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Well-Posedness for Quintic Energy Critical Wave in 3D Cylindrical Convex Domains

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Abstract

In this work, we prove that the quintic energy critical wave inside a cylindrical convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega \neq \varnothing$ is well-posed in energy space. The dispersive estimates found in [1] and the Strichartz estimates found in [2] are essential resources for demonstrating local well-posedness. We note that our findings on the local and global existence of the wave equation solution in the cylindrical domain setting interpolate between those in any bounded domains in $\mathbb{R}^3$ and in Euclidean space $\mathbb{R}^3$. Furthermore, when combined with the trace estimates and the nonconcentration of nonlinear effect in a small light cone, the result of the Strichartz estimates in our context is strong enough to allow us to extend local to global well-posedness.

Subject Areas

Partial Differential Equations

Keywords

Energy Critical Waves, Cylindrical Domains, Dispersive Estimates, Strichartz Estimates

1. Introduction

Let $\Omega = \{x \geq 0, (y,z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ be a convex domain with smooth boundary $\partial\Omega = \{x = 0\}$ and $\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2$ the Laplacian acting on functions with Dirichlet boundary condition. We examine the energy critical semilinear wave equation that follows.

$$\begin{aligned}&\left(\partial_t^2 - \Delta\right)u + u^5 = 0 \ \text{ in } \mathbb{R}_t \times \Omega,\\&u_{|t=0} = u_0,\ \partial_t u_{|t=0} = u_1,\ u_{|x=0} = 0.\end{aligned} \tag{1}$$

In this work, we focus on the questions of well-posedness of (1) for the initial condition

$$\left(u,\partial_t u\right)_{|t=0}=\left(u_0,u_1\right)\in H_0^1(\Omega)\times L^2(\Omega).$$

By using cylindrical coordinates (r,θ,z), with $r=1-\frac{x}{2}$, $\theta=y$, and $z=z$ see Remark 1.1 [3], the Riemannian manifold (Ω,Δ) with Laplacian $\Delta=\partial_x^2+(1+x)\partial_y^2+\partial_z^2$ can be locally viewed as a cylindrical domain in $\mathbb{R}^3$. In our cylindrical domain, the boundary is convex with zero curvature along the cylinder's axis, and the nonnegative radius of curvature depends on the incident angle and vanishes in certain directions. Here, we emphasize that our domain interpolates between the bounded domain in $\mathbb{R}^3$ [4] and the Euclidean space $\mathbb{R}^3$ [5].

The inspiration to consider the Laplacian Δ in our context comes from the Friedlander's model domain of the half space $\Omega_D=\{(x,y)\mid x>0,y\in\mathbb{R}\}$ with Laplace operator given by

$$\Delta_D=\partial_x^2+(1+x)\partial_y^2.$$

We note that the problem is reduced to the Friedlander's model [3] when there is no z variable in our Laplacian. Additionally, there is a nice property of the Laplacian Δ in our setting that enables explicit computations [1] [6] [7]. Moreover, the Laplacian Δ in our setting has a nice feature that allows explicit computations. Finally, let us identify a connection between the Laplacian Δ and the Laplace-Beltrami operator Δ_g. For a metric $g=\mathrm{d}x^2+(1+x)^{-1}\mathrm{d}y^2+\mathrm{d}z^2$, we see that the Laplace-Beltrami is given by $\Delta_g=(1+x)^{\frac{1}{2}}\partial_x(1+x)^{-\frac{1}{2}}\partial_x+(1+x)\partial_y^2+\partial_z^2$, which is a self-adjoint operator with the volume form $\sqrt{\det g}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=(1+x)^{-\frac{1}{2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z$. This implies that the difference $\Delta_g-\Delta=-\frac{1}{2}(1+x)^{-1}\partial_x$ is the first order differential operator. Therefore, it can be seen as a lower order perturbative term as long as we are working with local in time dispersive/Strichartz estimates for data near the boundary and proving local in time estimates for Δ implying the same set of estimates for Δ_g. Our model uses instead the Laplace operator associated with the Dirichlet form

$$\int|\nabla u|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int\left(|\partial_x u|^2+(1+x)|\partial_y u|^2+|\partial_z u|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

We now turn to see some key properties of the solution to (1). The solution u of (1) possesses the invariant property under the dilation symmetry

$$u(t,x)\mapsto u_\lambda(t,x):=\lambda^{\frac{1}{2}}u(\lambda t,\lambda x)$$

and

$$\partial_t u(t,x)\mapsto\partial_t u_\lambda(t,x):=\lambda^{\frac{3}{2}}\partial_t u(\lambda t,\lambda x).$$

Recall that the $H_0^1(\Omega)\times L^2(\Omega)$ is invariant under this dilation symmetry.

Indeed, one has

$$\left\|\left(u_\lambda(0),\partial_t u_\lambda(0)\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}=\left\|\left(u(0),\partial_t u(0)\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}$$

In addition, solution u to (1) satisfies an energy conservation law [8]:

$$E(u)(t)=\int_\Omega\left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2+\frac{1}{6}|u|^6\right)\mathrm{d}x=E(u)(0).$$

We remark that this explains why the exponent 5 in the nonlinear term u^5 is the H^1 critical wave since it fulfills $5=\frac{d+2}{d-2}$ with the space dimension $d=3$. According to the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^6(\Omega)$, we see that the energy $E(u)(0)$ is finite for any initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$.

For $\Omega=\mathbb{R}^3$, the global existence of smooth solutions for the energy critical wave (1) was demonstrated in [5], and for energy space solutions in [9] [10]. In [11], this result is extended to the exterior of convex obtacles. Lastly, [4] addressed the situation of bounded domains $\Omega\subset\mathbb{R}^3$ using Dirichlet condition.

Let us also review some results on dispersive and Strichartz estimates as they are powerful tools in the study of well-possedness of nonlinear problems.

Let u be the solution of the Cauchy problem for the wave equation in $\mathbb{R}\times\mathbb{R}^d$,

$$\left(\partial_t^2-\Delta_{\mathbb{R}^d}\right)u=0,\ u(0,x)=u_0(x),\ \partial_t u(0,x)=u_1(x)$$

It follows that u can be written as a sum of Fourier integral operators (see [12] [13])

$$\mathrm{e}^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}f(x):=(2\pi)^{-n}\int_{\mathbb{R}^n}\mathrm{e}^{i(x\cdot\xi)\pm t|\xi|}\hat{f}(\xi)\mathrm{d}\xi$$

where $\hat{f}(\xi)=\frac{1}{2}\left(\hat{u}_0(\xi)\pm i^{-1}|\xi|^{-1}\hat{u}_1(\xi)\right)$ which satisfies the dispersive estimates

$$\left\|\chi(hD_t)\mathrm{e}^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}\right\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)}\le Ch^{-d}\min\left\{1,\left(\frac{h}{|t|}\right)^{\frac{d-1}{2}}\right\},\tag{2}$$

where $\Delta_{\mathbb{R}^d}$ is the Laplace operator in $\mathbb{R}^d$. The function χ in this case and the sequel belongs to $C_0^\infty(]0,\infty[)$ and is equal to 1 on $[1,2]$ and $D_t=\frac{1}{i}\partial_t$.

Ivanovici *et al.* established the optimal (local in time) dispersive estimates for the wave equations inside strictly convex domains Ω_D of dimensions $d\ge 2$ in [3]. More specifically, they have demonstrated that

$$\left\|\chi(hD_t)\mathrm{e}^{\pm it\sqrt{-\Delta_D}}\right\|_{L^1(\Omega_D)\to L^\infty(\Omega_D)}\le Ch^{-d}\min\left\{1,\left(\frac{h}{|t|}\right)^{\frac{d-1}{2}-\frac{1}{4}}\right\},\tag{3}$$

where Δ_D is the Laplace operator on Ω_D.

In comparison to the free wave estimates (2), (3) causes a loss of $\frac{1}{4}$ powers of $(h/|t|)$ factor due to the caustics creation in arbitrarily small times.

The dispersive estimates were established by the authors in [14] [15] outside the unit ball or the cylinder $\Theta \subset \mathbb{R}^3$ as follows.

$$\left\| \chi\left(hD_t\right) \mathrm{e}^{\pm it\sqrt{-\Delta_D}} \right\|_{L^1(\Omega_D) \to L^\infty(\Omega_D)} \le Ch^{-3} \min\left\{1, \frac{h}{|t|}\right\}, \tag{4}$$

where Δ_D is the Laplace operator on $\Omega_D = \mathbb{R}^3 \setminus \Theta$.

The Strichartz estimates on Riemannian manifolds are given in [3]. Let (Ω, g) be a Riemannian manifold without boundary of dimensions $d \ge 2$. Local in time Strichartz estimates state that

$$\|u\|_{L^q\left((-T,T);L^r(\Omega)\right)} \le C_T\left(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)}\right), \tag{5}$$

where $\dot{H}^\beta$ denotes the homogeneous Sobolev space over Ω of order β and $2 \le q, r \le \infty$ satisfy

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta, \ \frac{1}{q} \le \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right).$$

Here $u = u(t,x)$ is a solution to the wave equation

$$\left(\partial_t^2 - \Delta_g\right)u = 0 \ \text{ in } (-T,T)\times\Omega, \ u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x),$$

where Δ_g denotes the Laplace-Beltrami operator on (Ω, g). The estimates (5) hold on $\Omega = \mathbb{R}^d$ and $g_{ij} = \delta_{ij}$.

The Strichartz estimates for the wave equation on (compact or noncompact) Riemannian manifolds with boundaries were established by Blair *et al.* in [16]. They demonstrated that the Strichartz estimates (5) hold if Ω is a compact manifold with boundary and (q, r, β) is a triple satisfying

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta \ \text{ for } \begin{cases} \dfrac{3}{q} + \dfrac{d-1}{r} \le \dfrac{d-1}{2}, & d \le 4, \\ \dfrac{1}{q} + \dfrac{1}{r} \le \dfrac{1}{2}, & d \ge 4. \end{cases}$$

Ivanovici *et al.* developed a local in time Strichartz estimate (5) in [3] from the optimal dispersive estimates inside strictly convex domains of dimensions $d \ge 2$ for a triple (d, q, β) satisfying

$$\frac{1}{q} \le \left(\frac{d-1}{2} - \frac{1}{4}\right)\left(\frac{1}{2} - \frac{1}{r}\right) \ \text{ and } \ \beta = d\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q}.$$

Compared to the result by Blair *et al.* in [16], this improves the range of indices for which sharp Strichartz estimates do hold for $d \ge 3$. On the other hand, the results in [16] are applicable to any manifolds or domains with boundary.

For pairings (q, r) such that

$$\frac{1}{q} \le \left(\frac{1}{2} - \frac{1}{9}\right)\left(\frac{1}{2} - \frac{1}{r}\right),$$

the most recent results on Strichartz estimates inside the Friedlander model domain have been obtained in [17]. For $d = 2$, this result improves the existing results for strictly convex domains, but [3] only provides a loss of $\frac{1}{4}$.

Here, we deal with the Equation (1) in $\Omega \subset \mathbb{R}^3$, where Ω is a locally cylindrical convex domain. This work's primary finding about local well-posedness is as follows.

Theorem 1. Let $\Omega = \{x \ge 0, (y,z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ and $\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2$. For any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the energy-critical wave Equation (1) is locally well-posed in the space

$$C^0\left((0,T); H_0^1(\Omega)\right) \cap C^1\left((0,T); L^2(\Omega)\right) \cap L^5\left((0,T); L^{10}(\Omega)\right).$$

We note that the nonlinearity is defocusing due to its sign, which does not play a role for the local existence of solutions to (1) and hence the result in Theorem 1 also hold in case of focusing quintic wave equation. While the sign of nonlinear term is crucial for global in time existence of solutions to (1). The main difficulty for proving global in time solutions to (1) is that one does not obtain a bound on $u \in L^5\left((0,T); L^{10}(\Omega)\right)$ and thus on $u^5 \in L^1\left((0,T); L^2(\Omega)\right)$. But in our domain, the Strichartz estimates in Theorem 4 allow us to control $L^5\left((0,T); L^{10}(\Omega)\right)$ of the solution to (1) by the energy norm of the initial data.

The Theorem 1 was established by Burq, Lebeau, Planchon in [4] for any bounded domains $\Omega_D \subset \mathbb{R}^3$. Their idea is based on the spectral projector established by Smith and Sogge in [18] to derive the optimal and scale invariant Strichartz estimates for the solution to the wave equation, while in our setting we will use the dispersive estimates obtained in [1] [6] [7] and the Strichartz estimates in [2] to establish local existence of solutions to (1).

We point out that the approach in [4] to get the Strichartz estimates to control the $L^5\left((0,1); W_0^{\frac{3}{10},5}(\Omega_D)\right)$ norm of the solution of the wave equation by the energy norm, assuming Ω is compact. This estimate allows them to extend local to global well-posedness for any initial data having finite energy, when combined with the estimate for the normal derivative and the nonconcentration estimate of nonlinear effect.

The global well-posedness in our setting reads as follows.

Theorem 2. Let $\Omega = \{x \ge 0, (y,z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ and $\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2$. For any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the energy-critical wave Equation (1) is globally well-posed in the space

$$C^0\left(\mathbb{R}_t; H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}_t; L^2(\Omega)\right) \cap L^5_{loc}\left(\mathbb{R}_t; L^{10}(\Omega)\right).$$

In this paper, for $s \ge 0$, let $\dot{H}^s(\Omega) = (-\Delta)^{-\frac{s}{2}} L^2(\Omega)$ be the homogeneous Sobolev space over Ω and $H_0^s(\Omega)$ is the closure in $\dot{H}^s(\Omega)$ of the set of smooth and compactly supported functions. We note that $\dot{H}^s(\Omega) = H_0^s(\Omega)$ for $0 \le s < \frac{3}{2}$, and that when $s = 1$, $H_0^1(\Omega)$ is a Hilbert space with the inner product of $H^1(\Omega)$. The notation $A \lesssim B$ means that there exists a constant C such that $A \le CB$ and this constant may change from line to line but is independent of all parameters. Similarly, $A \sim B$ means there exist constants C_1, C_2 such that

$C_1 B \le A \le C_2 B$.

2. Dispersive and Strichartz Estimates

The results of the local in time dispersive and Strichartz estimates for the solution to the linear wave equation in the cylindrical domain Ω with the previously defined Laplace Δ will be shown in this section.

Let us consider the Dirichlet wave equation inside the half space $\Omega = \{x \ge 0, (y,z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$

$$\begin{aligned} &(\partial_t^2 - \Delta) u = 0 \text{ in } \mathbb{R}_t \times \Omega, \\ &u_{|t=0} = \delta_a,\ \partial_t u_{|t=0} = 0,\ u_{|x=0} = 0. \end{aligned} \tag{6}$$

where the Dirac distribution $\delta_a = \delta_{x=a,y=0,z=0}$ with $(a,0,0) \in \Omega$, $a > 0$. In local coordinates a denotes the distance from the source point to the boundary of Ω. We assume that $0 < a \ll 1$ is small enough as we are interested only in highly reflected waves, which give us interesting phenomena such as caustics near the boundary.

The local in time dispersive estimates for the wave Equation (6) are established in [1] [6] [7]. Let $\chi \in C_0^\infty(]0,\infty[), \chi = 1$ on $[1,2]$ and $D_t = \frac{1}{i}\partial_t$. Let $\mathcal{G}_a$ be the Green function for (6).

Theorem 3. There exists C such that for every $h \in]0,1]$, every $t \in [-1,1]$ and every $a \in]0,1]$ the following holds:

$$\left\| \chi(hD_t) \mathcal{G}_a(t,x,y,z) \right\|_{L^1(\Omega) \to L^\infty(\Omega)} \le C h^{-3} \min\left\{ 1, \left(\frac{h}{|t|} \right)^{\frac{3}{4}} \right\}.$$

In our cylindrical domain with boundary, the light rays may no longer slightly distorted straight lines. There may be rays glancing near tangential direction of the boundary or rays gliding along a convex part of the cylinder boundary or combinations of both. We note that the interesting phenomena analyzed in [1] [6] [7] are the caustics (cusps and swallowtails) near the boundary and due to these caustics on the boundary, the dispersive estimates in Theorem 3 have a sharp loss of $\frac{1}{4}$ powers of $(h/|t|)$ factor compared to the free wave estimates in dimension 3. This is compatible with the intuition: near the boundary less dispersion occurs compared to the $\mathbb{R}^3$ case. Moreover, the geometry analysis of the wave front set allows us to track the swallowtail type singularities in the Green function originating at $x = a$ in the (x,t) plane. The singular points are $\left(a, 4N\sqrt{a(1+a)/(1-\delta^2)}\right)$ for $|N| \lesssim 1/\sqrt{a}$, $|\delta| < 1$, where the times depend on the frequency of the source and its distance to the boundary. Let us mention that if $\delta = 0$, it is exactly where the swallowtail singularity for the Friedlander's model in [3] occurs.

Now, we state the homogeneous Strichartz estimates inside cylindrical convex domains in dimension 3 established in [2].

Theorem 4. Let $\Omega=\left\{x\ge 0,(y,z)\in\mathbb{R}^2\right\}\subset\mathbb{R}^3$ and $\Delta=\partial_x^2+(1+x)\partial_y^2+\partial_z^2$. Let u be a solution of the following wave equation on Ω:

$$\begin{aligned}&\left(\partial_t^2-\Delta\right)u=0 \text{ in } \mathbb{R}_t\times\Omega,\\ &u_{|t=0}=u_0,\ \partial_t u_{|t=0}=u_1,\ u_{|x=0}=0,\end{aligned} \tag{7}$$

for some initial data $(u_0,u_1)\in\dot{H}^{\beta}(\Omega)\times\dot{H}^{\beta-1}(\Omega)$. Then

$$\|u\|_{L^q((0,T);L^r(\Omega))}\le C_{\text{Str}}\left(\|u_0\|_{\dot{H}^{\beta}(\Omega)}+\|u_1\|_{\dot{H}^{\beta-1}(\Omega)}\right),$$

with

$$\frac{1}{q}\le\frac{3}{4}\left(\frac{1}{2}-\frac{1}{r}\right) \tag{8}$$

and

$$\beta=3\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{q}. \tag{9}$$

Remark that the Strichartz estimates in Theorem 4 in dimension 3 improves the range of indices for which the sharp Strichartz estimates hold compared to the result in [16]. However, the result in Theorem 4 is restricted to cylindrical domains, while [16] applies to any domains or manifolds with boundary.

We note also that from the result in Theorem 4, we have control of the $L^4((0,T);L^{12}(\Omega))$ norms and the $L^5((0,T);L^{10}(\Omega))$ of the solution of the wave equation in terms of the energy norm of the initial data, which are important estimates to prove that there is scattering for (1) when the domain is the compliment of a star-shaped obstacle [16].

Finally, the inhomogeneous Strichartz estimates follow from the homogeneous Strichartz estimates and the Christ-Kiselev lemma [19]. In the following corollary $(\tilde{q}',\tilde{r}')$ denotes the exponents conjugate to $(\tilde{q},\tilde{r})$.

Corollary 1. Let $\Omega=\left\{x\ge 0,(y,z)\in\mathbb{R}^2\right\}\subset\mathbb{R}^3$ and the Laplace operator $\Delta=\partial_x^2+(1+x)\partial_y^2+\partial_z^2$. Let u be a solution of the following wave equation on Ω:

$$\begin{aligned}&\left(\partial_t^2-\Delta\right)u=F \text{ in } \mathbb{R}_t\times\Omega,\\ &u_{|t=0}=u_0,\ \partial_t u_{|t=0}=u_1,\ u_{|x=0}=0.\end{aligned} \tag{10}$$

for some initial data $(u_0,u_1)\in\dot{H}^{\beta}(\Omega)\times\dot{H}^{\beta-1}(\Omega)$ and $F\in L^{\tilde{q}'}((0,T);L^{\tilde{r}'}(\Omega))$. Then

$$\begin{aligned}&\|u\|_{L^q((0,T);L^r(\Omega))}+\|u\|_{L^\infty((0,T);\dot{H}^{\beta}(\Omega))}+\|\partial_t u\|_{L^\infty((0,T);\dot{H}^{\beta-1}(\Omega))}\\ &\le C_{\text{Str}}\left(\|u_0\|_{\dot{H}^{\beta}(\Omega)}+\|u_1\|_{\dot{H}^{\beta-1}(\Omega)}+\|F\|_{L^{\tilde{q}'}((0,T);L^{\tilde{r}'}(\Omega))}\right),\end{aligned}$$

with

$$\frac{1}{q}\le\frac{3}{4}\left(\frac{1}{2}-\frac{1}{r}\right) \text{ and } \beta=3\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{q}=3\left(\frac{1}{2}-\frac{1}{\tilde{r}'}\right)-\frac{1}{\tilde{q}'}+2.$$

Proof. The Duhamel formula yields

$$u(t,.)=\cos\left(t\sqrt{-\Delta}\right)u_0+\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}u_1+\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s,.)\mathrm{d}s.$$

The contribution of (u_0,u_1) follows from Theorem 4. It remains to prove the bounds on $(u,\partial_t u)$ in $\dot{H}^{\beta}(\Omega)\times\dot{H}^{\beta-1}(\Omega)$ and that

$$\left\|\int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s,.)\mathrm{d}s\right\|_{L^q\left((0,T);L^r(\Omega)\right)} \lesssim \|F\|_{L^{\tilde{q}'}\left((0,T);L^{\tilde{r}'}(\Omega)\right)}.$$

By the Christ-Kiselev lemma, it suffices to show that for $q>\tilde{q}'$,

$$\left\|\int_0^T \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s,.)\mathrm{d}s\right\|_{L^q\left((0,T);L^r(\Omega)\right)} \lesssim \|F\|_{L^{\tilde{q}'}\left((0,T);L^{\tilde{r}'}(\Omega)\right)}. \tag{11}$$

Now, let $U(t)=\mathrm{e}^{it\sqrt{-\Delta}}:L^2\to L^2$ be the half wave operator. Recall that

$$\cos\left(t\sqrt{-\Delta}\right)=\frac{U(t)+U(-t)}{2},\ \sin\left(t\sqrt{-\Delta}\right)=\frac{U(t)-U(-t)}{2i}.$$

It follows from Theorem 4 that

$$\|U(t)f\|_{L^q\left((0,T);L^r(\Omega)\right)}\lesssim\|f\|_{\dot{H}^{\beta}(\Omega)}$$

holds for all (q,r,β) satisfying (8) and (9). For $\beta\in\mathbb{R}$ and (q,r) satisfying satisfying (8) and (9), we define the operator T_β by

$$T_\beta : L^2(\Omega)\to L^q\left((0,T);L^r(\Omega)\right)$$
$$f\mapsto T_\beta(f)=\left(\sqrt{-\Delta}\right)^{-\beta}\mathrm{e}^{it\sqrt{-\Delta}}f.$$

By the duality, it follows that the operator $T^*_{1-\beta}$ defined by

$$T^*_{1-\beta}:L^{\tilde{q}'}\left((0,T);L^{\tilde{r}'}(\Omega)\right)\to L^2(\Omega)$$
$$F(s,.)\mapsto T^*_{1-\beta}\left(F(s,.)\right)=\int_0^T\left(\sqrt{-\Delta}\right)^{-(1-\beta)}\mathrm{e}^{-is\sqrt{-\Delta}}F(s,.)\mathrm{d}s,$$

where $1-\beta=3\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)-\frac{1}{\tilde{q}}$. Hence, we get

$$\begin{aligned}\left\|\int_0^T U(t)U^*(s)(-\Delta)^{-\frac{1}{2}}F(s,.)\mathrm{d}s\right\|_{L^q\left((0,T);L^r(\Omega)\right)} &= \left\|T_\beta T^*_{1-\beta}F\right\|_{L^q\left((0,T);L^r(\Omega)\right)}\\ &\lesssim \|F\|_{L^{\tilde{q}'}\left((0,T);L^{\tilde{r}'}(\Omega)\right)},\end{aligned}$$

where (q,r) and $(\tilde{q},\tilde{r})$ satisfy (8) and (9) as a result of $\beta=3\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{q}$ and $1-\beta=3\left(\frac{1}{2}-\frac{1}{\tilde{r}}\right)-\frac{1}{\tilde{q}}$. But

$$\int_0^T U(t)U^*(s)(-\Delta)^{-\frac{1}{2}}F(s,.)\mathrm{d}s=\int_0^T\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s,.)\mathrm{d}s,$$

it follows that (11) holds. Observe that for all (q,r) and $(\tilde{q},\tilde{r})$ satisfy (8) and (9), we must have $q>\tilde{q}'$.

To this end, we consider the bound on

$$\|u(t,.)\|_{L^\infty\left((0,T);\dot{H}^{\beta}(\Omega)\right)}+\|\partial_t u(t,.)\|_{L^\infty\left((0,T);\dot{H}^{\beta-1}(\Omega)\right)}.$$

We have

$$\begin{aligned}&\left\|u(t,.)\right\|_{\dot{H}^{\beta}(\Omega)}+\left\|\partial_t u(t,.)\right\|_{\dot{H}^{\beta-1}(\Omega)}\\&=\left\|\cos\left(t\sqrt{-\Delta}\right)u_0+\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}u_1\right\|_{\dot{H}^{\beta}(\Omega)}\\&\quad+\left\|\left(-\sqrt{-\Delta}\right)\sin\left(t\sqrt{-\Delta}\right)u_0+\cos\left(t\sqrt{-\Delta}\right)u_1\right\|_{\dot{H}^{\beta-1}(\Omega)}\\&\le 2\left(\left\|u_0\right\|_{\dot{H}^{\beta}(\Omega)}+\left\|u_1\right\|_{\dot{H}^{\beta-1}(\Omega)}\right).\end{aligned}$$

It reduces to showing that it is uniformly in t such that

$$\left\|\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s,.)\mathrm{d}s\right\|_{\dot{H}^{\beta}(\Omega)}+\left\|\int_0^t\cos\left((t-s)\sqrt{-\Delta}\right)F(u(s,.))\mathrm{d}s\right\|_{\dot{H}^{\beta-1}(\Omega)}\lesssim\left\|F\right\|_{L^{\tilde{q}'}\left((0,T);L^{\tilde{r}'}(\Omega)\right)}.$$

or equivalently,

$$\left\|\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\left(\sqrt{-\Delta}\right)^{1-\beta}}F(s,.)\mathrm{d}s\right\|_{L^2(\Omega)}+\left\|\int_0^t\frac{\cos\left((t-s)\sqrt{-\Delta}\right)}{\left(\sqrt{-\Delta}\right)^{1-\beta}}F(s,.)\mathrm{d}s\right\|_{L^2(\Omega)}\lesssim\left\|F\right\|_{L^{\tilde{q}'}\left((0,T);L^{\tilde{r}'}(\Omega)\right)}. \tag{12}$$

But as consequence of the boundedness of the adjoint operator $T_{1-\beta}^{*}$, we obtain

$$\left\|\int_0^T\frac{\mathrm{e}^{i(t-s)\sqrt{-\Delta}}}{\left(\sqrt{-\Delta}\right)^{1-\beta}}F(s,.)\mathrm{d}s\right\|_{L^2(\Omega)}\lesssim\left\|F\right\|_{L^{\tilde{q}'}\left((0,T);L^{\tilde{r}'}(\Omega)\right)}.$$

and the Christ-Kiselev lemma yields

$$\left\|\int_0^t\frac{\mathrm{e}^{i(t-s)\sqrt{-\Delta}}}{\left(\sqrt{-\Delta}\right)^{1-\beta}}F(s,.)\mathrm{d}s\right\|_{L^2(\Omega)}\lesssim\left\|F\right\|_{L^{\tilde{q}'}\left((0,T);L^{\tilde{r}'}(\Omega)\right)}.$$

and (12) follows from Euler formula. This completes the proof. □

3. Well-Posedness

3.1. Local Existence

In this section, we establish the local existence for the solution to the wave Equation (1) by applying the Strichartz estimates in Theorem 4 and the fixed point argument.

Theorem 1 follows immediately from the following result.

Theorem 5. Let $\Omega=\left\{x\ge 0,(y,z)\in\mathbb{R}^2\right\}\subset\mathbb{R}^3$ and the Laplace operator $\Delta=\partial_x^2+(1+x)\partial_y^2+\partial_z^2$. For any $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$. Then there exists $T=T\left(\left\|(u_0,u_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}\right)>0$ such that the energy-critical wave Equation (1) is locally well-posed in $(0,T)$ and the unique solution u satisfies

$$u \in C^0\left((0,T);H_0^1(\Omega)\right) \cap C^1\left((0,T);L^2(\Omega)\right) \cap L^5\left((0,T);L^{10}(\Omega)\right).$$

Moreover, if $\left\|(u_0,u_1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} < \delta$ for small enough δ, the solution is globally defined.

Proof. We apply the Banach fixed point argument to prove this result. Let a space X_T be defined by

$$X_T = C^0\left((0,T);H_0^1(\Omega)\right) \cap C^1\left((0,T);L^2(\Omega)\right) \cap L^5\left((0,T);L^{10}(\Omega)\right).$$

We define a norm $\|.\|_T$ by

$$\|u\|_{X_T} = \sup_{0<t<T} \left\|(u,\partial_t u)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|u\|_{L^5\left((0,T);L^{10}(\Omega)\right)}.$$

We remark that the space $\left(X_T,\|.\|_T\right)$ is a Banach space. For any small constant $\varepsilon > 0$. We introduce a fixed-point space

$$B_T = \left\{u \in X_T : \left\|(u,\partial_t u)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} \le 2C_{\text{Str}}\delta, \|u\|_{L^5\left((0,T);L^{10}(\Omega)\right)} \le 2C_{\text{Str}}\varepsilon\right\}$$

with the metric

$$d(u,v) = \sup_{0<t<T} \left\|(u-v,\partial_t u-\partial_t v)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|u-v\|_{L^5\left((0,T);L^{10}(\Omega)\right)}.$$

Consider the solution map

$$\Phi(u) = \cos\left(t\sqrt{-\Delta}\right)u_0 + \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}u^5(s,.)\,\text{ds}. \quad (13)$$

Let denote

$$u_{\text{hom}} := \cos\left(t\sqrt{-\Delta}\right)u_0 + \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}u_1$$

and

$$u_{\text{inh}} := \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}u^5(s,.)\,\text{ds}.$$

Notice that $(q,r,\beta) = (5,10,1)$ satisfies the admissible condition as

$$\frac{1}{5} \le \frac{3}{4}\left(\frac{1}{2}-\frac{1}{10}\right),$$

and

$$1 = 3\left(\frac{1}{2}-\frac{1}{10}\right)-\frac{1}{5}.$$

It follows from the Strichartz estimates in Theorem 4 with the admissible triplet $q=5$, $r=10$, and $\beta=1$ that

$$\begin{aligned}&\|u\|_{L^5\left((0,T);L^{10}(\Omega)\right)} + \|u\|_{L^\infty\left((0,T);H_0^1(\Omega)\right)} + \|\partial_t u\|_{L^\infty\left((0,T);L^2(\Omega)\right)}\\ &\le C_{\text{Str}}\left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}\right).\end{aligned}$$

This yields

$$\|u_{\text{hom}}\|_{L^5\left((0,T);L^{10}(\Omega)\right)} \le C_{\text{Str}}\left(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}\right).$$

Therefore, if the norm of initial data $\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}<\delta$ for small enough δ, then we have

$$\|u_{\text{hom}}\|_{L^5((0,T);L^{10}(\Omega))}\le C_{\text{Str}}\varepsilon$$

holds for $T=\infty$; otherwise, the inequality holds for some small $T>0$ as a consequence of the dominated convergence theorem. It follows from Corollary 1 with admissible triplet $q=5$, $r=10$, and $\beta=1$ (so that $\tilde{q}'=1$ and $\tilde{r}'=2$) that

$$\|u_{\text{inh}}\|_{L^5((0,T);L^{10}(\Omega))}\le C_{\text{Str}}\|u^5\|_{L^1((0,T);L^2(\Omega))}.$$

We need to show that the operator Φ is well-defined on B_T and is a contraction map under the metric d. To do this, let $u\in B_T$ with $0<\varepsilon\ll 1$. Then, by Strichartz estimates we have

$$\begin{aligned}\|\Phi(u)\|_{L^5((0,T);L^{10}(\Omega))}&\le\|u_{\text{hom}}\|_{L^5((0,T);L^{10}(\Omega))}+\|u_{\text{inh}}\|_{L^5((0,T);L^{10}(\Omega))}\\&\le C_{\text{Str}}\varepsilon+C_{\text{Str}}\|u^5\|_{L^1((0,T);L^2(\Omega))}\\&\le 2C_{\text{Str}}\varepsilon,\end{aligned}$$

and

$$\begin{aligned}&\sup_{0<t<T}\|(\Phi(u),\partial_t\Phi(u))\|_{H_0^1(\Omega)\times L^2(\Omega)}\\&\le C_{\text{Str}}\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}+C_{\text{Str}}\|u^5\|_{L^1((0,T);L^2(\Omega))}\\&\le 2C_{\text{Str}}\|(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}\\&\le 2C_{\text{Str}}\delta,\end{aligned}$$

for small $T>0$. Therefore, $\Phi(u)\in B_T$. Now, we prove that Φ is a contraction map. Let $v_1,v_2\in B_T$. It follows from the Strichartz estimates and by choosing ε sufficiently small, we get

$$\begin{aligned}\|\Phi(v_1)-\Phi(v_2)\|_{L^5((0,T);L^{10}(\Omega))}&\le C_{\text{Str}}\|v_1^5-v_2^5\|_{L^1((0,T);L^2(\Omega))}\\&\le C_{\text{Str}}\|v_1-v_2\|_{L^5((0,T);L^{10}(\Omega))}\\&\quad\times\left(\|v_1\|^4_{L^5((0,T);L^{10}(\Omega))}+\|v_2\|^4_{L^5((0,T);L^{10}(\Omega))}\right)\\&\le\tilde{C}\varepsilon^4\|v_1-v_2\|_{L^5((0,T);L^{10}(\Omega))},\end{aligned}$$

and

$$\begin{aligned}&\sup_{0<t<T}\|(\Phi(v_1)-\Phi(v_2),\partial_t\Phi(v_1)-\partial_t\Phi(v_2))\|_{H_0^1(\Omega)\times L^2(\Omega)}\\&\le C_{\text{Str}}\|v_1^5-v_2^5\|_{L^1((0,T);L^2(\Omega))}\\&\le\tilde{C}\varepsilon^4\|v_1-v_2\|_{L^5((0,T);L^{10}(\Omega))}.\end{aligned}$$

We combine these two estimates to obtain

$$d(\Phi(v_1)-\Phi(v_2))\le\tilde{C}\varepsilon^4\|v_1-v_2\|_{L^5((0,T);L^{10}(\Omega))}\le\frac{1}{2}d(v_1,v_2).$$

We conclude from the fixed point theorem that there is a unique solution u to (1) on $(0,T)\times\Omega$. Hence, if δ is small enough, we get the global solution; otherwise, we have a local solution. □

3.2. Global Existence

In this section, we follow the ideas in [4] as well as [9] [10] to obtain the global solution to (1) for any data having finite energy in the context of our cylindrical convex domains as stated in Theorem 2. The key ingredients are the following stronger version of Strichartz estimates, trace estimates and the nonconcentration of nonlinear effect in a small light cones.

Theorem 6. Let $\Omega=\{x\geq 0,(y,z)\in\mathbb{R}^2\}\subset\mathbb{R}^3$ and $\Delta=\partial_x^2+(1+x)\partial_y^2+\partial_z^2$. If u,F satisfy

$$\left(\partial_t^2-\Delta\right)u=F,\ u_{|t=0}=u_0,\ \partial_t u_{|t=0}=u_1,\ u_{|x=0}=0,$$

then

$$\|u\|_{L^5\left((0,1);W_0^{\frac{3}{10},5}(\Omega)\right)}+\|u\|_{C^0\left((0,1);H_0^1(\Omega)\right)}+\|\partial_t u\|_{C^0\left((0,1);L^2(\Omega)\right)}$$
$$\leq C_{\mathrm{Str}}\left(\|u_0\|_{H_0^1(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|F\|_{L^{\frac{5}{4}}\left((0,1);W^{\frac{7}{10},\frac{5}{4}}(\Omega)\right)}\right).$$

Proof. We follow the streamline of the proof of Proposition 3.1 in [4] with some modifications for the homogeneous part. We have by Duhamel formula

$$u(t,.)=\cos\left(t\sqrt{-\Delta}\right)u_0+\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}u_1+\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s,.)\mathrm{d}s.$$

Notice that $(q,r,\beta)=\left(5,5,\frac{7}{10}\right)$ satisfies the admissible condition as

$$\frac{1}{5}\leq\frac{3}{4}\left(\frac{1}{2}-\frac{1}{5}\right),$$

and

$$\frac{7}{10}=3\left(\frac{1}{2}-\frac{1}{5}\right)-\frac{1}{5}.$$

The contribution of (u_0,u_1) follows from the Strichartz estimates in Theorem 4 with the admissible triplet $q=5$, $r=5$, and $\beta=\frac{7}{10}$ that

$$\left\|e^{it\sqrt{-\Delta}}f\right\|_{L^5((0,1)\times\Omega)}\lesssim\|f\|_{\dot{H}^{\frac{7}{10}}(\Omega)}. \tag{14}$$

Then if we apply this inequality to Δf and we use the L^p elliptic regularity in [20], we get

$$\left\|e^{it\sqrt{-\Delta}}f\right\|_{L^5\left((0,1);W^{2,5}(\Omega)\cap W_0^{1,5}(\Omega)\right)}\lesssim\|f\|_{\dot{H}^{\frac{27}{10}}(\Omega)}. \tag{15}$$

Consequently, the interpolation between (14) and (15) gives

$$\left\| e^{it\sqrt{-\Delta}} f \right\|_{L^5\left((0,1);W_0^{\frac{3}{10},5}(\Omega)\right)} \lesssim \| f \|_{\dot{H}^1(\Omega)}.$$

We can conclude that

$$\left\| \cos\left(t\sqrt{-\Delta}\right)u_0 + \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} u_1 \right\|_{L^5\left((0,1);W_0^{\frac{3}{10},5}(\Omega)\right)} \lesssim \|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}.$$

The bound for

$$\|u\|_{C^0\left((0,1);H_0^1(\Omega)\right)} + \|\partial_t u\|_{C^0\left((0,1);L^2(\Omega)\right)}$$

is similar to that of in Theorem 4. Finally, the bound of

$$\int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}} F(s,.)\mathrm{d}s$$

follows that same line as in the proof of Proposition 3.1 in [4] by TT^* argument and Christ-Kiselev lemma. □

Notice that one has the control for the nonlinear term (see [4]) by the following estimate

$$\left\|u^5\right\|_{L^{\frac{5}{4}}\left((0,1);W^{\frac{7}{10},\frac{5}{4}}(\Omega)\right)} \lesssim \|u\|^4_{L^5\left((0,1);L^{10}(\Omega)\right)} \|u\|^{\frac{3}{10}}_{L^\infty\left((0,1);L^6(\Omega)\right)} \|u\|^{\frac{7}{10}}_{L^\infty\left((0,1);H^1(\Omega)\right)}.$$

The idea now is to localize these estimates on small light cones and use the fact that the $L_t^\infty\left(L_x^6\right)$ norm is small in such a small cones [4] [9] [10]. Let us review the key results to obtain the global existence to the (1) as follows.

Proposition 7 (L^6-nonconcentration). Let $x_0 \in \bar{\Omega}$. Then for any solution u to (1) in the space $X_{<t_0} = C^0\left((0,t_0);H_0^1(\Omega)\right) \cap L_{loc}^5\left((0,t_0);L^{10}(\Omega)\right) \times C^0\left((0,t_0);L^2(\Omega)\right)$, there holds

$$\lim_{t\to t_0^-} \int_{x\in\Omega\cap\{|x-x_0|<t-t_0\}} u^6(t,x)\mathrm{d}x = 0$$

For the proof of this result see [4]. We point out here three important ingredients.

- The normal derivative estimate

$$\|\partial_\nu u\|_{L^2\left((0,t_0)\times\Omega\right)} \le CE(u)^{\frac{1}{2}}$$

is satisfied uniformly for $0 < T < t_0$, where $\partial_\nu u$ is the trace to the boundary of the outward unit normal of u. This follows from the estimate (see Proposition 3.2 in [4])

$$\int_0^T \int_\Omega \left[\left(\partial_t^2 - \Delta\right) Zu(t,x) - Z\left(\partial_t^2 - \Delta\right)u(t,x)\right] u(t,x)\mathrm{d}x\mathrm{d}t \lesssim E(u)$$

uniformly in T, where Z is a smooth vector filed on Ω which coincides with ∂_ν on $\partial\Omega$. Notice that such an estimate follows by using integration by parts and the energy conservation.

- The flux identity derived in [4]

$$\int_{x\in\Omega,|x|<-T}\left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2+\frac{1}{6}|u|^6\right)(x,T)\,\mathrm{d}x+\mathrm{Flux}\left(u,M_S^T\right)$$
$$=\int_{x\in\Omega,|x|<-S}\left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2+\frac{1}{6}|u|^6\right)(x,S)\,\mathrm{d}x$$
$$=E_{loc}(S).$$

Here,

$$\mathrm{Flux}\left(u,M_S^T\right):=\int_{M_S^T}\langle e(u),\nu\rangle\,\mathrm{d}\sigma(t,x)$$

where $M_S^T:=\{x:|x|=-t,S<t<T\}$, ν the unit outward normal to M_S^T, $\mathrm{d}\sigma(t,x)$ the induced measure on M_S^T, and the vector field e is given by

$$e(u)=\left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2+\frac{1}{6}|u|^6,-\partial_t u\nabla u\right).$$

The authors in [4] showed that

$$\lim_{S\to 0^-}\mathrm{Flux}\left(u,M_S^0\right)=0.$$

- A Morawetz type inequality is formally derived in [4] and integrate the identity over $K_S^T=\{(t,x);|x|<-t,S<t<T\}\cap\Omega$ combined with Hölder's inequality and the conservation of energy. More precisely, they proved that

$$\int_{x\in\Omega,|x|<-S}\frac{|u|^6}{6}(S,x)\,\mathrm{d}x\le|S|E(u)+C\,\mathrm{Flux}\left(u,M_S^0\right)+C\,\mathrm{Flux}\left(u,M_S^0\right)^{\frac{1}{3}}\underset{S\to 0^-}{\to}0.$$

Finally, the next proposition shows a localizing space-time estimates.

Proposition 8 ([4]). For any $\varepsilon>0$, there exists $t<0$ such that

$$\|u\|_{(L^5;L^{10})(K_t^0)}<\varepsilon,$$

where

$$\|u\|_{(L^5;L^{10})(K_t^0)}=\left(\int_{s=t}^0\left(\int_{\{|x|<-s\}\cap\Omega}|u|^{10}(s,x)\,\mathrm{d}x\right)^{\frac{5}{10}}\mathrm{d}s\right)^{\frac{1}{5}}.$$

With these results in hand, to prove the global existence it is sufficient to show that the local solution u satisfies $u\in L^5\left((s,0);L^{10}(\Omega)\right)$ since this allows us to show that $\lim_{s'\to 0^-}(u,\partial_s u)(s',.)$ exists in $H_0^1(\Omega)\times L^2(\Omega)$ and consequently can be extended for $s'>0=t_0$ small enough via the Duhamel formula and the Strichartz estimates in Theorem 6 together with conservation of energy.

4. Conclusions

In this work, we investigated the well-posedness of quintic energy critical wave equations within $3D$ cylindrical domains. The principal aim is to establish both local and global well-posedness in energy space for this context. First, by utilizing dispersive and Strichartz estimates, the existence, uniqueness, and stability of the local solution were established in suitable energy space. Then, applying the stronger version of Strichartz estimates, trace estimates and the concentration of nonlinear effect in small light cones, the global existence has been derived.

This work advances the understanding of the wave equation within the geometric domain interpolating between the bounded domain in $\mathbb{R}^3$ and the Euclidean space $\mathbb{R}^3$. The incorporation of dispersive and Strichartz estimates to extend local results to a global context paves the paths to further explorations of the wave equations in complex geometrical setting.

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Conflicts of Interest

The author declares no conflicts of interest.

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