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Normalized Solutions to Upper Critical Fractional Kirchhoff-Choquard Type Equations with Potentials

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Abstract

This paper is concerned with the normalized ground state solutions to the following Hardy-Littlewood-Sobolev upper critical fractional Kirchhoff-Choquard type equations under the constraint $\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=c$,

$$\left(a+b\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)(-\Delta)^s u+V(x)u$$
$$=\lambda u+\mu\left(I_\theta*|u|^p\right)|u|^{p-2}u+\left(I_\theta*|u|^{2^*_{\theta,s}}\right)|u|^{2^*_{\theta,s}-2}u\ \text{ in }\mathbb{R}^N,$$

where $s\in(0,1)$, $N\in(2s,4s)$, $\theta\in(0,N)$, $a,b,c,\mu>0$, $\lambda\in\mathbb{R}$, $p\in\left(\frac{N+\theta}{N},\frac{N+\theta+2s}{N}\right)$ and V is an external potential vanishing at infinity. Utilizing the perturbed Pohozaev constraint and Schwartz symmetrization rearrangements, we establish the existence of the normalized ground state solutions, and characterize the asymptotic behavior of solutions as $\mu\to 0^+$ in the autonomous case.

Subject Areas

Partial Differential Equation

Keywords

Fractional Kirchhoff-Choquard Equation, Normalized Solution, Upper Critical Exponent

1. Introduction

In this paper, we study the following upper critical fractional Kirchhoff-Choquard type equation

$$\begin{aligned}&\left(a+b\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)(-\Delta)^s u+V(x)u\\&=\lambda u+\mu\left(I_\theta*|u|^p\right)|u|^{p-2}u+\left(I_\theta*|u|^{2^*_{\theta,s}}\right)|u|^{2^*_{\theta,s}-2}u \text{ in } \mathbb{R}^N,\end{aligned} \tag{1.1}$$

with prescribed L^2-norm constraint

$$\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=c, \tag{1.2}$$

where $s\in(0,1)$, $N\in(2s,4s)$, $\theta\in(0,N)$, $a,b,c,\mu>0$, $p\in\left(\frac{N+\theta}{N},\frac{N+\theta+2s}{N}\right)$, $\frac{N+\theta+2s}{N}$ is the L^2-critical exponent, $\frac{N+\theta}{N}$ and $2^*_{\theta,s}:=\frac{N+\theta}{N-2s}$ are lower critical exponent and upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. Furthermore, λ appears as an unknown Lagrange multiplier, V is an external potential vanishing at infinity, the function $I_\theta:\mathbb{R}^N\to\mathbb{R}$ is called the Riesz potential and defined as follows,

$$I_\theta(x):=\frac{A_\theta}{|x|^{N-\theta}},\text{ where } A_\theta:=\frac{\Gamma\left(\frac{N-\theta}{2}\right)}{\Gamma\left(\frac{\theta}{2}\right)\pi^{\frac{N}{2}}2^\theta}$$

with Γ representing the Gamma function. For convenience, we drop A_θ in what follows. The symbol $(-\Delta)^s$ is the fractional Laplace operator defined as

$$(-\Delta)^s u(x)=C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y,\ u\in H^s(\mathbb{R}^N),$$

where the symbol P.V. means the Cauchy principal value of the singular integral and $C(N,s)$ is a dimensional constant, precisely given by $C(N,s):=\left(\int_{\mathbb{R}^N}\left((1-\cos\zeta_1)/\left(|\zeta|^{N+2s}\right)\right)\mathrm{d}\zeta\right)^{-1}$. $H^s(\mathbb{R}^N)$ is defined in detail below. The fractional Laplacian operator appears in diverse areas such as financial mathematics, optimization and minimal surfaces etc.

In recent years, significant attention has been paid to the following Choquard equation

$$-\Delta u+u=\left(I_\theta*|u|^p\right)|u|^{p-2}u,\ x\in\mathbb{R}^N, \tag{1.3}$$

which arises from several physical contexts. When $N=3$, $\theta=p=2$, (1.3) turns to be the well-known Choquard-Pekar equation by S. Pekar [1] in 1954 which was firstly introduced as a model in quantum theory of a polaron at rest. In 1976, (1.3) was used by Choquard [2] to study steady states of the one component plasma approximation to Hartree-Fock theory. In recent years, the existence of solutions to (1.3) was demonstrated mathematically in [3]-[5] via variational approaches.

Moroz and Schaftingen [6] proved the existence of the positive ground state solution to the Equation (1.3) with $p>1$, and established the regularity, positivity, monotonicity and decay asymptotics of the normalized ground state solutions. For $N\geq 3$, $p=\frac{N+\theta}{N-2}$ and $q\in\left(2,2+\frac{4}{N}\right)$, Li [7] considered the existence and orbital stability of the normalized ground states for the Choquard equation with a local perturbation $\mu|u|^{q-2}u$ under the L^2-norm constraint. Shang and Ma [8] considered the existence of positive solutions for the following Choquard equation

$$-\Delta u+u=\mu\left(I_\theta*|u|^p\right)|u|^{p-2}u+\left(I_\theta*|u|^{2_\theta^*}\right)|u|^{2_\theta^*-2}u,\ \ x\in\mathbb{R}^N, \tag{1.4}$$

where $N\geq 3$ and $\frac{N+\theta}{N}<p<2_\theta^*:=\frac{N+\theta}{N-2}$. Shang and Ma established a set of existence results and characterized the asymptotic behavior as $\mu\to 0^+$ by setting various assumptions on the parameter p. Meng only [9] considered the first case in (1.4) where $p\in\left(\frac{N+\theta}{N},\frac{N+\theta+2s}{N}\right)$, and extended it to the fractional setting. Long, Li and Rong [10] proved normalized ground state solutions to the following Choquard equations with weakly attractive potential under the L^2-norm constraint

$$-\Delta u+(V+\lambda)u=\mu\left(I_\theta*|u|^p\right)|u|^{p-2}u+\left(I_\theta*|u|^q\right)|u|^{q-2}u,\ \ x\in\mathbb{R}^N, \tag{1.5}$$

where $N\geq 3$, $p,q\in\left(\frac{N+\theta+2}{N},\frac{N+\theta}{N-2}\right)$ and the trapping potential V.

On the other hand, extensive research efforts have also been focused on exploring the normalized ground state solutions to Kirchhoff equations. The Kirchhoff equation was first introduced by Kirchhoff [11] in 1883 as an extension of classical D'Alembert's wave equation. For instance, Ye [12] analyzed the exclusion of dichotomy of the minimizing sequences for the related constrained minimization problem to prove the existence of solutions with L^2-norm constraint for the following Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u-\lambda u=|u|^{p-2}u,\ \ x\in\mathbb{R}^N,$$

where $N\leq 3$, $p\in\left(2,2^*\right)$, $2^*=6$ if $N=3$ and $2^*=+\infty$ if $N=1,2$. Especially, Ye obtained the sharp existence of global constraint minimizers for $2<p\leq 2+\frac{8}{N}$. Ji *et al.* [13] developed a perturbed Pohozaev constraint approach to consider the Kirchhoff equation under L^2-norm constraint as follows

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u-\lambda u=|u|^{p-2}u+\mu|u|^{q-2}u,\ \ x\in\mathbb{R}^3, \tag{1.6}$$

where $2<q<\frac{14}{3}<p\leq 6$ or $\frac{14}{3}<q<p\leq 6$, and proved several asymptotic results for the normalized ground state solutions obtained. Hu and Mao [14] par-

tially extended the results of (1.6) under different assumptions on the exponents $2<p<q<6$. They have studied the existence and nonexistence of the normalized ground state solutions based on the methods of constrained minimization and concentration compactness. Chen and Huang [15] proved the existence and nonexistence of solutions to the fractional Kirchhoff equation with an external potential V and doubly critical exponent.

Recently, some researchers have dedicated their efforts to exploring the existence of normalized ground state solutions of Kirchhoff-Choquard equation. In [16], Liu established the threshold values for the existence and nonexistence of solutions to the Kirchhoff-Choquard equation. Furthermore, the asymptotic behaviors of the corresponding Lagrange multipliers and energies as $c\to 0$ and $c\to\infty$ were obtained. Zhu *et al.* [17] developed a new perturbed Pohozaev constraint approach to prove the existence of two normalized ground state solutions for the following Kirchhoff-Choquard equation under L^2-norm constraint

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u-\lambda u=\left(I_\theta*|u|^p\right)|u|^{p-2}u+\left(I_\theta*|u|^q\right)|u|^{q-2}u,\ \ x\in\mathbb{R}^N, \quad (1.7)$$

where $N\geq 3$ and $\frac{N+\theta}{N}<q<\frac{N+4+\theta}{N}<p<2_\theta^*$ or $\frac{N+4+\theta}{N}<q<p<2_\theta^*$, and obtained the asymptotic behavior for the normalized ground state solutions as $\mu\to 0^+$. Liu and Sun [18] focused on the existence of normalized ground state solutions to the Kirchhoff-Choquard equation under L^2-norm constraint as follows

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+\left(V(x)+\lambda\right)u=f(x,u),\ \ x\in\mathbb{R}^N, \quad (1.8)$$

where the potential V satisfies different conditions.

The key difficulty in combining the upper critical exponent with a vanishing potential in the normalized framework is the possible loss of compactness of minimizing sequences, which may exhibit dichotomy rather than convergence or vanishing. Our analysis, particularly in Lemma 3.10 and Theorem 1.5, is designed to exclude this dichotomy. Based on these results, we prove the existence of normalized solutions to the fractional Kirchhoff-Choquard Equation (1.1) with an HLS upper critical exponent and a potential, using a perturbed Pohozaev constraint and variational methods. Furthermore, we establish the asymptotic properties of the normalized ground states as $\mu\to 0^+$ in the autonomous case.

$$\begin{aligned}E_\mu(u)=&\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)^2+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x\\&-\frac{\mu}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\&-\frac{1}{22_{\theta,s}^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_{\theta,s}^*}|u(y)|^{2_{\theta,s}^*}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y,\end{aligned} \quad (1.9)$$

under the L^2-norm constraint

$$\mathcal{S}_c=\left\{u\in H^s\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=c\right\}.$$

We now define several important types of spaces that will be essential in subsequent sections. To begin with, we define that $|\cdot|_p$ is the standard norm in Lebesgue space $L^p\left(\mathbb{R}^N\right)$ for $p\in[1,\infty)$. Subsequently, we define that

$$\|u\|:=\|u\|^2_{\mathcal{D}^{s,2}\left(\mathbb{R}^N\right)}=\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$

is the standard norm in Sobolev space

$$\mathcal{D}^{s,2}\left(\mathbb{R}^N\right)=\left\{u\in L^{2_s^*}\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y<+\infty\right\},$$

where $2_s^*:=\dfrac{2N}{N-2s}$. Finally and most importantly, we introduce the definition and some useful facts of the fractional Sobolev space $H^s\left(\mathbb{R}^N\right)$. As usual, the fractional Sobolev space $H^s\left(\mathbb{R}^N\right)$ is defined for any $s\in(0,1)$ as

$$H^s\left(\mathbb{R}^N\right)=\left\{u\in L^2\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}}\mathrm{d}y\in L^2\left(\mathbb{R}^N\right)\right\}.$$

Note from [19] that

$$2C^{-1}(N,s)\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

The space $H^s\left(\mathbb{R}^N\right)$ is a Hilbert space endowed with the following inner product and norm

$$\langle u,v\rangle_{H^s\left(\mathbb{R}^N\right)}=\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v\mathrm{d}x+\int_{\mathbb{R}^N}uv\mathrm{d}x,\forall u,v\in H^s\left(\mathbb{R}^N\right)$$

and

$$\|u\|_{H^s\left(\mathbb{R}^N\right)}=\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x\right)^{\frac{1}{2}},\forall u\in H^s\left(\mathbb{R}^N\right).$$

Now, to establish our main results, we introduce the following auxiliary function

$$W(x)=\frac{1}{2}\nabla V\cdot x,x\in\mathbb{R}^N.$$

and proceed to list the key assumptions on $W(x)$ and $V(x)$.

(A1): $V(x)\in C^1\left(\mathbb{R}^N\right)\setminus\{0\}$, $V_\infty:=\lim_{|x|\to\infty}V(x)=0$, $\sup_{x\in\mathbb{R}^N}V(x)=0$ and there exists $\sigma_1\in[0,1)$ such that $\int_{\mathbb{R}^N}|V|u^2\le\sigma_1\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x$ for any $u\in H^s\left(\mathbb{R}^N\right)$.

(A2): $W(x) \in C^1(\mathbb{R}^N)$, $\lim_{|x|\to\infty} W(x) = 0$, $V(x) + \frac{1}{s}W(x) \le 0$, $\forall x \in \mathbb{R}^N$.

(A3): $V(x)$ is a radial function, non-decreasing with respect to $r = |x|$.

Remark 1.1. *Let*

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |u|^{2_s^*} \mathrm{d}x\right)^{\frac{2}{2_s^*}}}.$$

If $|V|_{\frac{N}{2s}} < S$, then there exists $\sigma_1 := S^{-1}|V|_{\frac{N}{2s}} < 1$ and

$$\int_{\mathbb{R}^N} |V| u^2 \mathrm{d}x \le |V|_{\frac{N}{2s}} |u|_{2_s^*}^2 \le S^{-1}|V|_{\frac{N}{2s}} \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x = \sigma_1 \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x.$$

For a sufficiently small $\varepsilon > 0$, the function $V(x) = -\frac{\varepsilon}{1+|x|^{2s}}$ on $\mathbb{R}^N$ satisfies the above conditions. Now we recall the definition of normalized ground state solutions.

Definition 1.2. *For any fixed* $c > 0$*, we say that* $u_0 \in \mathcal{S}_c$ *is a normalized ground state solution to* (1.1) *under the constraint* (1.2) *if*

$$E'_\mu\big|_{\mathcal{S}_c}(u_0) = 0 \quad \text{and} \quad E_\mu(u_0) = \inf\left\{E_\mu(u) : u \in \mathcal{S}_c, E'_\mu\big|_{\mathcal{S}_c}(u) = 0\right\}.$$

Prior to studying the nonautonomous Equation (1.1), we introduce the following fractional Kirchhoff-Choquard equation in the autonomous case, *i.e.* $V(x) \equiv V_\infty = 0$

$$\begin{aligned} &\left(a + b\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x\right)(-\Delta)^s u \\ &= \lambda u + \mu\left(I_\theta * |u|^p\right)|u|^{p-2} u + \left(I_\theta * |u|^{2_{\theta,s}^*}\right)|u|^{2_{\theta,s}^*-2} u \text{ in } \mathbb{R}^N, \end{aligned} \tag{1.10}$$

under the L^2-norm constraint $\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = c$. Firstly, we state the existence and the asymptotic properties of the normalized ground state solutions in the autonomous case. This is accomplished in the following Theorem 1.3 and 1.4.

Theorem 1.3. *Assume that* $N \in (2s, 4s)$ *and* $p \in \left(\frac{N+\theta}{N}, \frac{N+\theta+2s}{N}\right)$. *For any* $\mu > 0$*, there exists a constant* $c_\infty := c_\infty(\mu) > 0$ *such that for any* $c \in (0, c_\infty)$*, problem* (1.10) *under the* L^2*-norm constraint* $\mathcal{S}_c$ *possesses a normalized ground state solution* $\tilde{u}_{c_\infty,\mu} \in H^s(\mathbb{R}^N)$ *such that* $E_{\infty,\mu}(\tilde{u}_{c_\infty,\mu}) < 0$ *and* $\lambda_{c_\infty,\mu} < 0$.

Theorem 1.4. *Assume that* $N \in (2s, 4s)$ *and* $p \in \left(\frac{N+\theta}{N}, \frac{N+\theta+2s}{N}\right)$. *For any fixed* $c \in (0, c_\infty)$*, let* $u_{c_\infty,\mu} \in \mathcal{S}_c$ *be the corresponding normalized ground state solution for* $\mu > 0$ *sufficiently small, then*

$$\left\|u_{c_\infty,\mu}\right\|^2 \to 0 \quad \text{and} \quad E_{\infty,\mu}(u_{c_\infty,\mu}) \to 0 \quad \text{as} \quad \mu \to 0^+.$$

These results from the autonomous case play a crucial role in addressing our main problem (1.1). Next, we state the existence of the normalized ground state solutions in the nonautonomous cases.

Theorem 1.5. *Assume that* $N \in (2s, 4s)$, $p \in \left(\frac{N+\theta}{N}, \frac{N+\theta+2s}{N} \right)$ *and* V *satisfies* **(A1)**, **(A2)** *and* **(A3)**. *For any* $\mu > 0$, *there exists a constant* $c_0 = c_0(\mu) > 0$ *such that for any* $c \in (0, c_0)$, *problem* (1.1) *under the* L^2*-norm constraint* $\mathcal{S}_c$ *possesses a normalized ground state solution* $u_{c,\mu} \in H^s(\mathbb{R}^N)$ *such that* $E_\mu(u_{c,\mu}) < 0$ *and* $\lambda_{c,\mu} < 0$.

This paper is organized as follows. In Section 2, we give some preliminary results which will be used in proving main results. In Section 3, we study existence and the asymptotic properties of the normalized ground state solutions in the autonomous case and prove Theorem 1.3 and Theorem 1.4. In Section 4, we focus on the existence of the normalized ground state solutions in the nonautonomous case and prove Theorem 1.5.

Throughout this paper, we adopt the following notations.

- $B_r(y) := \{ x \in \mathbb{R}^N : |x - y| < r \}$.
- C denotes any positive constants possibly different in different places.
- The symbol $\rightharpoonup$ denotes weak convergence and the symbol $\to$ denotes strong convergence.
- $o_n(1)$ denotes a real sequence tending to 0 as $n \to \infty$.
- $H^{-s}(\mathbb{R}^N)$ is the dual space of $H^s(\mathbb{R}^N)$.

2. Preliminaries

In Section 2, we give several preliminary results which are significant for the subsequent development of our work. Especially, for convenience, we define

$$A(u) := \int_{\mathbb{R}^N} \left(I_\theta * |u|^p \right) |u|^{p-2} u, \quad B(u) := \int_{\mathbb{R}^N} \left(I_\theta * |u|^{2^*_{\theta,s}} \right) |u|^{2^*_{\theta,s}-2} u.$$

Then, according to (1.9), we have

$$E_\mu(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) u^2 \mathrm{d}x - \frac{\mu}{2p} A(u) - \frac{1}{22^*_{\theta,s}} B(u) \tag{2.1}$$

and

$$E_{\infty,\mu}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\mu}{2p} A(u) - \frac{1}{22^*_{\theta,s}} B(u) \tag{2.2}$$

Firstly, let us recall the well-known Hardy-Littlewood-Sobolev inequality and fractional Gagliardo-Nirenberg inequality.

Lemma 2.1. ([20]) *Let* $r, t > 1, \theta \in (0, N)$ *satisfy* $\frac{1}{r} + \frac{1}{t} = \frac{N+\theta}{N}$, $f \in L^r(\mathbb{R}^N)$ *and* $g \in L^t(\mathbb{R}^N)$. *Then there exists a sharp constant* $C(N, \theta, r, t) > 0$, *independent of* f *and* g, *such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\right|\le C(N,\theta,r,t)|f|_r|g|_t.$$

If $r=t=\frac{2N}{N+\theta}$, then

$$C(N,\theta,r,t)=C(N,\theta)=\pi^{\frac{N-\theta}{2}}\frac{\Gamma\left(\frac{\theta}{2}\right)}{\Gamma\left(\frac{N+\theta}{2}\right)}\left[\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right]^{-\frac{\theta}{N}}.$$

Lemma 2.2. ([9]) *Let* $N>2s$ *and* $\frac{N+\theta}{N}<p<2^*_{\theta,s}$. *Then there exists a constant* $C(N,\theta,s,p)>0$ *such that*

$$\int_{\mathbb{R}^N}\left(I_\theta*|u|^p\right)|u|^p\,\mathrm{d}x\le C(N,\theta,s,p)\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{2p\gamma_{p,s}}|u|_2^{2p(1-\gamma_{p,s})},\forall u\in H^s\left(\mathbb{R}^N\right),\quad(2.3)$$

where $\gamma_{p,s}:=\frac{Np-N-\theta}{2ps}$.

Let S_θ be best constant

$$S_\theta:=\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\left(I_\theta*|u|^{2^*_{\theta,s}}\right)|u|^{2^*_{\theta,s}}\,\mathrm{d}x\right)^{\frac{1}{2^*_{\theta,s}}}},\quad(2.4)$$

it is well-known that S_θ is achieved if and only if u is the form

$$C\left(\frac{\varepsilon}{\varepsilon^2+|x-x_0|^2}\right)^{\frac{N-2s}{2}},\quad x\in\mathbb{R}^N,$$

for some $x_0\in\mathbb{R}^N$, $C>0$ and $\varepsilon>0$ see [21].

We give classical Brezis-Lieb lemma for the nonlinear local term.

Lemma 2.3. *Let* $\Omega\subseteq\mathbb{R}^N$ *be a domain,* $p\in[1,\infty)$ *and* $\{u_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^r(\Omega)$. *If* $u_n\to u$ *a.e. on* Ω *as* $n\to\infty$, *then for every* $p\in[1,r]$,

$$\lim_{n\to\infty}\int_\Omega\left||u_n|^p-|u_n-u|^p-|u|^p\right|^{\frac{r}{p}}\mathrm{d}x=0.$$

We give Brezis-Lieb lemma for the convolution term of the functional.

Lemma 2.4. ([6]) *Let* $N>2s$, $p\in[1,\infty)$ *and* $\{u_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^{\frac{2Np}{N+\theta}}\left(\mathbb{R}^N\right)$. *If* $u_n\to u$ *a.e. on* $\mathbb{R}^N$ *as* $n\to\infty$, *then*

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\left(I_\theta*|u_n|^p\right)|u_n|^p\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(I_\theta*|u_n-u|^p\right)|u_n-u|^p\,\mathrm{d}x\right)$$
$$=\int_{\mathbb{R}^N}\left(I_\theta*|u|^p\right)|u|^p\,\mathrm{d}x.$$

Lemma 2.5. ([22]) *Let* $R>0$ *and* $p\in\left[2,2_s^*\right)$. *If* $\{u_n\}$ *is bounded in* $H^s\left(\mathbb{R}^N\right)$ *and*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}\left|u_n(x)\right|^p\,\mathrm{d}x=0,$$

then $u_n\to 0$ in $L^t\left(\mathbb{R}^N\right)$ as $n\to\infty$ for any $t\in\left(2,2_s^*\right)$.

Lemma 2.6. *Assume that* V *satisfies* **(A1)**, **(A2)** and **(A3)**. *If* $\{u_n\}\subset H^s\left(\mathbb{R}^N\right)$ *satisfies* $u_n\rightharpoonup u$ *in* $H^s\left(\mathbb{R}^N\right)$ *and* $u_n(x)\to u(x)$ *a.e. in* $\mathbb{R}^N$ *for some* $u\in H^s\left(\mathbb{R}^N\right)$ *as* $n\to\infty$, *there holds that*

$$E_\mu(u_n)=E_\mu(u)+E_\mu(u_n-u)+\frac{b}{2}\|u\|^2\|u_n-u\|^2+o_n(1).$$

Proof: Since $u_n\rightharpoonup u$ as $n\to\infty$ in $H^s\left(\mathbb{R}^N\right)$, in view of Lemma 2.3, it implies that

$$\|u_n\|_{H^s(\mathbb{R}^N)}^2=\|u_n-u\|_{H^s(\mathbb{R}^N)}^2+\|u\|_{H^s(\mathbb{R}^N)}^2+o_n(1) \tag{2.5}$$

and

$$\int_{\mathbb{R}^N}|u_n|^2\,\mathrm{d}x=\int_{\mathbb{R}^N}|u_n-u|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x+o_n(1). \tag{2.6}$$

Combining (2.5) with (2.6), we have

$$\|u_n\|^2=\|u_n-u\|^2+\|u\|^2+o_n(1).$$

Hence, we can get that

$$\begin{aligned}\frac{a}{2}\|u_n\|^2&=\frac{a}{2}\|u_n-u\|^2+\frac{a}{2}\|u\|^2+o_n(1)\\ \frac{b}{4}\|u_n\|^2&=\frac{b}{4}\|u_n-u\|^2+\frac{b}{4}\|u\|^2+\frac{b}{2}\|u\|^2\|u_n-u\|^2+o_n(1).\end{aligned} \tag{2.7}$$

Next, we let $f_n:=|V|^{1/2}u_n$. It follows (A_1) implies that $\{f_n\}$ is bounded in $L^2\left(\mathbb{R}^N\right)$. Since $u_n\to u$ a.e. as $n\to\infty$, we have $f_n\to|V|^{1/2}u=:f$ a.e. as $n\to\infty$. Applying Lemma 2.3 with $p=2$ and $r=2$ to the sequence $\{f_n\}$, we deduce that

$$\int_{\mathbb{R}^N}\left||f_n|^2-|f_n-f|^2-|f|^2\right|\mathrm{d}x\to 0.$$

Substituting back $f_n=|V|^{1/2}u_n$ and $f=|V|^{1/2}u$, and noting that $V\le 0$ (so $|V|=-V$), we obtain

$$\int_{\mathbb{R}^N}V(x)|u_n|^2\,\mathrm{d}x=\int_{\mathbb{R}^N}V(x)|u_n-u|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}V(x)|u|^2\,\mathrm{d}x+o_n(1). \tag{2.8}$$

It follows Lemma 2.4, (2.8) and (2.7) that

$$E_\mu(u_n)=E_\mu(u)+E_\mu(u_n-u)+\frac{b}{2}\|u\|^2\|u_n-u\|^2+o_n(1).$$

Hence, the proof is complete. □

Now, we introduce the Pohozaev manifold

$$\mathcal{P}_{c,\mu}:=\left\{u\in\mathcal{S}_c:P_\mu(u)=0\right\},$$

with

$$P_\mu(u)=as\|u\|^2+bs\|u\|^4-\int_{\mathbb{R}^N}W(x)u^2\mathrm{d}x-\mu s\gamma_{p,s}A(u)-sB(u),$$

and it is easy to prove $\mathcal{P}_{c,\mu}$ is a smooth manifold.

Lemma 2.7. *Assume that* V *satisfies* **(A1)**, **(A2)** and **(A3)**. *Suppose that* $u\in H^s(\mathbb{R}^N)$ *is a weak solution of problem* (1.1) *under the* L^2 *-norm constraint* $\mathcal{S}_c$, *then* $u\in\mathcal{P}_{c,\mu}$.

Proof: Since u is a solution of problem (1.1) under the L^2-norm constraint $\mathcal{S}_c$, by [6] and [23], we know that u satisfies the Pohozaev identity

$$\begin{aligned}&\frac{N-2s}{2s}a\|u\|^2+\frac{N-2s}{2s}b\|u\|^4\\&=-\frac{1}{s}\int_{\mathbb{R}^N}W(x)u^2\mathrm{d}x-\frac{N}{2s}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x\\&\quad+\frac{N}{2s}\lambda\int_{\mathbb{R}^N}u^2\mathrm{d}x+\mu\frac{N+\theta}{2ps}A(u)+\frac{N+\theta}{22^*_{\theta,s}s}B(u)\\&=a\|u\|^2+b\|u\|^4-\frac{1}{s}\int_{\mathbb{R}^N}W(x)u^2\mathrm{d}x-\mu\gamma_{p,s}A(u)-B(u),\end{aligned}\tag{2.9}$$

where we have used the fact that

$$\lambda=\frac{a\|u\|^2+b\|u\|^4+\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-\mu A(u)-B(u)}{|u|_2^2}.$$

Thus, one has

$$a\|u\|^2+b\|u\|^4-\frac{1}{s}\int_{\mathbb{R}^N}W(x)u^2\mathrm{d}x-\mu\gamma_{p,s}A(u)-B(u)=0$$

which means that $u\in\mathcal{P}_{c,\mu}$. □

For $u\in\mathcal{S}_c$ and $t\in\mathbb{R}$, we set

$$(t\star u)(x):=e^{\frac{Nt}{2}}u(e^tx),\quad x\in\mathbb{R}^N,\tag{2.10}$$

then $t\star u\in\mathcal{S}_c$. We define the map,

$$\begin{aligned}I_\mu(t)&:=E_\mu(t\star u)\\&=\frac{a}{2}e^{2st}\|u\|^2+\frac{b}{4}e^{4st}\|u\|^4+\frac{1}{2}\int_{\mathbb{R}^N}V(e^{-t}x)u^2\mathrm{d}x\\&\quad-\frac{\mu}{2p}e^{2p\gamma_{p,s}st}A(u)-\frac{1}{22^*_{\theta,s}}e^{22^*_{\theta,s}st}B(u).\end{aligned}\tag{2.11}$$

Then we easily obtain that

$$\begin{aligned}(I_\mu)'(t)&=ase^{2st}\|u\|^2+bse^{4st}\|u\|^4-\int_{\mathbb{R}^N}W(e^{-t})u^2\mathrm{d}x\\&\quad-\mu\gamma_{p,s}e^{2p\gamma_{p,s}st}A(u)-se^{22^*_{\theta,s}st}B(u)\\&=P_\mu(t\star u).\end{aligned}$$

As a result, we get the following conclusion.

Lemma 2.8. *Let* $u\in\mathcal{S}_c$ *and* V *satisfies* **(A1)**, **(A2)** and **(A3)**. *Then* $t\in\mathbb{R}$ *is critical point of* $I_\mu(t)$ *if and only if* $t\star u\in\mathcal{P}_{c,\mu}$.

Lemma 2.9. *Assume that* V *satisfies* **(A1)**, **(A2)** and **(A3)**. *The function*

$c \to e(c)$ *is continuous for any* $c \in (0, c_0)$.

Proof: For any $c \in (c, c_0)$ and $\{c_n\} \subset (0, c_0)$ such that $c_n \to c$ as $n \to \infty$. From the definition of $e(c_n)$ and Lemma 2.12, for every n there exists $u_n \in U(c_n)$ such that

$$E_\mu(u_n) < e(c_n) + \frac{1}{n} \text{ and } E_\mu(u_n) < 0. \tag{2.12}$$

We set $v_n := \sqrt{\frac{c}{c_n}} u_n$, and hence $\{v_n\} \subset \mathcal{S}_c$. Since $E_\mu(u) < 0$, in view of (2.16) and Lemma 2.11, one has

$$\|v_n\|^2 = \frac{c}{c_n}\|u_n\|^2 < \frac{c}{c_n} \cdot \frac{c_n}{c_0} r_0 < r_0,$$

which implies that $v_n \in U(c)$. Thus, we have

$$\begin{aligned} e(c) &\le E_\mu(v_n) \\ &= E_\mu(u_n) + \frac{a}{2}\left(\frac{c}{c_n} - 1\right)\|u_n\|^2 + \frac{b}{4}\left[\left(\frac{c}{c_n}\right)^2 - 1\right]\|u_n\|^4 \\ &\quad + \frac{1}{2}\left(\frac{c}{c_n} - 1\right)\int_{\mathbb{R}^N} V(x) u^2 \mathrm{d}x - \frac{\mu}{2p}\left[\left(\frac{c}{c_n}\right)^p - 1\right] A(u) \\ &\quad - \frac{1}{22^*_{\theta,s}}\left[\left(\frac{c}{c_n}\right)^{2^*_{\theta,s}} - 1\right] B(u) \\ &= E_\mu(u_n) + o_n(1). \end{aligned} \tag{2.13}$$

From $\{u_n\} \subset U(c_n)$, Gagliardo-Nirenberg inequality (2.3) and (2.4), we get the boundedness of $A(u)$ and $B(u)$. Then, combining (2.12) with (2.13), we have

$$e(c) \le \liminf_{n\to\infty} e(c_n). \tag{2.14}$$

On the other hand, let $\{w_n\} \subset U(c)$ be a minimizing sequence for $e(c)$ with $E_\mu(w_n) < 0$. Set $z_n := \sqrt{\frac{c_n}{c}} w_n$, then $\{z_n\} \subset U(c_n)$. Therefore, we obtain that

$$e(c_n) \le E_\mu(z_n) = E_\mu(w_n) + o_n(1) = e(c) + o_n(1),$$

which means that

$$\limsup_{n\to\infty} e(c_n) \le e(c). \tag{2.15}$$

From (2.14) and (2.15), we can easily get that

$$e(c) \le \liminf_{n\to\infty} e(c_n) \le \limsup_{n\to\infty} e(c_n) \le e(c),$$

which leads to $e(c_n) \to e(c)$ as $n \to \infty$. □

Lemma 2.10. *For* $u \in \mathcal{S}_c$ *and* V *satisfies* **(A1)**, **(A2)** and **(A3)**. *Then,* $E_\mu(t \star u) \to 0^-$ *as* $t \to -\infty$ *and* $E_\mu(t \star u) \to -\infty$ *as* $t \to +\infty$.

Proof: By the condition **(A1)** and (2.11), we have

$$E_\mu(t\star u)\geq\frac{a-\sigma_1}{2}e^{2st}\|u\|^2+\frac{b}{4}e^{4st}\|u\|^2-\frac{\mu}{2p}e^{2p\gamma_{p,s}st}A(u)-\frac{1}{22^*_{\theta,s}}e^{22^*_{\theta,s}st}B(u)$$

and

$$E_\mu(t\star u)\leq\frac{a}{2}e^{2st}\|u\|^2+\frac{b}{4}e^{4st}\|u\|^2-\frac{\mu}{2p}e^{2p\gamma_{p,s}st}A(u)-\frac{1}{22^*_{\theta,s}}e^{22^*_{\theta,s}st}B(u).$$

Since $0<2p\gamma_{p,s}<2$ and $22^*_{\theta,s}>4$, it is easy to get that the conclusion holds.

Next, we consider the constrained functional $E_\mu|_{\mathcal{S}_c}$. For every $u\in\mathcal{S}_c$, by **(A1)**, Gagliardo-Nirenberg inequality (2.3) and (2.4), we have

$$\begin{aligned}E_\mu(u)&=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-\frac{\mu}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\&\quad-\frac{1}{22^*_{\theta,s}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_{\theta,s}}|u(y)|^{2^*_{\theta,s}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\&\geq\frac{a-\sigma_1}{2}\|u\|^2-\frac{\mu}{2p}C(N,\theta,s,p)\|u\|^{2p\gamma_{p,s}}c^{p(1-\gamma_{p,s})}-\frac{1}{22^*_{\theta,s}S_\theta^{2^*_{\theta,s}}}\|u\|^{22^*_{\theta,s}}\\&=\|u\|^2\left[\frac{a-\sigma_1}{2}-\frac{\mu}{2p}C(N,\theta,s,p)\|u\|^{2p\gamma_{p,s}-2}c^{p(1-\gamma_{p,s})}-\frac{1}{22^*_{\theta,s}S_\theta^{2^*_{\theta,s}}}\|u\|^{22^*_{\theta,s}-2}\right]\end{aligned}\tag{2.16}$$

To analyze the geometry of the functional $E_\mu(u)$, for each $c>0$, we define the function $h:\mathbb{R}^+\to\mathbb{R}$,

$$h_c(r)=\frac{a-\sigma_1}{2}-\frac{\mu}{2p}C(N,\theta,s,p)c^{p(1-\gamma_{p,s})}r^{p\gamma_{p,s}-1}-\frac{1}{22^*_{\theta,s}S_\theta^{2^*_{\theta,s}}}r^{2^*_{\theta,s}-1}.\tag{2.17}$$

Lemma 2.11. *Let* $\mu>0$ *and* V *satisfies* **(A1)**, **(A2)** and **(A3)**. *There exists a constant* $c_0=c_0(\mu)>0$ *such that for each* $c\in(0,c_0)$, *the function* $h_c(r)$ *has a unique global maximum at positive level. Moreover, there exists* $r_0>0$ *such that* $h_c(r)\geq 0$ *for any* $r\in\left[\frac{c}{c_0}r_0,r_0\right]$.

Proof: Since $p\gamma_{p,s}<1$ and $2^*_{\theta,s}>2$, we get

$$h_c(r)\to-\infty\text{ as }r\to0^+\text{ and }h_c(r)\to-\infty\text{ as }r\to+\infty.$$

It is not difficult to see that $h_c(r)$ has a unique global maximum point at

$$r_c=\left[\frac{\mu(1-p\gamma_{p,s})C(N,\theta,s,p)2^*_{\theta,s}S_\theta^{2^*_{\theta,s}}}{p(2^*_{\theta,s}-1)}\right]^{\frac{1}{2^*_{\theta,s}-p\gamma_{p,s}}}c^{\frac{N-2s}{N}}.\tag{2.18}$$

Moreover, the maximum level is

$$h_c(r_c):=\frac{a-\sigma_1}{2}-Qc^{\frac{\theta+2s}{N}},$$

where

$$Q=\frac{1}{2}\left(\frac{\mu}{p}C(N,\theta,s,p)\right)^{\frac{2^*_{\theta,s}-1}{2^*_{\theta,s}-p\gamma_{p,s}}}\left[\frac{(1-p\gamma_{p,s})2^*_{\theta,s}S_\theta^{2^*_{\theta,s}}}{2^*_{\theta,s}-1}\right]^{\frac{p\gamma_{p,s}-1}{2^*_{\theta,s}-p\gamma_{p,s}}}\left(\frac{2^*_{\theta,s}-p\gamma_{p,s}}{2^*_{\theta,s}-1}\right).$$

Define

$$c_0=\left(\frac{2Q}{a-\sigma_1}\right)^{-\frac{N}{\theta+2s}},$$

then

$$\max_{r>0}h_{c_0}(r)=h_{c_0}(r_{c_0})=0.$$

For each $r\in(0,\infty)$, we easily obtain that $c\to h_c(r)$ is a nonincreasing function. Then, we get

$$\max_{r>0}h_c(r)>0 \text{ for each } c\in(0,c_0). \tag{2.19}$$

Let us define $r_0:=r_{c_0}$, which is given by (2.18). Thus, for any $c\in(0,c_0]$, we infer to

$$h_c(r_0)\ge h_{c_0}(r_0)=0. \tag{2.20}$$

On the other hand, we can also obtain that

$$h_c\left(\frac{c}{c_0}r_0\right)\ge h_{c_0}(r_0)=0. \tag{2.21}$$

By using (2.20)-(2.21) and the geometry of the function $r\to h_c(r)$, we get

$$h_c(r)\ge 0 \text{ for } r\in\left[\frac{c}{c_0}r_0,r_0\right].$$

So the proof is finished.

Let

$$U(c)=\left\{u\in\mathcal{S}_c:\|u\|^2<r_0\right\} \text{ and } \partial U(c)=\left\{u\in\mathcal{S}_c:\|u\|^2=r_0\right\}.$$

For any $c\in(0,c_0)$, we set

$$e(c):=\min_{u\in U(c)}E_\mu(u),$$

where $E_\mu(u)$ is defined in (2.1).

Lemma 2.12. *Assume that* V *satisfies* **(A1)**, **(A2)** and **(A3)**. *For any* $c\in(0,c_0)$, *then*

$$e(c)<0<\min_{u\in\partial U(c)}E_\mu(u).$$

Proof: For any $u\in\partial U(c)$, we get $\|u\|^2=r_0$. By using (2.16) and (2.20), we obtain that

$$E_\mu(u)\ge r_0h_c(r_0)>r_0h_{c_0}(r_0)=0,$$

which means $\inf_{u\in\partial U(c)}E_\mu(u)>0$. For $u\in\mathcal{S}_c$, in view of Lemma 2.10, there holds $E_\mu(t\star u)\to 0^-$ as $t\to-\infty$. So there exists $t_0\ll -1$ such that

$\|t_0 \star u\|^2 = e^{2st_0}\|u\|^2 < r_0$ and $E_\mu(t_0 \star u) = I_\mu(t_0) < 0$. Combining $\|t_0 \star u\|^2 < r_0$ with $|t_0 \star u|_2^2 = c$, we can easily get that $t_0 \star u \in U(c)$, which implies that $e(c) < 0$.

3. The Autonomous Case

In section 3, assuming that $V(x) \equiv V_\infty = 0$, we prove the existence and the asymptotic properties of the normalized ground state solutions in the autonomous case. Similar to the proof in Section 2.2, we obtain the Lemma 3.1 through Lemma 3.7, together with definitions of $\Lambda(c)$, $\partial\Lambda(c)$, $\Upsilon(c)$, etc.

Consider the Pohozaev manifold

$$\mathcal{P}_{c,\mu}^\infty := \left\{u \in \mathcal{S}_c : P_\mu^\infty(u) = 0\right\},$$

with

$$P_\mu^\infty(u) = as\|u\|^2 + bs\|u\|^4 - \mu s\gamma_{p,s}A(u) - sB(u).$$

Lemma 3.1. *Suppose that* $u \in H^s(\mathbb{R}^N)$ *is a weak solution of problem* (1.10) *under the* L^2*-norm constraint* $\mathcal{S}_c$*, then* $u \in \mathcal{P}_{c,\mu}^\infty$.

For $u \in \mathcal{S}_c$ and $t \in \mathbb{R}$, we also set $(t \star u)(x) = e^{\frac{Nt}{2}}u(e^t x)$, then $t \star u \in \mathcal{S}_c$. Similarly to (2.11), we introduce the following map

$$\begin{aligned} I_\mu^\infty(t) &:= E_{\mu,\infty}(t \star u) \\ &= \frac{a}{2}e^{2st}\|u\|^2 + \frac{b}{4}e^{4st}\|u\|^4 - \frac{\mu}{2p}e^{2p\gamma_{p,s}st}A(u) - \frac{1}{22_{\theta,s}^*}e^{22_{\theta,s}^* st}B(u). \end{aligned}$$

Then we easily obtain that

$$\begin{aligned} \left(I_\mu^\infty\right)'(t) &:= ase^{2st}\|u\|^2 + bse^{4st}\|u\|^4 - \mu\gamma_{p,s}e^{2p\gamma_{p,s}st}A(u) - se^{22_{\theta,s}^* st}B(u), \\ &= as\|t \star u\|^2 + bs\|t \star u\|^4 - \mu\gamma_{p,s}sA(t \star u) - sB(t \star u) \\ &= P_\mu^\infty(t \star u). \end{aligned}$$

As a result, we get the following conclusion.

Lemma 3.2. *Let* $u \in \mathcal{S}_c$*. Then* $t \in \mathbb{R}$ *is critical point of* $I_\mu^\infty(t)$ *if and only if* $t \star u \in \mathcal{P}_{c,\mu}^\infty$.

Lemma 3.3. *If* $\{u_n\} \subset H^s(\mathbb{R}^N)$ *satisfies* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^N)$ *and* $u_n(x) \to u(x)$ *a.e. in* $\mathbb{R}^N$ *for some* $u \in H^s(\mathbb{R}^N)$ *as* $n \to \infty$*, there holds that*

$$E_{\mu,\infty}(u_n) = E_{\mu,\infty}(u) + E_{\mu,\infty}(u_n - u) + \frac{b}{2}\|u\|^2\|u_n - u\|^2 + o_n(1).$$

Lemma 3.4 *For* $u \in \mathcal{S}_c$*. Then,* $E_{\mu,\infty}(t \star u) \to 0^-$ *as* $t \to -\infty$ *and* $E_{\mu,\infty}(t \star u) \to -\infty$ *as* $t \to +\infty$.

We consider the constrained functional $E_{\mu,\infty}|_{\mathcal{S}_c}$. For every $u \in \mathcal{S}_c$, by Gagliardo-Nirenberg inequality (2.3) and (2.4), we have

$$\begin{aligned}E_{\mu,\infty}(u) &= \frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{\mu}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\ &\quad -\frac{1}{22^*_{\theta,s}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_{\theta,s}}|u(y)|^{2^*_{\theta,s}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\ &\ge \frac{a}{2}\|u\|^2-\frac{\mu}{2p}C(N,\theta,s,p)\|u\|^{2p\gamma_{p,s}}c^{p(1-\gamma_{p,s})}-\frac{1}{22^*_{\theta,s}S_\theta^{2^*_{\theta,s}}}\|u\|^{22^*_{\theta,s}}\\ &= \|u\|^2\left[\frac{a}{2}-\frac{\mu}{2p}C(N,\theta,s,p)\|u\|^{2p\gamma_{p,s}-2}c^{p(1-\gamma_{p,s})}-\frac{1}{22^*_{\theta,s}S_\theta^{2^*_{\theta,s}}}\|u\|^{22^*_{\theta,s}-2}\right].\end{aligned} \tag{3.1}$$

To analyze the geometry of the functional $E_{\mu,\infty}(u)$, for each $c>0$, we define the function $h:\mathbb{R}^+\to\mathbb{R}$,

$$h_c(r)=\frac{a}{2}-\frac{\mu}{2p}C(N,\theta,s,p)c^{p(1-\gamma_{p,s})}r^{p\gamma_{p,s}-1}-\frac{1}{22^*_{\theta,s}S_\theta^{2^*_{\theta,s}}}r^{2^*_{\theta,s}-1}. \tag{3.2}$$

Lemma 3.5. *Let* $\mu>0$ *. There exists an* $c_\infty:=\left(\frac{2Q}{a}\right)^{-\frac{N}{\theta+s}}=c_\infty(\mu)>0$ *such that for each* $c\in(0,c_\infty)$*, the function* $h_c(r)$ *has a unique global maximum at positive level. Moreover, there exists* $r_\infty:=r_{c_\infty}\ge r_0>0$ *such that* $h_{c_\infty}(r)\ge 0$ *for any* $r\in\left[\frac{c}{c_\infty}r_\infty,r_\infty\right]$ *and* $h_{c_\infty}(r_\infty)=0.$

Let

$$\Lambda(c)=\left\{u\in\mathcal{S}_c:\|u\|^2<r_\infty\right\}\text{ and }\partial\Lambda(c)=\left\{u\in\mathcal{S}_c:\|u\|^2=r_\infty\right\}.$$

For any $c\in(0,c_\infty)$, we set

$$\Upsilon(c):=\min_{u\in\Lambda(c)}E_{\mu,\infty}(u),$$

where $E_{\mu,\infty}(u)$ is defined in (2.2). It is not difficult to show that $U(c)\subseteq\Lambda(c)$ and $\partial U(c)\subset\partial V(c)$.

Lemma 3.6. *For any* $c\in(0,c_\infty)$*, then*

$$\Upsilon(c)<0<\min_{u\in\partial\Lambda(c)}E_{\mu,\infty}(u).$$

Lemma 3.7. *The function* $c\to\Upsilon(c)$ *is continuous for any* $c\in(0,c_\infty)$*.*

Lemma 3.8. *Let* $c_1,c_2>0$ *satisfy* $c_1+c_2=c\in(0,c_\infty)$*, then*

$$\Upsilon(c)\le\Upsilon(c_1)+\Upsilon(c_2).$$

In particular, if $\Upsilon(c_1)$ or $\Upsilon(c_2)$ is achieved, then

$$\Upsilon(c)<\Upsilon(c_1)+\Upsilon(c_2).$$

Proof: Without loss of generality, we may assume $c_1<c_2$. In view of the definition of $\Upsilon(c)$ and Lemma 3.6, for any $\varepsilon>0$ small enough, there exists $u\in\Lambda(c_1)$ such that

$$E_{\mu,\infty}(u)\le\Upsilon(c_1)+\varepsilon\text{ and }E_{\mu,\infty}(u)<0. \tag{3.3}$$

Together (3.1) with Lemma 3.5, we get that

$$\|u\|^2 < \frac{c_1}{c_\infty} r_\infty.$$

Set $\zeta := \frac{c_2}{c_1} > 1$ and $\hat{u}(x) := u\left(\zeta^{-\frac{1}{N}} x\right)$. Then, $|\hat{u}|_2^2 = c_2$ and

$$\|\hat{u}\|^2 = \zeta^{1-\frac{2s}{N}} \|u\|^2 < \zeta \|u\|^2 < \frac{c_2}{c_1} \cdot \frac{c_1}{c_\infty} r_\infty < r_\infty.$$

Thus, $\hat{u} \in \Lambda(c_2)$. In view of (3.3), we obtain that

$$\begin{aligned}
\Upsilon(c_2) &\le E_{\mu,\infty}(\hat{u}) \\
&= \zeta^{1-\frac{2s}{N}} \frac{a}{2}\|u\|^2 + \zeta^{2-\frac{4s}{N}} \frac{b}{4}\|u\|^4 - \zeta^{\frac{N+\theta}{N}} \frac{\mu}{2p} A(u) - \zeta^{\frac{N+\theta}{N}} \frac{1}{22_{\theta,s}^*} B(u) \\
&= \zeta E_{\mu,\infty}(u) + \left(\zeta^{1-\frac{2s}{N}} - \zeta\right)\frac{a}{2}\|u\|^2 + \left(\zeta^{2-\frac{4s}{N}} - \zeta\right)\frac{b}{4}\|u\|^4 \\
&\quad - \left(\zeta^{\frac{N+\theta}{N}} - \zeta\right)\frac{\mu}{2p} A(u) - \left(\zeta^{\frac{N+\theta}{N}} - \zeta\right)\frac{1}{22_{\theta,s}^*} B(u) \\
&< \zeta E_{\mu,\infty}(u) < \zeta\left(\Upsilon(c_1) + \varepsilon\right).
\end{aligned}$$

Let $\varepsilon \to 0$, we get

$$\Upsilon(c_2) \le \frac{c_2}{c_1} \Upsilon(c_1). \tag{3.4}$$

Similarly as above, one has

$$\Upsilon(c) = \Upsilon(c_1 + c_2) \le \frac{c_1 + c_2}{c_2} \Upsilon(c_2). \tag{3.5}$$

Combining (3.4) with (3.5), we get

$$\Upsilon(c_1 + c_2) \le \Upsilon(c_1) + \Upsilon(c_2).$$

In particular, if $\Upsilon(c_1)$ is achieved, taking $\varepsilon = 0$, we can obtain that

$$\Upsilon(c_2) < \frac{c_2}{c_1} \Upsilon(c_1).$$

Furthermore, we get that

$$\Upsilon(c_1 + c_2) < \Upsilon(c_1) + \Upsilon(c_2).$$

Similarly, if $e(c_2)$ is achieved, the strict inequality also holds. □

Lemma 3.9

$$\Upsilon(c) = \min_{u \in \mathcal{P}_{c,\mu}^\infty} E_{\mu,\infty}(u).$$

Proof: For any $u \in \mathcal{S}_c$, in view of Lemma 3.4 and (2.10), we have

$$I_\mu^\infty(t) = E_{\mu,\infty}(t \star u) \to 0^- \text{ and } \|t \star u\| \to 0 \text{ as } t \to -\infty$$

and

$$I_\mu^\infty(t)=E_{\mu,\infty}(t\star u)\to-\infty \text{ as } t\to+\infty.$$

By Lemma 3.6, we obtain that $I_\mu^\infty(t)=E_{\mu,\infty}(t\star u)>0$ when $t\star u\in\partial\Lambda(c)$. It follows that $I_\mu^\infty(t)$ has at least two critical points $t_{u,1}$ and $t_{u,2}$, such that $t_{u,1}$ is a local minimum point with $I_\mu^\infty(t_{u,1})=E_{\mu,\infty}(t_{u,1}\star u)<0$ and $t_{u,2}$ is a maximum point with $E_{\mu,\infty}(t_{u,2}\star u)\ge\inf_{u\in\partial\Lambda(c)}E_{\mu,\infty}(u)>0$. In view of Lemma 3.2, one has

$$t_{u,1}\star u\in\mathcal{P}_{c,\mu}^{\infty,-}:=\left\{u\in\mathcal{P}_{c,\mu}^{\infty}:E_{\mu,\infty}(u)<0\right\}$$

and

$$t_{u,2}\star u\in\mathcal{P}_{c,\mu}^{\infty,+}:=\left\{u\in\mathcal{P}_{c,\mu}^{\infty}:E_{\mu,\infty}(u)>0\right\}.$$

In particular, since $\|t\star u\|$ is monotonically increasing with respect to t and $\|t\star u\|\to0$ as $t\to-\infty$, we obtain that $t_{u,1}\star u\in\Lambda(c)$. Next, we prove that $I_\mu^\infty(t)$ has no other critical points. Indeed, as $\left(I_\mu^\infty\right)'(t)=0$, we define

$$g(t)=\mu\gamma_{p,s}sA(u).$$

Hence, we can obtain that

$$g(t)=ase^{2st-2p\gamma_{p,s}st}\|u\|^2+bse^{4st-2p\gamma_{p,s}st}\|u\|^4-se^{22_{\theta,s}^*st-2p\gamma_{p,s}st}B(u).$$

It is not difficult to obtain that the monotonicity of $g(t)$ is the same as that of $\left(I_\mu^\infty\right)'(t)=0$. Next, similarly to the above method, we define

$$w(t)=as\|u\|^2.$$

As a consequence, we get that

$$w(t)=bse^{2st}\|u\|^4-se^{22_{\theta,s}^*st-2st}B(u).$$

The monotonicity of $w(t)$ is the same as that of $g(t)$. Clearly, we can get $w(t)$ has only a unique critical point, which is a global maximum at a positive level. Hence $I_\mu^\infty(t)$ has at most two critical points. It follows the above consideration that

$$\Upsilon(c)\le\min_{u\in\mathcal{P}_{c,\mu}^{\infty,-}}E_{\mu,\infty}(u)=\min_{u\in\mathcal{P}_{c,\mu}^{\infty}}E_{\mu,\infty}(u)$$

On the other hand, in view of Lemma 3.1 and Lemma 3.6, we get any minimizer v for $E_{\mu,\infty}(u)$ on $\Lambda(c)$ must belong to $\mathcal{P}_{c,\mu}^{\infty,-}$. Hence, we obtain that

$$\Upsilon(c)=E_{\mu,\infty}(v)\ge\min_{u\in\mathcal{P}_{c,\mu}^{\infty,-}}E_{\mu,\infty}(u)=\min_{u\in\mathcal{P}_{c,\mu}^{\infty}}E_{\mu,\infty}(u).$$

So the proof is completed. □

Lemma 3.10. *Assume that* $\{u_n\}\subset\Lambda(c)$ *be a minimizing sequence for* $\Upsilon(c)$. *Then one of the following alternatives holds:*

(i)

$$\limsup_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}|u_n|^2\,dx=0;$$

(ii) There exist $u\in\Lambda(c)$ and a family $\{y_n\}\subset\mathbb{R}^N$ such that $u_n(\cdot-y_n)\to u$ in $H^s(\mathbb{R}^N)$ as $n\to\infty$. In particular, $E_{\mu,\infty}(u)=\Upsilon(c)$.

Proof: Let $\{u_n\} \subset \Lambda(c)$ such that

$$E_{\mu,\infty}(u_n) \to \Upsilon(c) \text{ as } n \to \infty. \tag{3.6}$$

Then, we know the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^N)$. By contradiction, we suppose that the conclusion (i) does not hold. Hence, we need only prove that conclusion (ii) is valid. Since $\{u_n\} \subset \Lambda(c)$, there exists a family $\{y_n\} \subset \mathbb{R}^N$, up to a subsequence, we have

$$0 < \lim_{n\to\infty} \int_{B_1(0)} |u_n(x - y_n)|^2 \, \mathrm{d}x \le c, \tag{3.7}$$

Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, we get $u_n(\cdot - y_n)$ is bounded in $H^s(\mathbb{R}^N)$. Then, there exists $u \in H^s(\mathbb{R}^N)$, up to a subsequence, we have as $n \to \infty$

$$\begin{aligned} &u_n(\cdot - y_n) \rightharpoonup u \text{ in } H^s(\mathbb{R}^N), \\ &u_n(\cdot - y_n) \to u \text{ in } L^p_{\text{loc}}(\mathbb{R}^N) \quad \forall p \in [2, 2_s^*), \\ &u_n(\cdot - y_n) \to u \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.8}$$

Together (3.7) with (3.8), we get $u \ne 0$ and $|u|_2 > 0$. Define $v_n := u_n(\cdot - y_n) - u$, by (3.8), we get $v_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. Thus, by Lemma 3.3, we have

$$|u_n|_2^2 = |v_n|_2^2 + |u|_2^2 + o_n(1); \tag{3.9}$$

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o_n(1); \tag{3.10}$$

$$E_{\mu,\infty}(u_n) = E_{\mu,\infty}(u) + E_{\mu,\infty}(v_n) + \frac{b}{2}\|v_n\|^2 \|u\|^2 + o_n(1). \tag{3.11}$$

Next, we prove

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_1(z)} |v_n|^2 \, \mathrm{d}x = 0. \tag{3.12}$$

Assume by contradiction that (3.12) is not valid. Similarly as above, by the boundedness of $\{v_n\}$, there exists $\{z_n\} \subset \mathbb{R}^N$ and $v \in H^s(\mathbb{R}^N)$, up to a subsequence, such that as $n \to \infty$

$$\begin{aligned} &v_n(\cdot - z_n) \rightharpoonup v \text{ in } H^s(\mathbb{R}^N), \\ &v_n(\cdot - z_n) \to v \text{ in } L^p_{\text{loc}}(\mathbb{R}^N) \quad \forall p \in [2, 2_s^*), \\ &v_n(\cdot - z_n) \to v \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.13}$$

Moreover, $v \ne 0$ and $|v|_2 > 0$. Set $w_n = v_n(\cdot - z_n) - v$, similarly by Lemma 3.3, we obtain that

$$|v_n|_2^2 = |w_n|_2^2 + |v|_2^2 + o_n(1); \tag{3.14}$$

$$\|v_n\|^2 = \|w_n\|^2 + \|v\|^2 + o_n(1); \tag{3.15}$$

$$E_{\mu,\infty}(v_n) = E_{\mu,\infty}(w_n) + E_{\mu,\infty}(v) + \frac{b}{2}\|w_n\|^2 \|v\|^2 + o_n(1). \tag{3.16}$$

Combining (3.9)-(3.11) with (3.14)-(3.16), we get that

$$|u_n|_2^2 = |w_n|_2^2 + |v|_2^2 + |u|_2^2 + o_n(1). \tag{3.17}$$

$$\|u_n\|^2 = \|w_n\|^2 + \|v\|^2 + \|u\|^2 + o_n(1). \tag{3.18}$$

$$E_{\mu,\infty}(u_n) = E_{\mu,\infty}(w_n) + E_{\mu,\infty}(v) + E_{\mu,\infty}(u) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 + o_n(1). \tag{3.19}$$

Denote

$$m = |u|_2^2,\ n = |v|_2^2,\ l = c - m - n.$$

Note that $\{u_n\} \subset \Lambda(c)$, in view of (3.10), one has

$$\|u\|^2 = \|u_n\|^2 - \|v_n\|^2 + o_n(1) < r_\infty.$$

Thus, we get $u \in \Lambda(m)$. in view of (3.18), we obtain that $\|v\|^2 = \|u_n\|^2 - \|u\|^2 - \|w_n\|^2 + o_n(1) < r_\infty$. Thus, we get $v \in \Lambda(n)$. By (3.17), we get that

$$\lim_{n\to\infty} |w_n|_2^2 = l \geq 0.$$

In what follows, we distinguish the proof into two cases.

Case (i): $l > 0$.

From (3.18),

$$\|w_n\|^2 = \|u_n\|^2 - \|u\|^2 - \|v\|^2 + o_n(1) < r_\infty. \tag{3.20}$$

Thus, we get $w_n \in \Lambda\left(|w_n|_2^2\right)$ and $E_{\mu,\infty}(w_n) \geq \Upsilon\left(|w_n|_2^2\right)$. Then, in light of (3.6), (3.19), Lemma 3.7 and Lemma r><, we obtain that

$$\begin{aligned}
\Upsilon(c) &= E_{\mu,\infty}(u_n) + o_n(1) \\
&\geq E_{\mu,\infty}(u) + E_{\mu,\infty}(v) + \Upsilon\left(|w_n|_2^2\right) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 + o_n(1) \\
&\geq \Upsilon(m) + \Upsilon(n) + \Upsilon(l) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\
&\geq \Upsilon(c) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\
&\geq \Upsilon(c).
\end{aligned}$$

Therefore, u and v are local minimizers with respect to $\Upsilon(m)$ and $\Upsilon(n)$. Moreover, in view of Lemma 3.8, we get

$$\Upsilon(m) + \Upsilon(n) > \Upsilon(m+n).$$

Hence, we deduce that

$$\begin{aligned}
\Upsilon(c) &\geq \Upsilon(m) + \Upsilon(n) + \Upsilon(l) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\
&> \Upsilon(m+n) + \Upsilon(l) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\
&\geq \Upsilon(m+n+l) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\
&= \Upsilon(c) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\
&\geq \Upsilon(c),
\end{aligned}$$

which is impossible.

Case (ii): $l = 0$.

Owing to $l = 0$, $l = c - m - n$ and (3.17), we have

$$c = m + n \text{ and } \lim_{n\to\infty} |w_n|_2^2 = 0.$$

Since the boundedness of $\{w_n\}$ and Gagliardo-Nirenberg inequality (2.3), we get that

$$\lim_{n\to\infty} A(u) = 0. \tag{3.21}$$

In view of (2.4), (3.20) and (3.21), we infer that

$$\begin{aligned} E_{\mu,\infty}(w_n) &\ge \|w_n\|^2 \left[\frac{a}{2} - \frac{1}{22_{\theta,s}^* S_\theta^{2_{\theta,s}^*}} \|w_n\|^{22_{\theta,s}^*-2} \right] - \frac{\mu}{2p} A(u) \\ &> \|w_n\|^2 \left[\frac{a}{2} - \frac{1}{22_{\theta,s}^* S_\theta^{2_{\theta,s}^*}} r_\infty^{2_{\theta,s}^*-1} \right] - \frac{\mu}{2p} A(u) \\ &> \|w_n\|^2 h_{c_\infty}(r_\infty) + o_n(1), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} E_{\mu,\infty}(w_n) \ge 0. \tag{3.22}$$

In view of (3.19), (3.22) and Lemma 3.8, we have

$$\begin{aligned} \Upsilon(c) &\ge E_{\mu,\infty}(u) + E_{\mu,\infty}(v) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\ &\ge \Upsilon(m) + \Upsilon(n) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\ &\ge \Upsilon(c) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\ &\ge \Upsilon(c). \end{aligned}$$

Therefore, u and v are local minimizers with respect to $\Upsilon(m)$ and $\Upsilon(n)$. Moreover, by Lemma 3.8, we get

$$\Upsilon(m) + \Upsilon(n) > \Upsilon(c)$$

and hence

$$\begin{aligned} \Upsilon(c) &\ge E_{\mu,\infty}(u) + E_{\mu,\infty}(v) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\ &\ge \Upsilon(m) + \Upsilon(n) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\ &> \Upsilon(c) + \frac{b}{2}\|w_n\|^2\|v\|^2 + \frac{b}{2}\|v_n\|^2\|u\|^2 \\ &\ge \Upsilon(c), \end{aligned}$$

which yields a contradiction. Consequently, (3.12) holds. In light of Lemma 2.5, we have

$$v_n \to 0 \text{ in } L^p(\mathbb{R}^N) \text{ with } p \in (2, 2_s^*).$$

Then, it follows from Lemma 2.1 that

$$\lim_{n\to\infty} A(v_n) = 0. \tag{3.23}$$

Next, we claim that

$$\lim_{n\to\infty} \|v_n\| = 0. \tag{3.24}$$

Firstly, we prove

$$\lim_{n\to\infty} |v_n|_2 = 0. \tag{3.25}$$

By $|u|_2^2 = m > 0$, $|u_n|_2^2 = c > 0$ and (3.9), to prove (3.25) holds. We need only show that $m = c$. For the sake of contradiction, we suppose $m < c$. By applying (3.10), we have $\|v_n\|^2 \le \|u_n\|^2 \le r_\infty$. Thus, $v_n \in \Lambda(|v_n|_2^2)$ and $E_{\mu,\infty}(v_n) \ge \Upsilon(|v_n|_2^2)$. In view of (3.6) and (3.11), we have

$$\begin{aligned}\Upsilon(c) &= E_{\mu,\infty}(u_n) + o_n(1) \\ &= E_{\mu,\infty}(u) + E_{\mu,\infty}(v_n) + \frac{b}{2}\|v_n\|^2\|u\|^2 + o_n(1). \\ &\ge E_{\mu,\infty}(u) + \Upsilon(|v_n|_2^2) + o_n(1)\end{aligned}$$

Then, by Lemma 3.7 and (3.9), we get that

$$\Upsilon(c) \ge E_{\mu,\infty}(u) + \Upsilon(c-m). \tag{3.26}$$

Since $u \in \Lambda(m)$, then $E_{\mu,\infty}(u) \ge \Upsilon(m)$. If $E_{\mu,\infty}(u) > \Upsilon(m)$, in view of (3.26) and Lemma r><, we get that

$$\Upsilon(c) > \Upsilon(m) + \Upsilon(c-m) \ge \Upsilon(c),$$

which is impossible. Hence, we obtain that $E_{\mu,\infty}(u) = \Upsilon(m)$ and u is a local minimizer respect to $\Upsilon(m)$. By (3.26) and Lemma r><, we have

$$\Upsilon(c) \ge \Upsilon(m) + \Upsilon(c-m) > \Upsilon(c),$$

which is impossible. Thus, (3.25) holds and $|u|_2^2 = c$. Moreover, $u \in \Lambda(c)$ and $E_{\mu,\infty}(u) \ge \Upsilon(c)$. It follows from (3.6) and (3.11) that

$$\limsup_{n\to\infty} E_{\mu,\infty}(v_n) \le 0 - \frac{b}{2}\|v_n\|^2\|u\|^2 \le 0. \tag{3.27}$$

Due to $h_{c_\infty}(r_\infty) = 0$, $\|v_n\|^2 < r_\infty$ and (2.4), we get that

$$\frac{a}{2}\|v_n\|^2 + \frac{b}{4}\|v_n\|^4 - \frac{1}{22^*_{\theta,s}}B(u) > \|v_n\|^2\left(\frac{a}{2} - \frac{1}{22^*_{\theta,s}S_\theta^{2^*_{\theta,s}}}r_\infty^{2^*_{\theta,s}-1}\right) := \beta_0\|v_n\|^2, \tag{3.28}$$

where $\beta_0 = \frac{\mu}{2p}C(N,\theta,s,p)c_0^{p(1-\gamma_{p,s})}r_\infty^{p\gamma_{p,s}-1} > 0$. In view of (3.23), (3.27) and (3.28), we have

$$\limsup_{n\to\infty} \beta_0\|v_n\|^2 \le \limsup_{n\to\infty}\left(E_{\mu,\infty}(v_n) + \frac{\mu}{p}A(u)\right) \le 0,$$

which implies $\lim_{n\to\infty}\|v_n\| = 0$. Consequently, $u_n(\cdot - y_n) \to u$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$. In particular, $E_{\mu,\infty}(u) = \Upsilon(c)$. □

Proof of Theorem 1.3. By the translation invariance of the problem (*i.e.*, both $E_{\mu,\infty}$ and $\Lambda(c)$ are invariant under $u(\cdot)\mapsto u(\cdot-y)$), we may apply Lemma 3.10 to obtain a sequence $\{y_n\}\subset\mathbb{R}^N$ such that $u_n(\cdot-y_n)\to u_{c_\infty,\mu}$ strongly in $H^s(\mathbb{R}^N)$ as $n\to\infty$. Therefore, $u_{c_\infty,\mu}$ is a minimizer for $E_{\mu,\infty}\big|_{\Lambda(c)}$ at level $\Upsilon(c)$.

Let $u_n\subset\Lambda(c)$ be a minimizing sequence for $\Upsilon(c)$. In view of Lemma 3.6, we get that $\Upsilon(c)<0$. It can be ready shown that

$$\limsup_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}|u_n|^2\,\mathrm{d}x=0 \tag{3.29}$$

can not happen. Otherwise, by Lemma 2.5, we have $u_n\to 0$ in $L^t(\mathbb{R}^N)$ as $n\to\infty$ for any $t\in(2,2_s^*)$. Together this and Gagliardo–Nirenberg inequality (2.3), we get that $\lim_{n\to\infty}A(u_n)=0$. Following the proof of (3.22), we obtain that

$$\Upsilon(c)=\lim_{n\to\infty}E_{\mu,\infty}(u_n)\ge 0,$$

which contradicts to $\Upsilon(c)<0$. Now, since the vanishing case (3.29) is impossible and the dichotomy case is ruled out by Lemma 3.8, it follows from the concentration-compactness principle that the minimizing sequence must be compact. Since the energy functional $E_{\mu,\infty}$ and the constraint manifold $\Lambda(c)$ are invariant under the translation $u(\cdot)\mapsto u(\cdot-y)$ for any $y\in\mathbb{R}^N$, Lemma 3.10 yields a sequence $\{y_n\}\subset\mathbb{R}^N$ such that $u_n(\cdot-y_n)\to u_{c_\infty,\mu}$ strongly in $H^s(\mathbb{R}^N)$ as $n\to\infty$. Consequently, $u_{c_\infty,\mu}$ is a minimizer for $E_{\mu,\infty}\big|_{\Lambda(c)}$ at level $\Upsilon(c)$. Let $\tilde{u}_{c_\infty,\mu}$ denote the Schwarz rearrangement of $u_{c_\infty,\mu}$, then

$$|\tilde{u}_{c_\infty,\mu}|_2^2=|u_{c_\infty,\mu}|_2^2=c,\quad \|\tilde{u}_{c_\infty,\mu}\|_2^2=\|u_{c_\infty,\mu}\|_2^2<r_\infty.$$

By using Riesz's rearrangement inequality [24], we get that

$$A(\tilde{u}_{c_\infty,\mu})>A(u_{c_\infty,\mu}),\quad B(\tilde{u}_{c_\infty,\mu})>B(u_{c_\infty,\mu}).$$

Therefore, we obtain

$$\tilde{u}_{c_\infty,\mu}\in\Lambda(c)\ \text{ and }\ \Upsilon(c)\le E_{\mu,\infty}(\tilde{u}_{c_\infty,\mu})\le E_{\mu,\infty}(u_{c_\infty,\mu})=\Upsilon(c),$$

which implies $E_{\mu,\infty}(\tilde{u}_{c_\infty,\mu})=\Upsilon(c)$. As a consequence, $\Upsilon(c)$ is reached by a radially symmetric function $\tilde{u}_{c_\infty,\mu}$, that satisfies problem (1.10) under the L^2-norm constraint $\mathcal{S}_c$. By the Lagrange multiplier theorem, there exists a $\lambda_{c_\infty,\mu}\in\mathbb{R}^N$ such that

$$\begin{aligned}&\left(a+b\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}\tilde{u}_{c_\infty,\mu}\right|^2\mathrm{d}x\right)(-\Delta)^s\tilde{u}_{c_\infty,\mu}\\&=\lambda_{c_\infty}\tilde{u}_{c_\infty,\mu}+\mu\left(I_\theta*|\tilde{u}_{c_\infty,\mu}|^p\right)|\tilde{u}_{c_\infty,\mu}|^{p-2}\tilde{u}_{c_\infty,\mu}+\left(I_\theta*|\tilde{u}_{c_\infty,\mu}|^{2_{\theta,s}^*}\right)|\tilde{u}_{c_\infty,\mu}|^{2_{\theta,s}^*-2}\tilde{u}_{c_\infty,\mu},\end{aligned}$$

which implies that $P_\mu^\infty(u)$. Combining $P_\mu^\infty(u)=0$, we get that

$$\begin{aligned}\lambda_{c_\infty,\mu}c&=a\|\tilde{u}_{c_\infty,\mu}\|^2+b\|\tilde{u}_{c_\infty,\mu}\|^4-\mu A(\tilde{u}_{c_\infty,\mu})-B(\tilde{u}_{c_\infty,\mu})\\&=\mu\gamma_{p,s}A(\tilde{u}_{c_\infty,\mu})-\mu A(\tilde{u}_{c_\infty,\mu})+B(\tilde{u}_{c_\infty,\mu})-B(\tilde{u}_{c_\infty,\mu})\\&=\mu(\gamma_{p,s}-1)A(\tilde{u}_{c_\infty,\mu}).\end{aligned}$$

Since $\gamma_{p,s} < 1$, we obtain that $\lambda_{c_\infty,\mu} < 0$. Furthermore, by Lemma 2.12 and Lemma 3.9, $\tilde{u}_{c_\infty,\mu}$ is a normalized ground state solution of problem (1.10) under the L^2-norm constraint $\mathcal{S}_c$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.4. Since $c_\infty \to +\infty$ as $\mu \to 0^+$, for a fixed $c > 0$, there exists $u_{c_\infty,\mu} \in \mathcal{S}_c$ for any $\mu > 0$ sufficiently small, be a normalized ground state solutions of problem (1.10) under the L^2-norm constraint $\mathcal{S}_c$. By using Lemma 3.5, we get that

$$u_{c_\infty,\mu} \in \mathcal{P}_{c,\mu}^\infty \text{ with } \left\|u_{c_\infty,\mu}\right\|^2 < r_\infty,$$

where r_∞ is defined by Lemma 3.5. It is not difficult to see that $r_\infty \to 0$ as $\mu \to 0^+$, then

$$\left\|u_{c_\infty,\mu}\right\|^2 \to 0 \text{ as } \mu \to 0^+.$$

By Gagliardo-Nirenberg inequality (2.3), we derive that $\mu A\left(u_{c_\infty,\mu}\right) \to 0$ as $\mu \to 0^+$. Hence, in view of $u_{c_\infty,\mu} \in \mathcal{P}_{c,\mu}^\infty$, we obtain that

$$\begin{aligned}\lim_{\mu\to 0} E_{\infty,\mu}\left(u_{c_\infty,\mu}\right) &= \lim_{\mu\to 0}\left[\left(\frac{a}{2}-\frac{a}{22_{\theta,s}^*}\right)\left\|u_{c_\infty,\mu}\right\|^2+\left(\frac{b}{4}-\frac{b}{22_{\theta,s}^*}\right)\left\|u_{c_\infty,\mu}\right\|^4\right.\\ &\quad\left.-\frac{\mu}{2p}\left(1-\frac{2p\gamma_{p,s}}{22_{\theta,s}^*}\right)A\left(u_{c_\infty,\mu}\right)\right]\\ &= 0.\end{aligned}$$

4. The Nonautonomous Case

In Section 4, V satisfies **(A1)**, **(A2)** and **(A3)**. We prove the existence of the normalized ground state solutions in the nonautonomous case.

Lemma 4.1. *Assume that* V *satisfies* **(A1)**, **(A2)** *and* **(A3)**. *Define*

$$e_\infty(c) := \inf_{u\in U(c)} E_{\mu,\infty}(u),$$

where $E_{\mu,\infty}(u)$ is defined in (2.2). Then $e(c) < e_\infty(c)$ for $c \in (c, c_0)$.

Proof: Similar to Theorem 1.2, we know that there exists a local minimizer $u_0 \in U(c)$ satisfying $E_{\mu,\infty}(u_0) = e_\infty(c)$ for $c \in (0, c_0)$. By using **(A1)**, we can obtain that

$$e(c) \le E_\mu(u_0) = E_{\mu,\infty}(u_0) + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u_0|^2 \,\mathrm{d}x < E_{\mu,\infty}(u_0) = e_\infty(c).$$

Therefore, we get $e(c) < e_\infty(c)$ for $c \in (c, c_0)$.

Lemma 4.2. *Assume that* V *satisfies* **(A1)**, **(A2)** *and* **(A3)**. *If* $0 < c_1 < c_2$ *and* $c_1, c_2 \in (0, c_0)$, *then* $e(c_2) < \frac{c_2}{c_1} e(c_1)$.

Proof: In view of the definition of $e(c)$ and Lemma 2.12, for any $\varepsilon > 0$ small enough, there exists $u \in U(c_1)$ such that

$$E_\mu(u) \le e(c_1) + \varepsilon \text{ and } E_\mu(u) < 0. \tag{4.1}$$

Together (2.16) with Lemma 2.11, we get that $\|u\|^2 < \frac{c_1}{c_0} r_0$. Set $\eta := \frac{c_2}{c_1} > 1$ and

$\hat{u}(x) := u\left(\eta^{-\frac{1}{N}}x\right)$. Then, $|\hat{u}|_2^2 = c_2$ and

$$\|\hat{u}\|^2 = \eta^{1-\frac{2s}{N}}\|u\|^2 < \eta\|u\|^2 < \frac{c_2}{c_1}\cdot\frac{c_1}{c_0}r_0 < r_0.$$

Thus, $\hat{u} \in U(c_2)$. It follows from **(A3)** and $\eta^{\frac{1}{N}} > 1$ that

$$V\left(\eta^{\frac{1}{N}}|x|\right) < V(|x|),\ \forall x \in \mathbb{R}^N. \tag{4.2}$$

Then, in view of (4.1) and (4.2), we obtain that

$$\begin{aligned} e(c_2) &\le E_\mu(\hat{u}) \\ &= \eta^{1-\frac{2s}{N}}\frac{a}{2}\|u\|^2 + \eta^{2-\frac{4s}{N}}\frac{b}{4}\|u\|^4 + \frac{\eta}{2}\int_{\mathbb{R}^N} V\left(\eta^{\frac{1}{N}}x\right)|u(x)|^2\,\mathrm{d}x \\ &\quad -\eta^{\frac{N+\theta}{N}}\frac{\mu}{2p}A(u) + \eta^{\frac{N+\theta}{N}}\frac{1}{22_{\theta,s}^*}B(u) \\ &= \eta E_\mu(u) + \left(\eta^{1-\frac{2s}{N}} - \eta\right)\frac{a}{2}\|u\|^2 + \left(\eta^{2-\frac{4s}{N}} - \eta\right)\frac{b}{4}\|u\|^4 \\ &\quad + \frac{\eta}{2}\int_{\mathbb{R}^N}\left[V\left(\eta^{\frac{1}{N}}x\right) - V(x)\right]|u(x)|^2\,\mathrm{d}x \\ &\quad -\left(\eta^{\frac{N+\theta}{N}} - \eta\right)\frac{\mu}{2p}A(u) - \left(\eta^{\frac{N+\theta}{N}} - \eta\right)\frac{1}{22_{\theta,s}^*}B(u) \\ &< \eta E_\mu(u) < \eta\left(e(c_1) + \varepsilon\right). \end{aligned}$$

Let $\varepsilon \to 0$, we get

$$e(c_2) \le \frac{c_2}{c_1}e(c_1). \tag{4.3}$$

Proof of Theorem 1.5. Let $\{u_n\} \subset U(c)$ be a minimizing sequence with respect to $e(c)$, it is evident that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Up to a subsequence, there exists a $u \in H^s(\mathbb{R}^N)$ such that as $n \to \infty$

$$\begin{aligned} &u_n(\cdot - y_n) \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^N), \\ &u_n(\cdot - y_n) \to u \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N)\quad \forall p \in [2, 2_s^*), \\ &u_n(\cdot - y_n) \to u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{4.4}$$

Case (i): $u = 0$. In view of **(A1)**, $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, the boundedness of $V(x)$ in $\mathbb{R}^N$ and the boundedness of $\{u_n\}$ in $L^2(\mathbb{R}^N)$ hold, so we obtain that

$$\int_{\mathbb{R}^N} V(x)|u_n|^2\,\mathrm{d}x = o_n(1).$$

Accordingly, it can observe that

$$e_\infty(c) + o_n(1) \le E_{\mu,\infty}(u_n) + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u_n|^2\,\mathrm{d}x + o_n(1) = E_\mu(u_n) = e(c) + o_n(1),$$

which implies $e_\infty(c) \le e(c)$. However, it is in contradiction to Lemma 4.1.

Therefore, $u \neq 0$.

Case (ii): $u \neq 0$. Define $v_n := u_n - u$. In view of Lemma 2.6, we get that

$$|u_n|_2^2 = |v_n|_2^2 + |u|_2^2 + o_n(1); \tag{4.5}$$

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o_n(1); \tag{4.6}$$

$$E_\mu(u_n) = E_\mu(v_n) + E_\mu(u) + \frac{b}{2}\|u\|^2\|v_n\|^2 + o_n(1). \tag{4.7}$$

Now we claim that $|v_n|_2^2 = o_n(1)$. In order to prove this, let $|u|_2^2 = c_1$, we get $0 < c_1 \le c$ from (4.5). If $c_1 = c$, it follows from (4.5) that $|v_n|_2^2 = o_n(1)$. If $c_1 \in (0,c)$, by (4.5) and (4.6), we deduce that

$$|v_n|_2^2 < c \text{ and } \|v_n\|^2 \le \|u_n\|^2 < r_0$$

for n large enough. Therefore, we can get that $\{v_n\} \in U\left(|v_n|_2^2\right)$ and $E(v_n) \ge e\left(|v_n|_2^2\right)$. We also get that $u \in U(c_1)$ by the weak limit. In view of (4.7), Lemma 4.2 and $u \in U(c_1)$, we obtain that

$$\begin{aligned}
e(c) + o_n(1) = E_\mu(u_n) &= E_\mu(v_n) + E_\mu(u) + \frac{b}{2}\|u\|^2\|v_n\|^2 + o_n(1) \\
&\ge e\left(|v_n|_2^2\right) + e(c_1) + \frac{b}{2}\|u\|^2\|v_n\|^2 + o_n(1) \\
&\ge \frac{|v_n|_2^2}{c} e(c) + e(c_1) + \frac{b}{2}\|u\|^2\|v_n\|^2 + o_n(1) \\
&\ge \frac{|v_n|_2^2}{c} e(c) + e(c_1) + o_n(1).
\end{aligned}$$

Then, owing to (4.5) and Lemma 4.2, we have that

$$e(c) \ge \frac{c - c_1}{c} e(c) + e(c_1) > \frac{c - c_1}{c} e(c) + \frac{c_1}{c} e(c) = e(c),$$

which is impossible. Hence, we infer that $u \in U(c)$. It follows from $u_n \in U(c)$, $u \in U(c)$ and (4.5) that we have

$$|v_n|_2^2 = o_n(1). \tag{4.8}$$

Let us prove that $\|v_n\|^2 = o_n(1)$, this will prove $v_n \to 0$ in $H^s(\mathbb{R}^N)$ and $E_\mu(u) = e(c)$. To complete this purpose, owing to (4.6) and $u \neq 0$, we observe that

$$\|v_n\|^2 \le \|u_n\|^2 < r_0 \text{ for } n \text{ large enough.} \tag{4.9}$$

In view of Gagliardo-Nirenberg inequality (2.3), (4.8) and (4.9), we have $\lim_{n\to\infty} A(u) = 0$. Due to $h_{c_0}(r_0) = 0$, $\|v_n\|^2 < r_0$ and (2.4), we get that

$$\begin{aligned}
E_\mu(v_n) &= \frac{a}{2}\|v_n\|^2 + \frac{b}{4}\|v_n\|^4 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x - \frac{\mu}{2p}A(u) - \frac{1}{22_{\theta,s}^*}B(u) \\
&\ge \|v_n\|^2\left(\frac{a-\sigma_1}{2} - \frac{1}{22_{\theta,s}^* S_\theta^{2_{\theta,s}^*}} r^{2_{\theta,s}^*-1}\right) + o_n(1) \qquad (4.10) \\
&:= \eta_0\|v_n\|^2 + o_n(1),
\end{aligned}$$

where $\eta_0 = \frac{\mu}{2p} C(N,\theta,s,p) c_0^{p(1-\gamma_{p,s})} r_0^{p\gamma_{p,s}-1} > 0$. Furthermore, by virue of $u \in U(c)$, (4.7) and (4.10), we have

$$
\begin{aligned}
e(c)+o_n(1) = E_\mu(u_n) &= E_\mu(u)+E_\mu(v_n)+\frac{b}{2}\|u\|^2\|v_n\|^2+o_n(1) \\
&\geq e(c)+\eta_0\|v_n\|^2+o_n(1),
\end{aligned}
$$

which leads to

$$
e(c) \geq e(c)+\eta_0 \lim_{n\to\infty}\|v_n\|^2,
$$

so

$$
\|v_n\|^2 = o_n(1). \tag{4.11}
$$

Together with (4.7), (4.8) and (4.11), we have

$$
u_n \to u \text{ in } H^s(\mathbb{R}^N) \text{ and } E_\mu(u) = e(c).
$$

In what follows, we note that $u := u_{c,\mu}$ is a minimizer for $E_\mu(u)$ on $U(c)$. By the Lagrange multiplier rules, there exists $\lambda_{c,\mu} \in \mathbb{R}$ such that

$$
\begin{aligned}
&\left(a+b\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right|^2\mathrm{d}x\right)(-\Delta)^s u_{c,\mu}+V(x)u_{c,\mu} \\
&= \lambda_{c,\mu}u_{c,\mu}+\mu\left(I_\theta*|u_{c,\mu}|^p\right)|u_{c,\mu}|^{p-2}u_{c,\mu}+\left(I_\theta*|u_{c,\mu}|^{2^*_{\theta,s}}\right)|u_{c,\mu}|^{2^*_{\theta,s}-2}u_{c,\mu} \text{ in } \mathbb{R}^N,
\end{aligned} \tag{4.12}
$$

which means that $P_\mu(u_{c,\mu}) = 0$. By using $P_\mu(u_{c,\mu}) = 0$, we get that

$$
\begin{aligned}
\lambda_{c,\mu}c &= a\|u_{c,\mu}\|^2+b\|u_{c,\mu}\|^4+\int_{\mathbb{R}^N}V(x)u_{c,\mu}^2\mathrm{d}x-\mu A(u_{c,\mu})-B(u_{c,\mu}) \\
&= \left[\int_{\mathbb{R}^N}V(x)u_{c,\mu}^2\mathrm{d}x+\frac{1}{s}\int_{\mathbb{R}^N}W(x)u_{c,\mu}^2\mathrm{d}x\right] \\
&\quad +\left[\mu\gamma_{p,s}A(u_{c,\mu})-\mu A(u_{c,\mu})\right]+\left[B(u_{c,\mu})-B(u_{c,\mu})\right] \\
&= \int_{\mathbb{R}^N}\left[V(x)+\frac{1}{s}W(x)\right]u_{c,\mu}^2\mathrm{d}x+\mu(\gamma_{p,s}-1)A(u_{c,\mu}).
\end{aligned}
$$

Since $\gamma_{p,s} < 1$ and the condition of **(A2)**, we obtain that $\lambda_{c,\mu} < 0$. The proof is completed.

In this paper, we establish the existence of normalized ground state solutions for a class of upper critical fractional Kirchhoff-Choquard equations with potentials. Both the autonomous and nonautonomous cases are considered. By applying the concentration-compactness principle, we rule out the possibilities of vanishing and dichotomy, which leads to the compactness of minimizing sequences. Translation invariance then guarantees that a suitable translation of the sequence remains minimizing and converges strongly to a ground state. Furthermore, in the autonomous case, we analyze the asymptotic behavior of these solutions as $\mu \to 0^+$.

Conflicts of Interest

The author declares no conflicts of interest.

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