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On Tensor Products of Bounded Linear Operators

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Abstract

This paper studies tensor products of bounded linear operators on Hilbert spaces. It establishes bilinearity of the tensor product mapping and shows that key operator properties, including rank-one, finite-rank, compactness, and injectivity, are preserved under tensorization. Spectral behavior is analyzed at the level of eigenvalues, where multiplicativity is obtained. In addition, a structural decomposition for tensor products involving bounded and diagonal operators is derived. The paper also shows that compactness is preserved under tensor products when both factor operators are compact, and in particular when one factor is compact and the other acts on a finite-dimensional space. These results provide a coherent and systematic framework for understanding tensor products of bounded linear operators and lay a foundation for further developments in spectral theory.

Subject Areas

Algebra, Applied Physics

Keywords

Hilbert Spaces, Tensor Product of Operators, Bounded Linear Operators, Compact Operators, Finite-Rank Operators

1. Introduction

In study of theory of composite systems of Hilbert spaces, the theory of tensor products of bounded linear operators forms the center of attention. Considering Hilbert spaces, say H and K, and operators $D \in B(H)$ and $E \in B(K)$, the product $D \otimes E$ is a bounded operator on $H \otimes K$, and the properties of operators can be applied to product spaces. Tensor products have been widely studied in spectral theory, operator spaces, and multilinear analysis (Ichinose, [1]; Paulsen & Smith [2]; Blecher & Paulsen [3]; Kubrusly & Vieira [4]. These studies establish that tensor products preserve fundamental properties such as boundedness and continuity, and provide tools for analyzing operator behavior in product spaces.

In addition, tensor products play a significant role in understanding operator norms, numerical ranges, and spectral characteristics. Several works have examined inequalities and structural properties associated with tensor products (Gau, Wang, & Wu [5]; Saito [6]; Dash [7], highlighting their importance in both matrix analysis and operator theory. Related studies on numerical radius and operator inequalities (Shebrawi & Albadawi [8]; Bhunia & Paul [9]; Bhunia, Paul, & Sen [10]) further demonstrate the rich interplay between tensor structures and operator behavior.

Despite these advances, a unified structural characterization of tensor products of bounded linear operators particularly concerning preservation of operator classes and structural decomposition remains limited. While previous work has addressed specific aspects such as spectral inequalities or numerical radius bounds (Hirzallah, Kittaneh, & Shebrawi [11]; Ismailov & Ipek [12]), a systematic framework integrating algebraic, analytical, and structural properties under tensorization is still lacking.

Motivated by this gap, this paper develops a systematic framework for understanding tensor products of bounded linear operators on Hilbert spaces. Specifically, we establish bilinearity of the tensor product mapping and show that tensor products preserve key operator classes, including rank-one, finite-rank, and compact operators. We derive spectral properties at the level of eigenvalues, demonstrating their multiplicativity, and prove that injectivity is preserved under tensor products. Define the algebraic tensor product and the completed Hilbert tensor product separately at first mention. Let $H \odot K$ denote the algebraic tensor product, consisting of finite linear combinations of elementary tensors, and let $H \otimes K$ denote its Hilbert space completion under the induced inner product.

Then state explicitly that for $D \in B(H)$ and $E \in B(K)$, the operator $D \otimes E$ is first defined on the algebraic tensor product $H \odot K$ by

$$(D \otimes E)(x \otimes y) = (Dx) \otimes (Ey), \ x \in H, \ y \in K,$$

Extended linearly to all of $H \odot K$, and then extended uniquely by continuity to a bounded linear operator on the completed tensor product $H \otimes K$. Finally, we provide a detailed characterization of rank-one operators and their tensor products, including explicit norm computations.

These results provide a unified framework that integrates algebraic, structural, and analytical properties of tensor products of operators, forming a foundation for further investigations in spectral theory.

2. Preliminaries

Assume that the Hilbert space H and K are complex. The Banach algebras of bounded operators on H and K are denoted as $B(H)$ and $B(K)$ respectively, equipped with the operator norm, $\|D\| = \sup_{\|x\|=1} \|Dx\|$, $D \in B(H)$. And the space of compact operators denoted by $\mathbb{K}(H)$.

2.1. Tensor Product of Hilbert Spaces

Let H and K be complex Hilbert spaces.

We first distinguish between the algebraic tensor product and the completed Hilbert tensor product. The algebraic tensor product, denoted by $H \odot K$, consists of all finite linear combinations of elementary tensors of the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i, \; x_i \in H, \; y_i \in K.$$

Define an inner product on elementary tensors by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle_H \langle y_1, y_2 \rangle_K,$$

and extend it linearly to $H \odot K$.

The Hilbert tensor product $H \otimes K$ is then defined as the completion of $H \odot K$ with respect to the norm induced by this inner product (Ryan [13]). This construction yields a Hilbert space suitable for the study of composite operator systems.

2.2. Tensor Product of Operators

Let $D \in B(H)$ and $E \in B(K)$. The tensor product operator $D \otimes E$ is first defined on the algebraic tensor product $H \odot K$ by

$$(D \otimes E)(x \otimes y) = (Dx) \otimes (Ey), \; x \in H, y \in K,$$

And extended linearly to all of $H \odot K$ (Ryan [13]).

Moreover, the operator satisfies the norm inequality

$$\|D \otimes E\| \le \|D\| \|E\| \quad \text{(Gau, Wang, \& Wu [5])},$$

which shows that $D \otimes E$ is bounded on $H \odot K$.

Therefore, $D \otimes E$ extends uniquely by continuity to a bounded linear operator on the completed Hilbert tensor product $H \otimes K$ (Ryan [13]; Kubrusly & Vieira [4]).

2.3. Fundamental Properties

The tensor product of bounded linear operators satisfies the following properties:

(i) Bilinearity

The mapping $(D, E) \mapsto D \otimes E$ is linear in each argument (Paulsen & Smith, [2]).

(ii) Adjoint Compatibility

$(D \otimes E)^* = D^* \otimes E^*$ (Kubrusly & Vieira, [4]).

(iii) Compactness Preservation

If $D \in \mathbb{K}(H)$ and $E \in \mathbb{K}(K)$, then $D \otimes E \in \mathbb{K}(H \otimes K)$.

2.4. Rank-One Operators

An operator that is rank one between a normed space, X, and Y is defined as $P(x)=\phi(x)y$, where $\phi\in X^*$ and $y\in Y$.

$P(x)=\phi(x)y$, where $\phi\in X^*$ and $y\in Y$. Such operators are bounded and satisfy $\|P\|=\|\phi\|\|y\|$ (Bhunia & Paul, [9]).

3. Main Results

Proposition 3.1

Suppose $D\in B(H)$ and $E\in B(K)$. Define the mapping $\Phi:B(H)\times B(K)\to B(H\otimes K)$, $\Phi(D,E)=D\otimes E$. Then Φ is bilinear.

Proof

To establish bilinearity, we show that the mapping is linear in each argument separately.

Linearity in the First Variable

Fix $E\in B(K)$. Define the operator $P_{(D,E)}$ on elementary tensors by

$$P_{(D,E)}(x\otimes y)=(Dx)\otimes(Ey),\ x\in H,\ y\in K.$$

Let $D_2\in B(H)$ and $\alpha,\beta\in\mathbb{C}$. Then for each elementary tensor $x\otimes y\in H\odot K$, we get

$$((\alpha D_1+\beta D_2)\otimes E)(x\otimes y)=((\alpha D_1+\beta D_2)x)\otimes(Ey).$$

Using linearity of operators,

$$\alpha(D_1x)\otimes(Ey)+\beta(D_2x)\otimes(Ey)=\alpha(D_1\otimes E)(x\otimes y)+\beta(D_2\otimes E)(x\otimes y).$$

Hence, $(\alpha D_1+\beta D_2)\otimes E=\alpha(D_1\otimes E)+\beta(D_2\otimes E)$ on elementary tensors. By linearity, this identity extends to finite sums in $H\odot K$, and by continuity to completed tensor product space $H\otimes K$.

Furthermore, since norm satisfies

$$\|D\otimes E\|\le\|D\|\|E\|,$$

the operator extends continuously. Therefore, the mapping $D\mapsto D\otimes E$ is linear.

Linearity in the Second Variable

Now fix $D\in B(H)$. Let $E_1,E_2\in B(K)$ and $\alpha,\beta\in\mathbb{C}$. For each $(x\otimes y\in H\odot K)$,

$$(D\otimes(\alpha E_1+\beta E_2))(x\otimes y)=Dx\otimes((\alpha E_1+\beta E_2)y).$$

Using linearity in *K*,

$$\begin{aligned}&=Dx\otimes(\alpha E_1y+\beta E_2y)\\&=\alpha(Dx\otimes E_1y)+\beta(Dx\otimes E_2y)\\&=\alpha(D\otimes E_1)(x\otimes y)+\beta(D\otimes E_2)(x\otimes y).\end{aligned}$$

Thus, $D\otimes(\alpha E_1+\beta E_2)=\alpha(D\otimes E_1)+\beta(D\otimes E_2)$ on $H\odot K$, and by continuity, on $H\otimes K$. Given that the function is linear in both arguments, it follows that Φ is bilinear.

Proposition 3.2

Suppose $D\in B(H)$ and $E\in B(K)$. Let $x\in H$ and $y\in K$ are eigenvectors of D and E, respectively, such that $Dx=\lambda x$ and $Ey=\mu y$, for some scalars $\lambda,\mu\in\mathbb{C}$. Then $(D\otimes E)(x\otimes y)=(\lambda\mu)(x\otimes y)$, therefore $x\otimes y$ is an eigenvector of $D\otimes E$ with eigenvalue $\lambda\mu$.

Proof

According to the tensor product of operators definition, we have

$$(D\otimes E)(x\otimes y)=(Dx)\otimes(Ey).$$

Using the eigenvector relations $Dx=\lambda x$ and $Ey=\mu y$, it can be seen that

$$(D\otimes E)(x\otimes y)=(\lambda x)\otimes(\mu y).$$

By the bi-linearity of the tensor product, scalar multiplication can be factored out as

$$(\lambda x)\otimes(\mu y)=\lambda\mu(x\otimes y).$$

Therefore, $(D\otimes E)(x\otimes y)=\lambda\mu(x\otimes y)$, this demonstrates that $x\otimes y$ is an eigenvector of $D\otimes E$ with eigenvalue $\lambda\mu$.

Proposition 3.3

Let $D\in B(H)$ and $E\in B(K)$ be finite-rank operators. Then the tensor product operator $D\otimes E\in B(H\otimes K)$ is also finite-rank. Moreover, $\operatorname{rank}(D\otimes E)\le\operatorname{rank}(D)\operatorname{rank}(E)$.

Proof. Since D is a finite-rank operator, $\operatorname{Ran}(D)$ is finite-dimensional. Choose a basis

$$\{u_1,u_2,\cdots,u_m\}$$

for $\operatorname{Ran}(D)$, where $m=\operatorname{rank}(D)$. Similarly, since E is finite-rank, choose a basis

$$\{v_1,v_2,\cdots,v_n\}$$

for $\operatorname{Ran}(E)$, where $n=\operatorname{rank}(E)$.

Let $x\otimes y\in H\odot K$ be an elementary tensor. By definition of the tensor product operator,

$$(D\otimes E)(x\otimes y)=Dx\otimes Ey.$$

Since $Dx\in\operatorname{Ran}(D)$ and $Ey\in\operatorname{Ran}(E)$, there exist scalars $\alpha_i,\beta_j\in\mathbb{C}$ such that

$$Dx=\sum_{i=1}^{m}\alpha_i u_i,\ \ Ey=\sum_{j=1}^{n}\beta_j v_j.$$

Therefore,

$$(D\otimes E)(x\otimes y)=\left(\sum_{i=1}^{m}\alpha_i u_i\right)\otimes\left(\sum_{j=1}^{n}\beta_j v_j\right).$$

Using bilinearity of the tensor product, we obtain

$$(D\otimes E)(x\otimes y)=\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j\left(u_i\otimes v_j\right).$$

Hence,

$$(D\otimes E)(x\otimes y)\in \operatorname{span}\left\{u_i\otimes v_j:1\le i\le m,1\le j\le n\right\}.$$

By linearity, the same inclusion holds for every element of the algebraic tensor product $H\odot K$, since every element of $H\odot K$ is a finite linear combination of elementary tensors. Thus,

$$(D\otimes E)(H\odot K)\subseteq \operatorname{span}\left\{u_i\otimes v_j:1\le i\le m,1\le j\le n\right\}.$$

Let

$$M=\operatorname{span}\left\{u_i\otimes v_j:1\le i\le m,1\le j\le n\right\}.$$

Then M is finite-dimensional, and hence closed in $H\otimes K$. Since $H\odot K$ is dense in $H\otimes K$ and $D\otimes E$ is bounded, for every $z\in H\otimes K$ there exists a sequence $(z_r)\subset H\odot K$ such that $z_r\to z$. Therefore,

$$(D\otimes E)z_r\to(D\otimes E)z.$$

But each $(D\otimes E)z_r\in M$, and since M is closed, it follows that

$$(D\otimes E)z\in M.$$

Hence,

$$\operatorname{Ran}(D\otimes E)\subseteq M.$$

Consequently,

$$\operatorname{rank}(D\otimes E)=\dim\operatorname{Ran}(D\otimes E)\le\dim M\le mn=\operatorname{rank}(D)\operatorname{rank}(E).$$

Therefore, $D\otimes E$ is finite-rank.

$$\boxed{\operatorname{rank}(D\otimes E)\le\operatorname{rank}(D)\operatorname{rank}(E).}$$

Lemma 3.4

Let $D\in K(H)$ and $E\in K(K)$ be compact operators. Then the tensor product operator $D\otimes E\in\mathbb{K}(H\otimes K)$ is also compact.

Proof

Since D and E are compact operators, there exist sequences of finite-rank operators $D_n\subset B(H)$ and $E_m\subset B(K)$ such that

$$\|D_n-D\|\to 0\text{ and }\|E_m-E\|\to 0.$$

Consider the difference

$$D\otimes E-D_n\otimes E_m.$$

We decompose this expression as

$$D\otimes E-D_n\otimes E_m=D\otimes(E-E_m)+(D-D_n)\otimes E_m.$$

Using the operator norm inequality

$$\|A \otimes B\| \le \|A\|\|B\|,$$

we obtain

$$\|D \otimes E - D_n \otimes E_m\| \le \|D \otimes (E - E_m)\| + \|(D - D_n) \otimes E_m\|.$$

Hence,

$$\|D \otimes E - D_n \otimes E_m\| \le \|D\|\|E - E_m\| + \|D - D_n\|\|E_m\|.$$

Next, select m big enough to make $E - E_m$ arbitrarily small, and then select n big enough to make $D - D_n$ arbitrarily small. It follows that

$$D_n \otimes E_m \to D \otimes E$$

in the operator norm.

By Proposition 3.3, each operator E_m is finite-rank, and hence compact. The norm limit of compact operator is compact since the set $\mathbb{K}(H \otimes K)$ of compact operators is closed in the operator norm, it follows that the norm limit of compact operators is compact.

Therefore,

$$D \otimes E \in \mathbb{K}(H \otimes K).$$

Corollary 3.5

Let $\mathbb{K} \in K(H)$ be a compact operator and $D \in B(\mathbb{C}^n)$ be a diagonal operator on the finite-dimensional space $\mathbb{C}^n$. Then the tensor product operator $K \otimes D \in B(H \otimes \mathbb{C}^n)$ is compact.

Proof

Since $\mathbb{C}^n$ is finite-dimensional, every bounded linear operator on $\mathbb{C}^n$ is compact. Hence,

$$D \in \mathbb{K}(\mathbb{C}^n).$$

Given that $K \in \mathbb{K}(H)$, by Lemma 3.4 it implies that

$$K \otimes D \in \mathbb{K}(H \otimes \mathbb{C}^n).$$

Therefore, $K \otimes D$ is compact.

Remark 3.6.

Corollary 3.5 should not be interpreted as a necessary and sufficient compactness condition for diagonal operators on ℓ^2. It only states that if one operator is compact and the other acts on a finite-dimensional Hilbert space, then their tensor product is compact.

A full necessary and sufficient condition for compactness of diagonal operators on ℓ^2 requires a separate result, namely that a diagonal operator $D = \operatorname{diag}(\lambda_n)$ on ℓ^2 is compact if and only if $\lambda_n \to 0$.

Proposition 3.7.

Suppose $U \in B(H)$ and let $D \in B(K)$ be a diagonal operator with respect to an orthonormal basis $\{e_\alpha\}_{\alpha \in I}$ of K, that is,

$$De_\alpha = \mu_\alpha e_\alpha,\ \alpha \in I.$$

Then

$$H \otimes K = \bigoplus_{\alpha\in I}\left(H \otimes \operatorname{span}\{e_\alpha\}\right),$$

and, under the natural identification

$$H \otimes \operatorname{span}\{e_\alpha\} \cong H,\ x \otimes e_\alpha \mapsto x,$$

the restriction of $U \otimes D$ to $H \otimes \operatorname{span}\{e_\alpha\}$ corresponds to $\mu_\alpha U$. Consequently,

$$U \otimes D \cong \bigoplus_{\alpha\in I}\mu_\alpha U.$$

Proof

For each $\alpha \in I$, define the subspace

$$M_\alpha = H \otimes \operatorname{span}\{e_\alpha\}.$$

Given that $\{e_\alpha\}_{\alpha\in I}$ has an orthonormal basis of K, it follows that the subspaces $\{M_\alpha\}_{\alpha\in I}$ are mutually orthogonal in $H \otimes K$. Moreover, finite linear combinations of elementary tensors of the form $x \otimes e_\alpha$, with $x \in H$, are dense in $H \otimes K$. Therefore,

$$H \otimes K = \oplus_{\alpha\in I} M_\alpha.$$

Now consider an elementary tensor $x \otimes e_\alpha \in M_\alpha$. According to tensor product of operators definition,

$$(U \otimes D)(x \otimes e_\alpha) = (Ux) \otimes (De_\alpha).$$

Using the diagonal structure of D,

$$= (Ux) \otimes (\mu_\alpha e_\alpha) = \mu_\alpha (Ux \otimes e_\alpha).$$

Thus, each subspace M_α is invariant under $U \otimes D$, and the restriction satisfies

$$(U \otimes D)\big|_{M_\alpha}(x \otimes e_\alpha) = \mu_\alpha (Ux \otimes e_\alpha).$$

Under the natural isometric identification

$$H \otimes \operatorname{span}\{e_\alpha\} \cong H,\ x \otimes e_\alpha \mapsto x,$$

the restriction $(U \otimes D)\big|_{M_\alpha}$ corresponds to the operator $\mu_\alpha U$ on H.

Since the subspaces M_α are mutually orthogonal and their direct sum equals $H \otimes K$, it follows that

$$U \otimes D = \oplus_{\alpha\in I}\mu_\alpha U.$$

Proposition 3.8

Let $P = \lambda I_H \in B(H)$ be a scalar multiple of the identity operator, where $\lambda \in \mathbb{C}$. Let $D \in B(K)$. Then:

i) $P \otimes D = \lambda(I_H \otimes D)$,

ii) $\|P \otimes D\| = |\lambda|\|D\|$.

Proof

Scalar Factorization

For any $h \in H$ and $k \in K$, we compute on elementary tensors:

$$(P \otimes D)(h \otimes k) = P(h) \otimes D(k).$$

Since $P = \lambda I_H$, we have $P(h) = \lambda h$, and hence

$$(P \otimes D)(h \otimes k) = (\lambda h) \otimes D(k).$$

By bi-linearity of the tensor product,

$$(\lambda h) \otimes D(k) = \lambda(h \otimes D(k)) = \lambda(I_H \otimes D)(h \otimes k).$$

Thus,

$$P \otimes D = \lambda(I_H \otimes D)$$

on elementary tensors. Since elementary tensors are dense in $H \otimes K$, and both sides define bounded operators, the identity extends to all of $H \otimes K$.

Norm Equality

From part (i), we have

$$P \otimes D = \lambda(I_H \otimes D).$$

Taking norms and using homogeneity of the operator norm,

$$\|P \otimes D\| = |\lambda| \|I_H \otimes D\|.$$

We now compute $\|I_H \otimes D\|$. By the general inequality for tensor products,

$$\|I_H \otimes D\| \le \|I_H\| \|D\| = \|D\|.$$

To obtain the reverse inequality, let $\varepsilon > 0$. Choose $y \in K$ with $\|y\| = 1$ such that

$$\|Dy\| > \|D\| - \varepsilon.$$

Let $x \in H$, $\|x\| = 1$. Then

$$\|(I_H \otimes D)(x \otimes y)\| = \|x \otimes Dy\| = \|x\| \|Dy\| = \|Dy\| > \|D\| - \varepsilon.$$

Since $\|x \otimes y\| = 1$, it implies that

$$\|I_H \otimes D\| \ge \|D\| - \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$\|I_H \otimes D\| = \|D\|.$$

Therefore,

$$\|P \otimes D\| = |\lambda| \|D\|.$$

Proposition 3.9

Let $P_1 \in B(H_1)$ and $P_2 \in B(H_2)$ be injective operators on Hilbert spaces H_1 and H_2. Then their tensor product $P_1 \otimes P_2 \in B(H_1 \otimes H_2)$ is also injective.

Proof

We first work on the algebraic tensor product $H_1 \odot H_2$. By bi-linearity of tensor products (Proposition 3.1), we may factor

$$P_1 \otimes P_2 = \left(P_1 \otimes I_{H_2}\right)\left(I_{H_1} \otimes P_2\right)$$

on $H_1 \odot H_2$.

Kernel of $I_{H_1} \otimes P_2$

We claim that

$$\ker\left(I_{H_1} \otimes P_2\right) = H_1 \odot \ker\left(P_2\right).$$

Let

$$z = \sum_{i=1}^{n} x_i \otimes y_i \in H_1 \odot H_2,$$

and suppose

$$\left(I_{H_1} \otimes P_2\right)(z) = \sum_{i=1}^{n} x_i \otimes P_2 y_i = 0.$$

Using standard linear algebra arguments, choose bounded linear functional $\left\{\varphi_j\right\} \subset H_1^*$ that separate a linearly independent subset of $\left\{x_i\right\}$. Applying $\varphi_j \otimes I_{H_2}$, we obtain

$$\sum_{i=1}^{n} \varphi_j\left(x_i\right) P_2 y_i = 0 \text{ for all } j.$$

As a result of linear independence,

$$y_i \in \ker\left(P_2\right).$$

Hence $z \in H_1 \odot \ker\left(P_2\right)$, proving the claim.

If P_2 is injective, then $\ker\left(P_2\right) = \{0\}$, and therefore

$$\ker\left(I_{H_1} \otimes P_2\right) = \{0\}.$$

Thus $I_{H_1} \otimes P_2$ is injective on $H_1 \odot H_2$.

Kernel of $P_1 \otimes I_{H_2}$

Similarly, one shows that

$$\ker\left(P_1 \otimes I_{H_2}\right) = \ker\left(P_1\right) \odot H_2.$$

Since P_1 is injective, $\ker\left(P_1\right) = \{0\}$, hence

$$\ker\left(P_1 \otimes I_{H_2}\right) = \{0\}.$$

Thus $P_1 \otimes I_{H_2}$ is injective on $H_1 \odot H_2$.

Extension to $H_1 \otimes H_2$

The preceding argument shows injectivity on the algebraic tensor product $H_1 \odot H_2$. However, injectivity on a dense subspace does not, by itself, guarantee injectivity of the continuous extension to the completed Hilbert tensor product.

To justify the extension, we use the standard Hilbert-space tensor product result that if $A \in B\left(H_1\right)$ and $B \in B\left(H_2\right)$ are injective bounded operators, then the induced operator

$$A \otimes B \in B(H_1 \otimes H_2)$$

is also injective (Ryan [13]; Kubrusly & Vieira [4]).

Since $P_1 \in B(H_1)$ and $P_2 \in B(H_2)$ are injective, this result applies. Hence,

$$\ker(P_1 \otimes P_2) = \{0\}.$$

Therefore,

$$P_1 \otimes P_2$$

is injective on the completed Hilbert tensor product $H_1 \otimes H_2$.

Lemma 3.10

Given normed spaces X and Y. Fix a bounded linear functional $\phi \in X^*$ and a vector $y \in Y$. Let rank-one operator be defined by $P: X \to Y$, $P(x) = \phi(x)y$. Then P is bounded and satisfies $\|P\| = \|\phi\|\|y\|$.

Proof

For all $x \in X$, we estimate

$$\|P(x)\| = \|\phi(x)y\| = |\phi(x)|\|y\|.$$

Since ϕ is bounded, we get

$$|\phi(x)| \le \|\phi\|\|x\|,$$

and therefore

$$\|P(x)\| \le \|\phi\|\|x\|\|y\|.$$

Taking the supremum over $x \in X$ with $\|x\| = 1$, gives

$$\|P\| \le \|\phi\|\|y\|.$$

When we take the supremum over $x \in X$, $\|x\| = 1$, we obtain

$$\|P\| \le \|\phi\|\|y\|$$

To establish the reverse inequality, first note that if $\phi = 0$ or $y = 0$, then $P = 0$, and the result is immediate. Assume therefore that $\phi \ne 0$ and $y \ne 0$.

By the definition of the dual norm, for each $\varepsilon > 0$, there exists $x_\varepsilon \in X$ with $\|x_\varepsilon\| = 1$ such that

$$|\phi(x_\varepsilon)| > \|\phi\| - \varepsilon.$$

Next

$$\|P(x_\varepsilon)\| = |\phi(x_\varepsilon)|\|y\| > (\|\phi\| - \varepsilon)\|y\|.$$

Thus,

$$\|P\| \ge (\|\phi\| - \varepsilon)\|y\|.$$

Letting $\varepsilon \to 0$, we obtain

Combining both inequalities yields

$$\|P\| = \|\phi\|\|y\|.$$

Example 3.11

Let $a, y \in \mathbb{R}^n$, and define the operator $P:\mathbb{R}^n \to \mathbb{R}^n$, $P(x) = (a^\top x)y$, for each $x \in \mathbb{R}^n$. Then $\|P\| = \|a\|_2 \|y\|_2$.

Indeed, Let $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$. Then $\|P(x)\|_2 = \|(a^\top x)y\|_2 = |a^\top x|\|y\|_2$.

Using Cauchy–Schwarz inequality, $|a^\top x| \le \|a\|_2 \|x\|_2 = \|a\|_2$.

Hence, $\|P(x)\|_2 \le \|a\|_2 \|y\|_2$

Taking the supremum over x, $\|x\|_2 = 1$, we obtain $\|P\| \le \|a\|_2 \|y\|_2$.

To establish the reverse inequality, assume that $a \ne 0$ and $y \ne 0$. Define $x = \frac{a}{\|a\|_2}$, so that $\|x\|_2 = 1$. Then $a^\top x = \frac{a^\top a}{\|a\|_2} = \frac{\|a\|_2^2}{\|a\|_2} = \|a\|_2$.

Thus, $\|P(x)\|_2 = |a^\top x|\|y\|_2 = \|a\|_2 \|y\|_2$.

Hence, $\|P\| \ge \|a\|_2 \|y\|_2$.

Combining both inequalities, we conclude that

$$\|P\| = \|a\|_2 \|y\|_2.$$

Lemma 3.12

Let X_1, X_2, Y_1, Y_2 be normed spaces. Let $P_1 = \phi_1 \otimes y_1 : X_1 \to Y_1$, $P_2 = \phi_2 \otimes y_2 : X_2 \to Y_2$ be rank-one operators, where $P_1(x) = \phi_1(x)y_1$. $P_2(z) = \phi_2(z)y_2$ with $\phi_1 \in X_1^*$, $\phi_2 \in X_2^*$, $y_1 \in Y_1$, $y_2 \in Y_2$. Then on the algebraic tensor product $X_1 \odot X_2$, $P_1 \otimes P_2 = (\phi_1 \otimes \phi_2)(y_1 \otimes y_2)$, that is, $(P_1 \otimes P_2)(u) = (\phi_1 \otimes \phi_2)(u)(y_1 \otimes y_2)$, $u \in X_1 \odot X_2$. Consequently, $P_1 \otimes P_2$ is a rank-one operator with $\operatorname{Ran}(P_1 \otimes P_2) \subseteq \operatorname{span}\{y_1 \otimes y_2\} \subseteq Y_1 \otimes Y_2$.

Proof

Let $x \in X_1$ and $z \in X_2$. Then by definition of the tensor product of operators, $(P_1 \otimes P_2)(x \otimes z) = P_1(x) \otimes P_2(z)$.

Substituting the definitions of P_1 and P_2,
$(P_1 \otimes P_2)(x \otimes z) = (\phi_1(x)y_1) \otimes (\phi_2(z)y_2)$.

By bilinearity of the tensor product,
$(\phi_1(x)y_1) \otimes (\phi_2(z)y_2) = \phi_1(x)\phi_2(z)(y_1 \otimes y_2)$.

On the other hand, the tensor product functional satisfies
$(\phi_1 \otimes \phi_2)(x \otimes z) = \phi_1(x)\phi_2(z)$.

Therefore, $(P_1 \otimes P_2)(x \otimes z) = (\phi_1 \otimes \phi_2)(x \otimes z)(y_1 \otimes y_2)$.

By linearity, this identity extends to all $u \in X_1 \odot X_2$, so that
$(P_1 \otimes P_2)(u) = (\phi_1 \otimes \phi_2)(u)(y_1 \otimes y_2)$.

Thus, the image of every vector is a scalar multiple of $y_1 \otimes y_2$, and hence $\operatorname{Ran}(P_1 \otimes P_2) \subseteq \operatorname{span}\{y_1 \otimes y_2\}$.

It follows that $P_1 \otimes P_2$ is a rank-one operator.

Finally, since $\|P_1 \otimes P_2\| \le \|P_1\|\|P_2\| = \|\phi_1\|\|y_1\|\|\phi_2\|\|y_2\|$, the operator extends uniquely by continuity to the completed tensor product $X_1 \otimes X_2$.

4. Extension to Banach Spaces

The results established in the preceding sections for Hilbert spaces extend naturally, in part, to the setting of Banach spaces. We briefly indicate how tensor product operators behave under the projective tensor norm.

Let X and Y be Banach spaces. Denote by $X \odot Y$ the algebraic tensor product and by $X \hat{\otimes}_\pi Y$ its completion with respect to the projective tensor norm (Ryan [13]).

Proposition 4.1.

Let $D \in B(X)$ and $E \in B(Y)$. Then the operator $D \otimes E$, defined on $X \odot Y$ by

$$(D \otimes E)(x \otimes y) = Dx \otimes Ey, \ \ x \in X, y \in Y,$$

extends uniquely to a bounded linear operator on the projective tensor product $X \hat{\otimes}_\pi Y$, and satisfies

$$\|D \otimes E\| \le \|D\|\|E\|.$$

Proof.

By linearity, the operator $D \otimes E$ extends to $X \odot Y$. Let $u = \sum_{i=1}^n x_i \otimes y_i \in X \odot Y$. Then

$$(D \otimes E)(u) = \sum_{i=1}^n Dx_i \otimes Ey_i.$$

By definition of the projective tensor norm,

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^n \|x_i\|\|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

Hence,

$$\|(D \otimes E)(u)\|_\pi \le \sum_{i=1}^n \|Dx_i\|\|Ey_i\| \le \|D\|\|E\| \sum_{i=1}^n \|x_i\|\|y_i\|.$$

Taking the infimum over all such representations of u, we obtain

$$\|(D \otimes E)(u)\|_\pi \le \|D\|\|E\|\|u\|_\pi.$$

Thus, $D \otimes E$ is bounded on $X \odot Y$ and extends uniquely by continuity to $X \hat{\otimes}_\pi Y$ (Ryan [13]).

□

Remark 4.2.

In contrast to the Hilbert space case, equality $\|D \otimes E\| = \|D\|\|E\|$ does not necessarily hold for general Banach space tensor products, as it depends on the choice of tensor norm. This highlights a key distinction between Hilbert space tensor products and more general Banach space tensor constructions.

Proposition 4.3.

Let X and Y be Banach spaces, and let $D \in K(X)$ and $E \in K(Y)$ be compact operators. Then the tensor product operator

$$D \otimes E : X \hat{\otimes}_\pi Y \to X \hat{\otimes}_\pi Y$$

is compact.

Proof.

Since D and E are compact operators, there exist sequences of finite-rank operators $(D_n) \subset B(X)$ and $(E_n) \subset B(Y)$ such that

$$\|D_n - D\| \to 0, \ \|E_n - E\| \to 0.$$

For each n, the operators D_n and E_n are finite-rank, hence $D_n \otimes E_n$ is finite-rank on $X \odot Y$, and therefore extends to a finite-rank operator on $X \hat{\otimes}_\pi Y$.

We now show that $D_n \otimes E_n \to D \otimes E$ in operator norm. Observe that

$$D \otimes E - D_n \otimes E_n = D \otimes (E - E_n) + (D - D_n) \otimes E_n.$$

Using the norm estimate from Proposition 5.1, we obtain

$$\|D \otimes E - D_n \otimes E_n\| \le \|D\| \|E - E_n\| + \|D - D_n\| \|E_n\|.$$

Since $E_n \to E$, the sequence (E_n) is bounded, and therefore the right-hand side tends to zero. Hence,

$$D_n \otimes E_n \to D \otimes E$$

in operator norm.

Since each $D_n \otimes E_n$ is finite-rank (and hence compact), and the set of compact operators is closed in the operator norm, it follows that $D \otimes E$ is compact. □

5. Conclusion

This study develops a unified framework for tensor products of bounded linear operators, demonstrating the preservation of rank, compactness, and injectivity under tensorization. It provides specific insights into eigenvalue multiplicativity, compactness preservation under tensor products, and block decomposition of tensor products involving diagonal operators. These results enhance the structural understanding of tensor product operators. It is therefore recommended that further research be directed toward tensor product operators' spectral characteristics.

Conflicts of Interest

The authors declare no conflicts of interest.

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