in S_2S_1S_1 \to \{F\} = \{A\} \to A\{(B+1)C + (B+1)(C+2)\}$$
$$\to A(B+1)C + A(B+1)(C+2) \in S_3S_1S_2 + S_3S_2S_1$$
$$A(B-1)(C+1) \in S_1S_2S_1 \to \{F\} = \{B-1\} \to (B-1)\{AC + A(C+2)\}$$
$$\to A(B-1)C + A(B-1)(C+2) \in S_1S_3S_2 + S_2S_3S_1$$
$$\begin{aligned} H(1,1,1,K) &= A(B-1)(C-2) + A(B-1)C + A(B+1)(C-2) \\ &\quad + A(B+1)C + A(B-1)(C+2) + A(B+1)(C+2) \\ &= 6ABC \end{aligned}$$

3.1) $(K_1, K_2, \cdots, K_M)$ is permutable in $H(N_1, N_2, \cdots, N_q, K, S)$

3.2) $H(N_1, N_2, \cdots, N_q, K, S) = \prod_{i=1}^{M}(K_i + D_i)$

$$\to H(N_q, N_{q-1}, \cdots, N_1, K, S) = \prod_{i=1}^{M}(K_i - D_i)$$

3.3) In the section 2, $SUM(N, PY) = \sum A_i \binom{P}{M_i}$

A_i is generated by the form $(G_1 + K_1)(G_2 + K_2)\cdots(G_M + K_M)$. It can be understood asgenerated by 2-SET: $\{G_i\}$, $\{K_i\}$.

If $K_i = J_{1,i} + J_{2,i} + \cdots + J_{x,i}$, ($J_{x,i} > 0$, $J_{x,i} = 0$, $J_{x,i} < 0$ are allowed),

$$SUM(N, PY) = \sum B_i \binom{P}{M_i}$$

B_i can be understood asgenerated by $(X+1)$-SET: $\{G_i\}, \{J_1\}, \{J_2\}, \cdots, \{J_X\}$

3.4) $K_i = J_{1,i} + J_{2,i} + \cdots + J_{x,i}$, Theorem 2.3 has the similar promotion.

4. Conclusion

Review the whole process, $\binom{N}{M} + \binom{N}{M+1} = \binom{N+1}{M+1} \to$ Basic Shapes in [1] $\to$ More Shapes in this article.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

Acknowledgements

The author is very grateful to Mr. HanSan Z. (Department of Electronic Information, Nanjing University) for his suggestions on this paper.

References

[1] Peng, J. (2020) Shape of Numbers and Calculation Formula of Stirling Numbers. Open Access Library Journal, **7**: e6081. https://doi.org/10.4236/oalib.1106081