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Normalized Solutions of the Gross-Pitaevskii System with Inhomogeneous Interactions

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Abstract

This paper is devoted to the normalized solutions of the two-dimensional Gross-Pitaevskii system with a microwave field and inhomogeneous interactions. By investigating the relevant L^2-critical constrained variational problem, we get the existence and nonexistence of the normalized solutions under suitable assumptions about the interaction potentials. We establish the existence and nonexistence of minimizers of $e(\gamma, a)$ via a threshold a^*, where a^* is the square of L^2-norm of the unique positive solution of $\Delta u - u + u^3 = 0$ in $\mathbb{R}^2$. We also analyze the limiting behavior of the constraint minimizers as $a \nearrow 2a^*$ through overcoming the challenges associated with the sign-changing property of the logarithmic convolutions and the impact of inhomogeneous interactions.

Subject Areas

Mathematical Analysis

Keywords

Gross-Pitaevskii System, L^2-Critical Exponent, Logarithmic Convolution, Inhomogeneous Interaction

1. Introduction

In this paper, we focus on the following two-component Gross-Pitaevskii system coupled with a microwave field and inhomogeneous interactions in $\mathbb{R}^2$:

$$\begin{cases} -\Delta u_1 + |x|^2 u_1 - \phi u_2 = \lambda u_1 + ak(x)|u_2|^2 u_1 & \text{in } \mathbb{R}^2, \\ -\Delta u_2 + |x|^2 u_2 - \phi u_1 = \lambda u_2 + ak(x)|u_1|^2 u_2 & \text{in } \mathbb{R}^2, \\ -\Delta \phi = \gamma u_1 u_2 & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is the chemical potential, $a \in \mathbb{R}$ denotes the contact interaction strength between two components, $k(x)$ stands for the inhomogeneous interactions, and $\gamma > 0$ (resp. < 0) represents that the magnetic field is attractive (resp. repulsive). The system (1.1) originates from Bose-Einstein Condensates (BECs) interacting with a microwave field at low temperature. The microwave field can influence BECs by the local field effect which plays an important role in the process of forming electromagnetic-matter waves. Due to the physical significance of this system, our goal is to research the existence, nonexistence and limiting behavior of the complex-valued states in system (1.1).

Another purpose of studying (1.1) derives from recent research [1]-[5]. In these references [1]-[4], the authors investigate various models, including single-component attractive BECs, two-component attractive BECs and rotating attractive BECs. They research a range of properties such as existence, nonexistence, uniqueness, quality concentration, symmetry breaking, and refined spike profiles of ground states. However, there is relatively little research on planar multi-component BECs interacting with electromagnetic fields. Recently, Wang, Cai and Wang [5] explored (1.1) in the complex-valued case. They obtained results about the existence and nonexistence of central vortex steady states. Now, we add inhomogeneous interactions to (1.1) for further research. For $0 < k(x) \le 1$, we give the existence of minimizers of $e(\gamma, a)$ in the case of $a \in (0, 2a^*)$, where a^* is the square of L^2-norm of the unique positive solution of

$$-\Delta u + u = u^3,\ u \in H^1(\mathbb{R}^2). \tag{1.2}$$

In this paper, we are interested in the nontrivial solutions of (1.1) under the normalization constraint

$$\int_{\mathbb{R}^2} \left(u_1^2 + u_2^2\right) \mathrm{d}x = 1. \tag{1.3}$$

Towards this purpose, with the help of the fundamental solution of $-\Delta$, we define the energy functional

$$\begin{aligned} E_{\gamma,a}(u_1,u_2) := &\frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u_1|^2 + |\nabla u_2|^2 + |x|^2\left(u_1^2+u_2^2\right)\right]\mathrm{d}x \\ &+\frac{\gamma}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\,u_1(x)u_2(x)u_1(y)u_2(y)\,\mathrm{d}x\mathrm{d}y \\ &-\frac{a}{2}\int_{\mathbb{R}^2}k(x)u_1^2u_2^2\,\mathrm{d}x. \end{aligned} \tag{1.4}$$

Due to the appearance of the harmonic potential term and the logarithmic convolution term, $E_{\gamma,a}$ is not well-defined on $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. Inspired by [3] [6] [7], we work in the subspace $\mathscr{X} := H \times H$, where

$$H := \left\{ u \in H^1(\mathbb{R}^2) : |u|_* := \left(\int_{\mathbb{R}^2} |x|^2 u^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}} < \infty \right\}.$$

Define the norm on $\mathscr{X}$ by

$$\left\|(u_1,u_2)\right\|_{\mathscr{X}}^2 = \|u_1\|_H^2 + \|u_2\|_H^2 = \int_{\mathbb{R}^2}\left[|\nabla u_1|^2 + |\nabla u_2|^2 + \left(1+|x|^2\right)\left(u_1^2+u_2^2\right)\right]\mathrm{d}x.$$

According to ([2] lemma 2.1), we have

$$\text{The embedding } \mathcal{X} = H \times H \hookrightarrow L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \text{ is compact for all } q \geq 2. \quad (1.5)$$

We now define the constraint set

$$\mathcal{M} := \left\{ (u_1, u_2) \in \mathcal{X} : \int_{\mathbb{R}^2} (u_1^2 + u_2^2) \mathrm{d}x = 1 \right\},$$

and focus on studying the following constrained variational problem:

$$e(\gamma, a) := \inf_{(u_1, u_2) \in \mathcal{M}} E_{\gamma,a}(u_1, u_2). \quad (1.6)$$

Throughout the paper, we assume that the potential $-1 \leq k(x) \leq 1$ satisfies the following assumptions:

$$\lim_{x \to 0} k(x) = 1, \quad (1.7)$$

$$1 - \frac{a}{2a^*} k(x) \leq C|x|^b, \quad (1.8)$$

where $C > 0$ and $b \geq 2$ are some constants, and $k(x) \in C^{0,\alpha}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$.

Stimulated by [6] [8], we perform that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u_1(x) u_2(x) u_1(y) u_2(y) \mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1+|x-y|) u_1(x) u_2(x) u_1(y) u_2(y) \mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+|x-y|^{-1}\right) u_1(x) u_2(x) u_1(y) u_2(y) \mathrm{d}x\mathrm{d}y \\
&:= F_1(u_1, u_2) - F_2(u_1, u_2).
\end{aligned} \quad (1.9)$$

For simplicity, we use $|\cdot|_q$ to denote the standard Lebesgue norm on $L^q(\mathbb{R}^2)$ for $q \in [1, \infty]$. Since

$$\ln(1+|x-y|) \leq |x-y| \leq |x| + |y| \text{ for } x, y \in \mathbb{R}^2,$$

We deduce by Hölder inequality that

$$\begin{aligned}
|F_1(u_1, u_2)| &\leq \int_{\mathbb{R}^2} |x||u_1(x)||u_2(x)| \mathrm{d}x \int_{\mathbb{R}^2} |u_1(y)||u_2(y)| \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^2} |u_1(x)||u_2(x)| \mathrm{d}x \int_{\mathbb{R}^2} |y||u_1(y)||u_2(y)| \mathrm{d}y \\
&\leq 2|u_1|_* |u_1|_2 |u_2|_2^2 \text{ for } (u_1, u_2) \in \mathcal{X}, \\
&\leq |u_1|_* |u_2|_2 \text{ for } (u_1, u_2) \in \mathcal{M}.
\end{aligned} \quad (1.10)$$

Applying Hardy-Littlewood-Sobolev inequality (cf. ([9], Theorem 4.3)), we then derive that there exists a constant $C > 0$ such that

$$\begin{aligned}
|F_2(u_1, u_2)| &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u_1(x) u_2(x) u_1(y) u_2(y)|}{|x-y|} \mathrm{d}x\mathrm{d}y \\
&\leq C|u_1 u_2|_{\frac{4}{3}}^2 \leq C|u_1|_{\frac{8}{3}}^2 |u_2|_{\frac{8}{3}}^2 \text{ for } (u_1, u_2) \in L^{\frac{8}{3}}(\mathbb{R}^2) \times L^{\frac{8}{3}}(\mathbb{R}^2).
\end{aligned} \quad (1.11)$$

It follows from (1.9)-(1.11) that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u_1(x) u_2(x) u_1(y) u_2(y) \mathrm{d}x\mathrm{d}y$ is well-defined on $\mathcal{X}$. We can get that $E_{\gamma,a}$ is of class C^1.

The existence and nonexistence of minimizers of the constrained variational problem (1.6) depend strongly on the following Gagliardo-Nirenberg type inequality (cf. ([2], lemma A. 2])):

$$\int_{\mathbb{R}^2}\left(|u_1|^2+|u_2|^2\right)^2\mathrm{d}x \le \frac{2}{|Q|_2^2}\int_{\mathbb{R}^2}\left(|\nabla u_1|^2+|\nabla u_2|^2\right)\mathrm{d}x\int_{\mathbb{R}^2}\left(|u_1|^2+|u_2|^2\right)\mathrm{d}x, \quad (1.12)$$

where $(u_1,u_2)\in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$ and Q is the unique positive solution of (1.2). For any $\theta\in(0,2\pi)$, the equality in (1.12) can be attained at $(Q\sin\theta, Q\cos\theta)$. Note from ([10], Proposition 4.1) that Q decays exponentially as $|x|\to\infty$ in the sense that

$$|Q(x)|,|\nabla Q(x)| = O\left(|x|^{-\frac{1}{2}}\mathrm{e}^{-|x|}\right) \text{ as } |x|\to\infty. \quad (1.13)$$

Moreover, recall from [11] that Q is the achievement function of the equality in the following classical Gagliardo-Nirenberg inequality:

$$\int_{\mathbb{R}^2}|u|^4\mathrm{d}x \le \frac{2}{|Q|_2^2}\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x\int_{\mathbb{R}^2}|u|^2\mathrm{d}x,\ u\in H^1(\mathbb{R}^2). \quad (1.14)$$

We then conclude from (1.2) and (1.14) that

$$\int_{\mathbb{R}^2}|\nabla Q|^2\mathrm{d}x = \int_{\mathbb{R}^2}Q^2\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^2}Q^4\mathrm{d}x. \quad (1.15)$$

Based on the above results, we can establish the existence and nonexistence of minimizers for the constrained variational problem (1.6).

Theorem 1.1 *Let* $Q(x)=Q(|x|)>0$ *be the unique positive solution of* (1.2) *and* $a^*:=|Q|_2^2$, *assume that* $k(x)$ *always satisfies* (1.7) *and* (1.8).

1) If $-1\le k(x)\le 0$, then there exists at least one minimizer of $e(\gamma,a)$ for $\gamma\ne 0$ and $a>0$.

2) If $0<k(x)\le 1$, then there exists at least one minimizer of $e(\gamma,a)$ for $\gamma\ne 0$ and $0<a<2a^*$, and then there is no minimizer of $e(\gamma,a)$ for $\gamma\ne 0$, $a>2a^*$, or $\gamma>0$, $a=2a^*$.

Theorem 1.1 establishes the existence and nonexistence of minimizers of $e(\gamma,a)$. When $-1\le k(x)\le 0$, we can get that $E_{\gamma,a}$ is bounded from below based on the non-positivity of $k(x)$, then we can get the existence of minimizers of $e(\gamma,a)$. As for $0<k(x)\le 1$, then we can also get the existence of minimizers of $e(\gamma,a)$ by rewriting $E_{\gamma,a}$ into (2.8). Moreover, when $a\ge 2a^*$, we use appropriate test function to get that there is no minimizer of $e(\gamma,a)$. If $(u_1^a,u_2^a)\in\mathcal{M}$ is a minimizer of $e(\gamma,a)$ for $\gamma\ne 0$ and $0<a<2a^*$, then the variational theory shows that (u_1^a,u_2^a) solves

$$\begin{cases} -\Delta u_1^a+|x|^2u_1^a+\dfrac{\gamma}{2\pi}\displaystyle\int_{\mathbb{R}^2}\ln|x-y|u_1^a(y)u_2^a(y)\mathrm{d}y\,u_2^a=\lambda_a u_1^a+ak(x)|u_2^a|^2u_1^a \text{ in } \mathbb{R}^2,\\ -\Delta u_2^a+|x|^2u_2^a+\dfrac{\gamma}{2\pi}\displaystyle\int_{\mathbb{R}^2}\ln|x-y|u_1^a(y)u_2^a(y)\mathrm{d}y\,u_1^a=\lambda_a u_2^a+ak(x)|u_1^a|^2u_2^a \text{ in } \mathbb{R}^2,\end{cases} \quad (1.16)$$

where $\lambda_a\in\mathbb{R}$ is the Lagrange multiplier associated with (1.3) and satisfies that

$$\lambda_a = 2e(\gamma,a) - a\int_{\mathbb{R}^2} k(x)\left|u_1^a\right|^2\left|u_2^a\right|^2 \mathrm{d}x + \frac{\gamma}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u_1^a(x)u_2^a(x)u_1^a(y)u_2^a(y)\mathrm{d}x\mathrm{d}y. \tag{1.17}$$

The second result in this paper deals with the limiting behavior of minimizers of $e(\gamma,a)$ as $a \nearrow 2a^*$, where $\gamma > 0$ is fixed. We always denote by $\to$ and $\rightharpoonup$ the strong convergence and the weak convergence, respectively.

Theorem 1.2 *Let* $(u_1^a, u_2^a) \in \mathcal{M}$ *be a minimizer of* $e(\gamma,a)$, *where* $\gamma > 0$ *is fixed and* $a \in (0, 2a^*)$. *Assume that* $0 < k(x) \le 1$ *and* $k(x) \in C^{0,\alpha}(\mathbb{R}^2)$ *for some* $\alpha \in (0,1)$, *then there exist a sequence* $\{y_{\varepsilon_a}\} \subset \mathbb{R}^2$ *and a point* $x_0 \in \mathbb{R}^2$ *such that*

$$\varepsilon_a \left|u_i^a\left(\varepsilon_a x + \varepsilon_a y_{\varepsilon_a}\right)\right| \to \frac{Q(x-x_0)}{\sqrt{2a^*}} \quad \text{in } H^1(\mathbb{R}^2) \text{ as } a \nearrow 2a^*,$$

where $i = 1,2$ and

$$\varepsilon_a := \left[\int_{\mathbb{R}^2}\left(\left|\nabla u_1^a\right|^2 + \left|\nabla u_2^a\right|^2\right)\mathrm{d}x\right]^{-\frac{1}{2}} \to 0 \quad \text{as } a \nearrow 2a^*.$$

Moreover, $\lim_{a \nearrow 2a^*} \lambda_a \varepsilon_a^2 = -1$. Particularly, if $(u_1^a, u_2^a) \in \mathcal{M}$ is non-negative, then

$$\sqrt{\frac{8\pi(2a^*-a)}{\gamma a^*}}\, u_i^a\left(\sqrt{\frac{8\pi(2a^*-a)}{\gamma a^*}}\, x + x_i^a\right) \to \frac{Q(x)}{\sqrt{2a^*}} \quad \text{in } H \cap L^\infty(\mathbb{R}^2) \text{ as } a \nearrow 2a^*,$$

where x_i^a is the unique global maximum point of u_i^a and $x_i^a \to 0$ as $a \nearrow 2a^*$.

For any minimizer $(u_1^a, u_2^a) \in \mathcal{M}$, Theorem 1.2 provides the preliminary limiting behavior as $a \nearrow 2a^*$. Moreover, Theorem 1.2 also gives the more refined convergence information when the minimizer is non-negative. In order to prove Theorem 1.2, we shall show that

$$e(\gamma,a) \approx \frac{\gamma}{32\pi}\ln\frac{8\pi(2a^*-a)}{\gamma a^*} + C \quad \text{as } a \nearrow 2a^*.$$

The above Theorems show that the system (1.1) has a stable normalized state when $0 < a < 2a^*$, and the stable state disappears when $a \ge 2a^*$. The inhomogeneous interaction $k(x)$ only affects the range of stable states and does not alter the cole role of critical threshold. In short, this result reveals the stability conditions and critical behavior of two-component BECs under microwave field, providing a physical basis for the artificial control of BECs configuration by regulating the interaction strength.

The rest of this paper is organized as follows. In section 2, we prove Theorem 1.1 on the existence and nonexistence of minimizers. In section 3, we analyze the limiting behavior of minimizers and prove Theorem 1.2.

2. Existence and Nonexistence of Minimizers

In this section, we prove the existence and nonexistence of minimizers for

$e(\gamma,a)$, where $\gamma \neq 0$ and $a>0$. For the case that $0<k(x)\leq 1$, the crucial technique is that we rewrite the energy functional $E_{\gamma,a}(u_1,u_2)$ into a new form (2.8) and we use an important estimate (2.6).

Proof of Theorem 1.1. 1) For the case that $-1\leq k(x)\leq 0$, we prove the existence of minimizers of $e(\gamma,a)$, where $\gamma\neq 0$ and $a>0$. By Young inequality, we derive from (1.10) that

$$\begin{aligned} |F_1(u_1,u_2)| &\leq \frac{\pi}{|\gamma|}|u_1|_*^2+\frac{|\gamma|}{4\pi}|u_2|_2^2 \\ &\leq \frac{\pi}{|\gamma|}\int_{\mathbb{R}^2}|x|^2\left(u_1^2+u_2^2\right)\mathrm{d}x+\frac{|\gamma|}{4\pi} \text{ for } (u_1,u_2)\in\mathcal{M}. \end{aligned} \tag{2.1}$$

Recall from [11] the general Gagliardo-Nirenberg inequality: for any $v\in H^1(\mathbb{R}^2)$ and $q\geq 2$,

$$\int_{\mathbb{R}^2}|v|^q\,\mathrm{d}x\leq\frac{q}{2|Q_q|_2^{q-2}}\left(\int_{\mathbb{R}^2}|\nabla v|^2\,\mathrm{d}x\right)^{\frac{q-2}{2}}\int_{\mathbb{R}^2}|v|^2\,\mathrm{d}x, \tag{2.2}$$

where Q_q is the unique positive solution of

$$-\frac{q-2}{2}\Delta v+v=v^{q-1},\ v\in H^1(\mathbb{R}^2).$$

Combining (1.11) with (2.2) yields that there exists a constant $C>0$ such that

$$\begin{aligned} |F_2(u_1,u_2)| &\leq C\left(\int_{\mathbb{R}^2}|\nabla u_1|^2\,\mathrm{d}x\right)^{\frac14}\left(\int_{\mathbb{R}^2}|\nabla u_2|^2\,\mathrm{d}x\right)^{\frac14} \\ &\leq C\left[\int_{\mathbb{R}^2}\left(|\nabla u_1|^2+|\nabla u_2|^2\right)\mathrm{d}x\right]^{\frac12} \text{ for } (u_1,u_2)\in\mathcal{M}. \end{aligned} \tag{2.3}$$

Following (1.9), (1.12), (2.1) and (2.3), we infer that

$$\begin{aligned} E_{\gamma,a}(u_1,u_2) \geq{}& \frac12\int_{\mathbb{R}^2}\left(|\nabla u_1|^2+|\nabla u_2|^2\right)\mathrm{d}x+\frac14\int_{\mathbb{R}^2}|x|^2\left(u_1^2+u_2^2\right)\mathrm{d}x \\ &-C\left[\int_{\mathbb{R}^2}\left(|\nabla u_1|^2+|\nabla u_2|^2\right)\mathrm{d}x\right]^{\frac12}-\frac{\gamma^2}{16\pi^2} \text{ for } (u_1,u_2)\in\mathcal{M}, \end{aligned} \tag{2.4}$$

which implies that $E_{\gamma,a}$ is bounded from below on $\mathcal{M}$. Let $\{(u_{1,n},u_{2,n})\}\subset\mathcal{M}$ be a minimizing sequence of $e(\gamma,a)$. We get from (2.4) that $\int_{\mathbb{R}^2}\left(|\nabla u_{1,n}|^2+|\nabla u_{2,n}|^2\right)\mathrm{d}x$ and $\int_{\mathbb{R}^2}|x|^2\left(u_{1,n}^2+u_{2,n}^2\right)\mathrm{d}x$ are bounded uniformly with respect to n. Combining with the fact $|u_{1,n}|_2^2+|u_{2,n}|_2^2=1$, we then obtain that $\{(u_{1,n},u_{2,n})\}$ is bounded uniformly in $\mathcal{X}$. By (1.5), there is $(u_1,u_2)\in\mathcal{X}$ such that

$$(u_{1,n},u_{2,n})\rightharpoonup(u_1,u_2) \text{ in } \mathcal{X},$$

$$(u_{1,n},u_{2,n})\to(u_1,u_2) \text{ in } L^q(\mathbb{R}^2)\times L^q(\mathbb{R}^2) \text{ for any } q\in[2,\infty)$$

as $n\to\infty$, which indicates that $(u_1,u_2)\in\mathcal{M}$. By the weak lower semicontinuity, we have

$$\int_{\mathbb{R}^2}\left[|\nabla u_1|^2+|\nabla u_2|^2+|x|^2\left(u_1^2+u_2^2\right)\right]\mathrm{d}x$$
$$\le\liminf_{n\to\infty}\int_{\mathbb{R}^2}\left[|\nabla u_{1,n}|^2+|\nabla u_{2,n}|^2+|x|^2\left(u_{1,n}^2+u_{2,n}^2\right)\right]\mathrm{d}x.$$

We then claim that

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_{1,n}(x)u_{2,n}(x)u_{1,n}(y)u_{2,n}(y)\mathrm{d}x\mathrm{d}y\\&\to\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_1(x)u_2(x)u_1(y)u_2(y)\mathrm{d}x\mathrm{d}y\ \text{ as } n\to\infty.\end{aligned}\tag{2.5}$$

Actually, note that

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_{1,n}(x)u_{2,n}(x)u_{1,n}(y)u_{2,n}(y)\mathrm{d}x\mathrm{d}y\right.\\&\left.-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_1(x)u_2(x)u_1(y)u_2(y)\mathrm{d}x\mathrm{d}y\right|\\&\le\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\left(u_{1,n}(x)-u_1(x)\right)u_{2,n}(x)u_{1,n}(y)u_{2,n}(y)\mathrm{d}x\mathrm{d}y\right|\\&\quad+\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_1(x)\left(u_{2,n}(x)-u_2(x)\right)u_{1,n}(y)u_{2,n}(y)\mathrm{d}x\mathrm{d}y\right|\\&\quad+\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_1(x)u_2(x)\left(u_{1,n}(y)-u_1(y)\right)u_{2,n}(y)\mathrm{d}x\mathrm{d}y\right|\\&\quad+\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_1(x)u_2(x)u_1(y)\left(u_{2,n}(y)-u_2(y)\right)\mathrm{d}x\mathrm{d}y\right|\\&:=L_{1,n}+L_{2,n}+L_{3,n}+L_{4,n}.\end{aligned}$$

We denote the norm of any function u in $H^1(\mathbb{R}^2)$ by $\|u\|_1:=\left[\int_{\mathbb{R}^2}\left(|\nabla u|^2+u^2\right)\mathrm{d}x\right]^{\frac{1}{2}}$ for convenience. In view of

$$\left|\ln|x-y|\right|=\left|\ln\left(1+|x-y|\right)-\ln\left(1+|x-y|^{-1}\right)\right|\le|x|+|y|+|x-y|^{-1},$$

we then deduce that

$$\begin{aligned}L_{1,n}&\le\int_{\mathbb{R}^2}|x||u_{1,n}(x)-u_1(x)||u_{2,n}(x)|\mathrm{d}x\int_{\mathbb{R}^2}|u_{1,n}(y)||u_{2,n}(y)|\mathrm{d}y\\&\quad+\int_{\mathbb{R}^2}|u_{1,n}(x)-u_1(x)||u_{2,n}(x)|\mathrm{d}x\int_{\mathbb{R}^2}|y||u_{1,n}(y)||u_{2,n}(y)|\mathrm{d}y\\&\quad+\int_{\mathbb{R}^2}|u_{1,n}(x)-u_1(x)||u_{2,n}(x)|\int_{\mathbb{R}^2}\frac{|u_{1,n}(y)||u_{2,n}(y)|}{|x-y|}\mathrm{d}y\mathrm{d}x\\&\le|u_{1,n}-u_1|_2|u_{2,n}|_*|u_{1,n}|_2|u_{2,n}|_2+|u_{1,n}-u_1|_2|u_{2,n}|_2^2|u_{1,n}|_*\\&\quad+C\|u_{1,n}\|_1\|u_{2,n}\|_1|u_{1,n}-u_1|_2|u_{2,n}|_2\\&\le C|u_{1,n}-u_1|_2\to0\ \text{ as } n\to\infty,\end{aligned}$$

where we use the fact that there exists a constant $C>0$, independent of $n\in\mathbb{Z}^+$ and $x\in\mathbb{R}^2$, such that

$$\begin{aligned}&\int_{\mathbb{R}^2}\frac{|u_{1,n}(y)||u_{2,n}(y)|}{|x-y|}\mathrm{d}y\\&=\int_{|x-y|<1}\frac{|u_{1,n}(y)||u_{2,n}(y)|}{|x-y|}\mathrm{d}y+\int_{|x-y|\ge1}\frac{|u_{1,n}(y)||u_{2,n}(y)|}{|x-y|}\mathrm{d}y\\&\le\left(\int_{|x-y|<1}\frac{1}{|x-y|^{\frac{3}{2}}}\mathrm{d}y\right)^{\frac{2}{3}}\left(\int_{|x-y|<1}|u_{1,n}(y)|^3|u_{2,n}(y)|^3\mathrm{d}y\right)^{\frac{1}{3}}\\&\quad+\int_{|x-y|\ge1}|u_{1,n}(y)||u_{2,n}(y)|\mathrm{d}y\end{aligned}$$

$$\leq C|u_{1,n}|_6 |u_{2,n}|_6 + |u_{1,n}|_2 |u_{2,n}|_2 \leq C\|u_{1,n}\|_1 \|u_{2,n}\|_1 . \tag{2.6}$$

Similarly, we also have

$$L_{2,n} \leq C|u_{2,n} - u_2|_2 \to 0 \text{ as } n \to \infty,$$

$$L_{3,n} \leq C|u_{1,n} - u_1|_2 \to 0 \text{ as } n \to \infty,$$

$$L_{4,n} \leq C|u_{2,n} - u_2|_2 \to 0 \text{ as } n \to \infty.$$

The claim (2.5) is thus proved.

It follows from Hölder inequality that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} k(x) u_{1,n}^2 u_{2,n}^2 \mathrm{d}x - \int_{\mathbb{R}^2} k(x) u_1^2 u_2^2 \mathrm{d}x\right| \\
&\leq \int_{\mathbb{R}^2} u_{1,n}^2 |u_{2,n} + u_2||u_{2,n} - u_2| \mathrm{d}x + \int_{\mathbb{R}^2} u_2^2 |u_{1,n} + u_1||u_{1,n} - u_1| \mathrm{d}x \\
&\leq |u_{2,n} + u_2|_3 |u_{2,n} - u_2|_3 |u_{1,n}|_6^2 + |u_{1,n} + u_1|_3 |u_{1,n} - u_1|_3 |u_2|_6^2 \\
&\leq C\left(|u_{1,n} - u_1|_3 + |u_{2,n} - u_2|_3\right) \to 0 \text{ as } n \to \infty
\end{aligned}$$

then we can get that

$$\int_{\mathbb{R}^2} k(x) u_{1,n}^2 u_{2,n}^2 \mathrm{d}x \to \int_{\mathbb{R}^2} k(x) u_1^2 u_2^2 \mathrm{d}x \text{ as } n \to \infty.$$

We conclude from the above that

$$e(\gamma,a) \leq E_{\gamma,a}(u_1,u_2) \leq \liminf_{n\to\infty} E_{\gamma,a}(u_{1,n},u_{2,n}) = e(\gamma,a),$$

which means that $e(\gamma,a) = E_{\gamma,a}(u_1,u_2)$, and hence (u_1,u_2) is a minimizer of $e(\gamma,a)$.

2). For the case that $0 < k(x) \leq 1$, we prove the existence and nonexistence of minimizers of $e(\gamma,a)$.

Case 1: $\gamma \neq 0$ and $a \in (0, 2a^*)$. For any $a \in (0, 2a^*)$, there exists a constant $\beta > 0$ such that

$$\frac{a}{2} \leq \beta < a^*. \tag{2.7}$$

We rewrite the energy functional $E_{\gamma,a}(u_1,u_2)$ as

$$\begin{aligned}
E_{\gamma,a}(u_1,u_2) = &\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u_1|^2 + |\nabla u_2|^2\right)\mathrm{d}x - \frac{\beta}{4}\int_{\mathbb{R}^2}\left(\sqrt{k(x)}u_1^2 + \sqrt{k(x)}u_2^2\right)^2 \mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathbb{R}^2}|x|^2\left(u_1^2 + u_2^2\right)\mathrm{d}x \\
&+ \frac{\gamma}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y| u_1(x)u_2(x)u_1(y)u_2(y)\mathrm{d}x\mathrm{d}y \\
&+ \frac{\beta}{4}\int_{\mathbb{R}^2}\left(\sqrt{k(x)}u_1^2 - \sqrt{k(x)}u_2^2\right)^2 \mathrm{d}x + \frac{2\beta - a}{2}\int_{\mathbb{R}^2} k(x) u_1^2 u_2^2 \mathrm{d}x.
\end{aligned} \tag{2.8}$$

Following (1.9), (1.12) (2.1), (2.3), (2.7) and (2.8), we infer that

$$\begin{aligned}
E_{\gamma,a}(u_1,u_2) \geq &\frac{1}{2}\left(1 - \frac{\beta}{a^*}\right)\int_{\mathbb{R}^2}\left(|\nabla u_1|^2 + |\nabla u_2|^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2}|x|^2\left(u_1^2 + u_2^2\right)\mathrm{d}x \\
&- C\left[\int_{\mathbb{R}^2}\left(|\nabla u_1|^2 + |\nabla u_2|^2\right)\mathrm{d}x\right]^{\frac{1}{2}} - \frac{\gamma^2}{16\pi^2} \text{ for } (u_1,u_2) \in \mathcal{M},
\end{aligned} \tag{2.9}$$

which indicates that $E_{\gamma,a}$ is bounded from below on $\mathscr{M}$. Using the similar argument as the case that $-1\le k(x)\le 0$, one can obtain that $e(\gamma,a)$ admits at least one minimizer.

Case 2: $\gamma\neq 0$ and $a>2a^*$. For any $a>2a^*$, we next prove the nonexistence of minimizers of $e(\gamma,a)$. Letting $0\le\theta\le 1$, we consider the test function

$$\left(u_{1,\tau}(x),u_{2,\tau}(x)\right):=\left(\frac{\sqrt{\theta}\tau}{|Q|_2}Q(\tau x),\frac{\sqrt{1-\theta}\tau}{|Q|_2}Q(\tau x)\right)\text{ for }\tau>0.$$

Clearly, $\left(u_{1,\tau},u_{2,\tau}\right)\in\mathscr{M}$ for all $\tau>0$. It follows from (1.4) and (1.15) that

$$\begin{aligned}E_{\gamma,a}\left(u_{1,\tau},u_{2,\tau}\right)=&\left[\frac{1}{2}-\frac{a\theta(1-\theta)}{2\left(a^*\right)^2}\int_{\mathbb{R}^2}k\left(\frac{x}{\tau}\right)Q^4(x)\mathrm{d}x\right]\tau^2+\frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2Q^2(x)\mathrm{d}x\\&-\frac{\gamma\theta(1-\theta)}{4\pi}\ln\tau+\frac{\gamma\theta(1-\theta)}{4\pi\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y.\end{aligned}\tag{2.10}$$

Choose $\theta=\frac{1}{2}$, combining with Fatou's lemma, then we obtain from (1.7), (1.13) and (2.10) that

$$\begin{aligned}e(\gamma,a)\le&\left[\frac{1}{2}-\frac{a}{8\left(a^*\right)^2}\int_{\mathbb{R}^2}k\left(\frac{x}{\tau}\right)Q^4(x)\mathrm{d}x\right]\tau^2+\frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2Q^2(x)\mathrm{d}x\\&-\frac{\gamma}{16\pi}\ln\tau+\frac{\gamma}{16\pi\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\\le&\left(\frac{1}{2}-\frac{a}{4a^*}\right)\tau^2+\frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2Q^2(x)\mathrm{d}x\\&-\frac{\gamma}{16\pi}\ln\tau+\frac{\gamma}{16\pi\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\&\to-\infty\text{ as }\tau\to\infty,\end{aligned}\tag{2.11}$$

which implies that there is no minimizers of $e(\gamma,a)$ for $\gamma\neq 0$ and $a>2a^*$.

Case 3: $\gamma>0$ and $a=2a^*$. Combining with (1.8), choosing $\theta=\frac{1}{2}$, we can obtain from (2.10) that

$$\begin{aligned}e(\gamma,a)\le&\frac{\tau^2}{4a^*}\int_{\mathbb{R}^2}\left(1-\frac{a}{2a^*}k\left(\frac{x}{\tau}\right)\right)Q^4(x)\mathrm{d}x+\frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2Q^2(x)\mathrm{d}x\\&-\frac{\gamma}{16\pi}\ln\tau+\frac{\gamma}{16\pi\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\\le&C\frac{\tau^{2-b}}{4a^*}\int_{\mathbb{R}^2}|x|^bQ^4(x)\mathrm{d}x+\frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2Q^2(x)\mathrm{d}x\\&-\frac{\gamma}{16\pi}\ln\tau+\frac{\gamma}{16\pi\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\&\to-\infty\text{ as }\tau\to\infty.\end{aligned}\tag{2.12}$$

which implies that there is no minimizers of $e(\gamma,a)$ for $\gamma>0$ and $a=2a^*$.

This completes the proof of Theorem 1.1.

3. Limiting Behavior of Minimizers

In this section, we prove Theorem 1.2 on the limiting behavior of minimizers $\left(u_1^a,u_2^a\right)$ of $e(\gamma,a)$ as $a\nearrow 2a^*$, where $\gamma>0$ is fixed. We shall make full use of the rewritten form (3.7) of the energy functional (1.4). In order to analyze the blowing-up property of minimizers $\left(u_1^a,u_2^a\right)$, we define

$$\varepsilon_a:=\left[\int_{\mathbb{R}^2}\left(\left|\nabla u_1^a\right|^2+\left|\nabla u_2^a\right|^2\right)\mathrm{d}x\right]^{-\frac{1}{2}}. \tag{3.1}$$

Lemma 3.1 *Let* $\left(u_1^a,u_2^a\right)\in\mathcal{M}$ *be a minimizer of* $e(\gamma,a)$, *where* $\gamma>0$ *is fixed and* $a\in\left(0,2a^*\right)$, *together with* $0<k(x)\le 1$ *satisfies* (1.7) *and* (1.8). *Then*

1) $\varepsilon_a>0$ satisfies that

$$\varepsilon_a\to 0\quad\text{and}\quad \lambda_a\varepsilon_a^2\to -1\quad\text{as}\quad a\nearrow 2a^*; \tag{3.2}$$

2) There exist a sequence $\left\{y_{\varepsilon_a}\right\}\subset\mathbb{R}^2$, and constants $r_0>0,\alpha>0$ independent of $a\in\left(0,2a^*\right)$, such that

$$\liminf_{a\nearrow 2a^*}\int_{B_{r_0}(0)}\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_i^a\right|^2\mathrm{d}x\ge\alpha>0\ \text{ for } i=1,2, \tag{3.3}$$

where

$$v_i^a(x):=\varepsilon_a u_i^a\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\ \text{ for } i=1,2; \tag{3.4}$$

3) There exists a point $x_0\in\mathbb{R}^2$ such that

$$\lim_{a\nearrow 2a^*}\left|v_i^a(x)\right|=\frac{Q\left(x-x_0\right)}{\sqrt{2a^*}}\ \text{ in } H^1\left(\mathbb{R}^2\right)\ \text{ for } i=1,2. \tag{3.5}$$

Proof. 1) We first prove that

$$\varepsilon_a\to 0\ \text{ as } a\nearrow 2a^*. \tag{3.6}$$

Rewrite

$$\begin{aligned}E_{\gamma,a}\left(u_1^a,u_2^a\right)=&\frac{1}{2}\int_{\mathbb{R}^2}\left(\left|\nabla u_1^a\right|^2+\left|\nabla u_2^a\right|^2\right)\mathrm{d}x-\frac{a^*}{4}\int_{\mathbb{R}^2}\left(\sqrt{k(x)}\left|u_1^a\right|^2+\sqrt{k(x)}\left|u_2^a\right|^2\right)^2\mathrm{d}x\\&+\frac{1}{2}\int_{\mathbb{R}^2}|x|^2\left(\left|u_1^a\right|^2+\left|u_2^a\right|^2\right)\mathrm{d}x\\&+\frac{\gamma}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_1^a(x)u_2^a(x)u_1^a(y)u_2^a(y)\mathrm{d}x\mathrm{d}y\\&+\frac{a^*}{4}\int_{\mathbb{R}^2}\left(\sqrt{k(x)}\left|u_1^a\right|^2-\sqrt{k(x)}\left|u_2^a\right|^2\right)^2\mathrm{d}x\\&+\frac{2a^*-a}{2}\int_{\mathbb{R}^2}k(x)\left|u_1^a\right|^2\left|u_2^a\right|^2\mathrm{d}x.\end{aligned} \tag{3.7}$$

We derive from (1.9), (1.12), (2.1), (2.3) and (3.7) that

$$e(\gamma,a)\ge -C\varepsilon_a^{-1}-\frac{\gamma^2}{16\pi^2}. \tag{3.8}$$

Derive from (2.12), choosing $\tau=\left(2a^*-a\right)^{-\frac{1}{2}}$, we can obtain that

$$
\begin{aligned}
e(\gamma,a) &\le C\frac{\left(2a^*-a\right)^{\frac{b-2}{2}}}{4a^*}\int_{\mathbb{R}^2}|x|^b Q^4(x)\,dx+\frac{2a^*-a}{2a^*}\int_{\mathbb{R}^2}|x|^2 Q^2(x)\,dx \\
&\quad+\frac{\gamma}{32\pi}\ln\left(2a^*-a\right)+\frac{\gamma}{16\pi\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\,dxdy \\
&\to-\infty \text{ as } a\nearrow 2a^*.
\end{aligned}
\tag{3.9}
$$

It then yields from (3.8) and (3.9) that (3.6) holds true.

We then prove that $\lambda_a\varepsilon_a^2\to-1$ as $a\nearrow 2a^*$. It follows from (3.6) and (3.8) that

$$
\liminf_{a\nearrow 2a^*}\varepsilon_a^2 e(\gamma,a)\ge\liminf_{a\nearrow 2a^*}\left(-\frac{\gamma^2}{16\pi^2}\varepsilon_a^2-C\varepsilon_a\right)=0. \tag{3.10}
$$

In addition, observe from (2.1), (2.3) and (3.7) that

$$
\begin{aligned}
\varepsilon_a^2 e(\gamma,a)\ge{}&\frac{1}{2}-\frac{a^*\varepsilon_a^2}{4}\int_{\mathbb{R}^2}\left(\sqrt{k(x)}\left|u_1^a\right|^2+\sqrt{k(x)}\left|u_2^a\right|^2\right)^2 dx \\
&+\frac{\varepsilon_a^2}{4}\int_{\mathbb{R}^2}|x|^2\left(\left|u_1^a\right|^2+\left|u_2^a\right|^2\right)dx \\
&+\frac{a^*\varepsilon_a^2}{4}\int_{\mathbb{R}^2}\left(\sqrt{k(x)}\left|u_1^a\right|^2-\sqrt{k(x)}\left|u_2^a\right|^2\right)^2 dx \\
&+\frac{\left(2a^*-a\right)\varepsilon_a^2}{2}\int_{\mathbb{R}^2}k(x)\left|u_1^a\right|^2\left|u_2^a\right|^2 dx-\frac{\gamma^2}{16\pi^2}\varepsilon_a^2-C\varepsilon_a.
\end{aligned}
\tag{3.11}
$$

Using (1.12), we then obtain from (3.9) and (3.11) that

$$
\begin{aligned}
0&\ge\limsup_{a\nearrow 2a^*}\varepsilon_a^2 e(\gamma,a) \\
&\ge\limsup_{a\nearrow 2a^*}\frac{\varepsilon_a^2}{4}\int_{\mathbb{R}^2}|x|^2\left(\left|u_1^a\right|^2+\left|u_2^a\right|^2\right)dx+\lim_{a\nearrow 2a^*}\left(-\frac{\gamma^2}{16\pi^2}\varepsilon_a^2-C\varepsilon_a\right) \\
&\ge\liminf_{a\nearrow 2a^*}\frac{\varepsilon_a^2}{4}\int_{\mathbb{R}^2}|x|^2\left(\left|u_1^a\right|^2+\left|u_2^a\right|^2\right)dx\ge 0.
\end{aligned}
\tag{3.12}
$$

It follows from (3.10) and (3.12) that

$$
\lim_{a\nearrow 2a^*}\varepsilon_a^2 e(\gamma,a)=0 \text{ and } \lim_{a\nearrow 2a^*}\varepsilon_a^2\int_{\mathbb{R}^2}|x|^2\left(\left|u_1^a\right|^2+\left|u_2^a\right|^2\right)dx=0. \tag{3.13}
$$

Similarly, we also have

$$
\begin{aligned}
&\lim_{a\nearrow 2a^*}\varepsilon_a^2\int_{\mathbb{R}^2}\left(\sqrt{k(x)}\left|u_1^a\right|^2-\sqrt{k(x)}\left|u_2^a\right|^2\right)^2 dx=0, \\
&\lim_{a\nearrow 2a^*}\varepsilon_a^2\int_{\mathbb{R}^2}\left(\sqrt{k(x)}\left|u_1^a\right|^2+\sqrt{k(x)}\left|u_2^a\right|^2\right)^2 dx=\frac{2}{a^*},
\end{aligned}
\tag{3.14}
$$

which imply that

$$
\lim_{a\nearrow 2a^*}\varepsilon_a^2\int_{\mathbb{R}^2}k(x)\left|u_1^a\right|^2\left|u_2^a\right|^2 dx=\frac{1}{2a^*}. \tag{3.15}
$$

Due to (2.1), (2.3) and (3.13), we obtain that

$$
\left|\varepsilon_a^2 F_1\left(u_1^a,u_2^a\right)\right|\le\frac{\pi}{\gamma}\varepsilon_a^2\int_{\mathbb{R}^2}|x|^2\left(\left|u_1^a\right|^2+\left|u_2^a\right|^2\right)dx+\frac{\gamma}{4\pi}\varepsilon_a^2\to 0 \text{ as } a\nearrow 2a^*,
$$

$$\left|\varepsilon_a^2 F_2\left(u_1^a,u_2^a\right)\right| \le C\varepsilon_a \to 0 \text{ as } a\nearrow 2a^*,$$

that is,

$$\lim_{a\nearrow 2a^*} \varepsilon_a^2 F_1\left(u_1^a,u_2^a\right)=0 \text{ and } \lim_{a\nearrow 2a^*} \varepsilon_a^2 F_2\left(u_1^a,u_2^a\right)=0. \tag{3.16}$$

Hence we conclude from (1.17), (3.13), (3.15) and (3.16) that

$$\lim_{a\nearrow 2a^*} \lambda_a\varepsilon_a^2=-1.$$

2) Note from (3.4) that

$$\int_{\mathbb{R}^2}\left(\left|v_1^a\right|^2+\left|v_2^a\right|^2\right)\mathrm{d}x=1=\int_{\mathbb{R}^2}\left(\left|\nabla v_1^a\right|^2+\left|\nabla v_2^a\right|^2\right)\mathrm{d}x. \tag{3.17}$$

Thus $\{v_i^a\}(i=1,2)$ is bounded uniformly in $H^1(\mathbb{R}^2)$ and in $L^q(\mathbb{R}^2)$ for all $q\in[2,\infty)$. Following (3.14) and (3.15), we then obtain from (3.4) that

$$\begin{aligned}&\int_{\mathbb{R}^2} k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a\right|^2\left|v_2^a\right|^2\mathrm{d}x\\ &=\varepsilon_a^2\int_{\mathbb{R}^2} k(x)\left|u_1^a\right|^2\left|u_2^a\right|^2\mathrm{d}x\to\frac{1}{2a^*} \text{ as } a\nearrow 2a^*,\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_1^a\right|^2-\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_2^a\right|^2\right)^2\mathrm{d}x\\ &=\varepsilon_a^2\int_{\mathbb{R}^2}\left(\sqrt{k(x)}\left|u_1^a\right|^2-\sqrt{k(x)}\left|u_2^a\right|^2\right)^2\mathrm{d}x\to 0 \text{ as } a\nearrow 2a^*.\end{aligned} \tag{3.19}$$

For $i=1,2$, we denote

$$\overline{v}_i^a(x):=v_i^a\left(x-y_{\varepsilon_a}\right)=\varepsilon_a u_i^a\left(\varepsilon_a x\right).$$

To prove (3.3), it suffices to prove that there exist $\{y_{\varepsilon_a}\}\subset\mathbb{R}^2$, $r_0>0$ and $\alpha>0$ independent of $a\in(0,2a^*)$, such that

$$\liminf_{a\nearrow 2a^*}\int_{B_{r_0}(y_{\varepsilon_a})}\sqrt{k(\varepsilon_a x)}\left|\overline{v}_i^a\right|^2\mathrm{d}x\ge\alpha>0 \text{ for } i=1,2. \tag{3.20}$$

We first show (3.20) for $i=1$. Indeed, if it were false, then for any $r>0$, there exists a subsequence $\{\overline{v}_1^{a_k}\}$, where $a_k\nearrow 2a^*$ as $k\to\infty$, such that

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_r(y)}\sqrt{k(\varepsilon_a x)}\left|\overline{v}_1^{a_k}\right|^2\mathrm{d}x=0.$$

By ([12] Lemma 1.21), we have $k(\varepsilon_a x)^{\frac{1}{4}}\overline{v}_1^{a_k}\to 0$ in $L^q(\mathbb{R}^2)$ as $k\to\infty$ for any $q\in(2,\infty)$. Hence

$$\begin{aligned}&\int_{\mathbb{R}^2} k(\varepsilon_a x)\left|\overline{v}_1^{a_k}\right|^2\left|\overline{v}_2^{a_k}\right|^2\mathrm{d}x\\ &\le\left(\int_{\mathbb{R}^2} k(\varepsilon_a x)\left|\overline{v}_1^{a_k}\right|^4\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2} k(\varepsilon_a x)\left|\overline{v}_2^{a_k}\right|^4\mathrm{d}x\right)^{\frac{1}{2}}\\ &\le C\left(\int_{\mathbb{R}^2} k(\varepsilon_a x)\left|\overline{v}_1^{a_k}\right|^4\mathrm{d}x\right)^{\frac{1}{2}}\to 0 \text{ as } k\to\infty,\end{aligned}$$

which contradicts with (3.18).

We next show (3.20) for $i=2$. Let $\{y_{\varepsilon_a}\}\subset\mathbb{R}^2$, $r_0>0$ and $\alpha>0$ be obtained for $\overline{v}_1^a$ in (3.20). If (3.20) were false for $\overline{v}_2^a$, then there is a subsequence

$\left\{\overline{v}_2^{a_k}\right\}$, where $a_k \nearrow 2a^*$ as $k \to \infty$, such that

$$\limsup_{k\to\infty} \int_{B_{r_0}\left(y_{\varepsilon_k}\right)} \sqrt{k\left(\varepsilon_a x\right)} \left|\overline{v}_2^{a_k}\right|^2 \mathrm{d}x = 0,$$

where $\varepsilon_k := \varepsilon_{a_k}$ is defined by (3.1). Choose $q > 4$ and $0 < \zeta < 1$ such that $\frac{1}{4} = \frac{1-\zeta}{q} + \frac{\zeta}{2}$. We then deduce that

$$\begin{aligned}
&\int_{B_{r_0}\left(y_{\varepsilon_k}\right)} k\left(\varepsilon_a x\right)\left|\overline{v}_1^{a_k}\right|^2\left|\overline{v}_2^{a_k}\right|^2 \mathrm{d}x \\
&\le \left(\int_{B_{r_0}\left(y_{\varepsilon_k}\right)} k\left(\varepsilon_a x\right)\left|\overline{v}_1^{a_k}\right|^4 \mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B_{r_0}\left(y_{\varepsilon_k}\right)} k\left(\varepsilon_a x\right)\left|\overline{v}_2^{a_k}\right|^4 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C\left(\int_{B_{r_0}\left(y_{\varepsilon_k}\right)} k\left(\varepsilon_a x\right)\left|\overline{v}_2^{a_k}\right|^4 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C\left[\int_{B_{r_0}\left(y_{\varepsilon_k}\right)}\left(k\left(\varepsilon_a x\right)^{\frac{1}{4}}\left|\overline{v}_2^{a_k}\right|\right)^q \mathrm{d}x\right]^{\frac{2(1-\zeta)}{q}}\left[\int_{B_{r_0}\left(y_{\varepsilon_k}\right)}\left(k\left(\varepsilon_a x\right)^{\frac{1}{4}}\left|\overline{v}_2^{a_k}\right|\right)^2 \mathrm{d}x\right]^{\zeta} \\
&\le C\left(\int_{B_{r_0}\left(y_{\varepsilon_k}\right)} \sqrt{k\left(\varepsilon_a x\right)}\left|\overline{v}_2^{a_k}\right|^2 \mathrm{d}x\right)^{\zeta} \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Together with (3.20) for $i = 1$, we then get that

$$\begin{aligned}
&\int_{B_{r_0}\left(y_{\varepsilon_k}\right)}\left(\sqrt{k\left(\varepsilon_a x\right)}\left|\overline{v}_1^{a_k}\right|^2 - \sqrt{k\left(\varepsilon_a x\right)}\left|\overline{v}_2^{a_k}\right|^2\right)^2 \mathrm{d}x \\
&\ge \frac{1}{2}\int_{B_{r_0}\left(y_{\varepsilon_k}\right)} k\left(\varepsilon_a x\right)\left|\overline{v}_1^{a_k}\right|^4 \mathrm{d}x \ge \frac{1}{2\pi r_0^2}\left(\int_{B_{r_0}\left(y_{\varepsilon_k}\right)} \sqrt{k\left(\varepsilon_a x\right)}\left|\overline{v}_1^{a_k}\right|^2 \mathrm{d}x\right)^2 \\
&\ge \frac{\alpha^2}{2\pi r_0^2} > 0 \text{ as } k \to \infty,
\end{aligned}$$

which contradicts with (3.19).

3) Since $\left\{\left(v_1^a, v_2^a\right)\right\}$ is bounded uniformly in $H^1\left(\mathbb{R}^2\right)\times H^1\left(\mathbb{R}^2\right)$, up to a subsequence if necessary, there exists $\left(v_1^*, v_2^*\right) \in H^1\left(\mathbb{R}^2\right)\times H^1\left(\mathbb{R}^2\right)$ such that as $a \nearrow 2a^*$,

$$\begin{gathered}
\left(v_1^a, v_2^a\right) \rightharpoonup \left(v_1^*, v_2^*\right) \text{ in } H^1\left(\mathbb{R}^2\right)\times H^1\left(\mathbb{R}^2\right), \\
\left(v_1^a, v_2^a\right) \to \left(v_1^*, v_2^*\right) \text{ in } L^q_{loc}\left(\mathbb{R}^2\right)\times L^q_{loc}\left(\mathbb{R}^2\right) \text{ for } q \in [2, \infty), \\
\left(v_1^a, v_2^a\right) \to \left(v_1^*, v_2^*\right) \text{ a.e. on } \mathbb{R}^2.
\end{gathered}$$

It follows from Fatou's lemma and (3.17) that $\int_{\mathbb{R}^2}\left(\left|v_1^*\right|^2 + \left|v_2^*\right|^2\right)\mathrm{d}x \le 1$. Moreover, we obtain from (3.3) that $v_1^* \not\equiv 0$ and $v_2^* \not\equiv 0$. According to Brézis-Lieb lemma and (3.17), we derive that

$$\begin{aligned}
1 &= \int_{\mathbb{R}^2}\left(\left|v_1^a\right|^2 + \left|v_2^a\right|^2\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^2}\left(\left|v_1^*\right|^2 + \left|v_2^*\right|^2\right)\mathrm{d}x + \int_{\mathbb{R}^2}\left(\left|v_1^a - v_1^*\right|^2 + \left|v_2^a - v_2^*\right|^2\right)\mathrm{d}x + o(1),
\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
1 &= \int_{\mathbb{R}^2}\left(\left|\nabla v_1^a\right|^2+\left|\nabla v_2^a\right|^2\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^2}\left(\left|\nabla v_1^*\right|^2+\left|\nabla v_2^*\right|^2\right)\mathrm{d}x+\int_{\mathbb{R}^2}\left(\left|\nabla v_1^a-\nabla v_1^*\right|^2+\left|\nabla v_2^a-\nabla v_2^*\right|^2\right)\mathrm{d}x+o(1),
\end{aligned} \tag{3.22}
$$

where and below $o(1)$ represents the quantities tending to 0 as $a \nearrow 2a^*$. Additionally, there holds

$$
\begin{aligned}
&\int_{\mathbb{R}^2} k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a v_2^a-v_1^* v_2^*\right|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^2} k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a-v_1^*\right|^2\left|v_2^a-v_2^*\right|^2 \mathrm{d}x+o(1),
\end{aligned} \tag{3.23}
$$

where we use the facts that $a \nearrow 2a^*$,

$$\int_{\mathbb{R}^2}\left(k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^*\right|^2\left|v_2^a\right|^2-k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)v_1^a v_1^*\left|v_2^a\right|^2\right)\mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^2}\left(k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a\right|^2\left|v_2^*\right|^2-k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a\right|^2 v_2^a v_2^*\right)\mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^2}\left(k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)v_1^a v_1^* v_2^a v_2^*-k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)v_1^a v_1^*\left|v_2^a\right|^2\right)\mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^2}\left(k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)v_1^a v_1^* v_2^a v_2^*-k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a\right|^2 v_2^a v_2^*\right)\mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^2}\left(k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)v_1^a v_1^* v_2^a v_2^*-k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)v_1^a v_1^*\left|v_2^*\right|^2\right)\mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^2}\left(k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)v_1^a v_1^* v_2^a v_2^*-k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^*\right|^2 v_2^a v_2^*\right)\mathrm{d}x \to 0.$$

It follows from Brézis-Lieb lemma and (3.23) that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a\right|^2\left|v_2^a\right|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^2} k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^*\right|^2\left|v_2^*\right|^2 \mathrm{d}x \\
&\quad+\int_{\mathbb{R}^2} k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a-v_1^*\right|^2\left|v_2^a-v_2^*\right|^2 \mathrm{d}x+o(1),
\end{aligned}
$$

which jointly with

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a\right|^4+k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_2^a\right|^4\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^2}\left(k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^*\right|^4+k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_2^*\right|^4\right)\mathrm{d}x \\
&\quad+\int_{\mathbb{R}^2}\left(k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a-v_1^*\right|^4+k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_2^a-v_2^*\right|^4\right)\mathrm{d}x+o(1),
\end{aligned}
$$

yields that

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_1^a\right|^2+\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_2^a\right|^2\right)^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_1^*\right|^2+\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_2^*\right|^2\right)^2 \mathrm{d}x \\
&\quad+\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_1^a-v_1^*\right|^2+\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_2^a-v_2^*\right|^2\right)^2 \mathrm{d}x+o(1).
\end{aligned} \tag{3.24}
$$

Due to (3.14), we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_1^a\right|^2+\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_2^a\right|^2\right)^2 \mathrm{d}x\\&=\varepsilon_a^2\int_{\mathbb{R}^2}\left(\sqrt{k(x)}\left|u_1^a\right|^2+\sqrt{k(x)}\left|u_2^a\right|^2\right)^2 \mathrm{d}x\to\frac{2}{a^*}\ \text{as}\ a\nearrow 2a^*.\end{aligned}\tag{3.25}$$

Then we deduce from (1.12), (3.21), (3.22), (3.24) and (3.25) that

$$\begin{aligned}0=&\lim_{a\nearrow 2a^*}\left[\int_{\mathbb{R}^2}\left(\left|\nabla v_1^a\right|^2+\left|\nabla v_2^a\right|^2\right)\mathrm{d}x-\frac{a}{4}\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left(\left|v_1^a\right|^2+\left|v_2^a\right|^2\right)\right)^2\mathrm{d}x\right]\\
=&\lim_{a\nearrow 2a^*}\left[\int_{\mathbb{R}^2}\left(\left|\nabla v_1^*\right|^2+\left|\nabla v_2^*\right|^2\right)\mathrm{d}x+\int_{\mathbb{R}^2}\left(\left|\nabla v_1^a-\nabla v_1^*\right|^2+\left|\nabla v_2^a-\nabla v_2^*\right|^2\right)\mathrm{d}x\right.\\
&-\frac{a}{4}\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_1^*\right|^2+\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_2^*\right|^2\right)^2\mathrm{d}x\\
&\left.-\frac{a}{4}\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_1^a-v_1^*\right|^2+\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left|v_2^a-v_2^*\right|^2\right)^2\mathrm{d}x\right]\\
\geq&\frac{a^*}{2}\left[\left(\int_{\mathbb{R}^2}\left(\left|v_1^*\right|^2+\left|v_2^*\right|^2\right)\mathrm{d}x\right)^{-1}-1\right]\int_{\mathbb{R}^2}\left(\left|v_1^*\right|^2+\left|v_2^*\right|^2\right)^2\mathrm{d}x\\
&+\lim_{a\nearrow 2a^*}\left[1-\int_{\mathbb{R}^2}\left(\left|v_1^a-v_1^*\right|^2+\left|v_2^a-v_2^*\right|^2\right)\mathrm{d}x\right]\\
&\times\int_{\mathbb{R}^2}\left(\left|\nabla v_1^a-\nabla v_1^*\right|^2+\left|\nabla v_2^a-\nabla v_2^*\right|^2\right)\mathrm{d}x\\
\geq&\,0,\end{aligned}\tag{3.26}$$

which implies that

$$\int_{\mathbb{R}^2}\left(\left|v_1^*\right|^2+\left|v_2^*\right|^2\right)\mathrm{d}x=1,\tag{3.27}$$

$$\lim_{a\nearrow 2a^*}\int_{\mathbb{R}^2}\left(\left|\nabla v_1^a-\nabla v_1^*\right|^2+\left|\nabla v_2^a-\nabla v_2^*\right|^2\right)\mathrm{d}x=0.$$

Therefore, we conclude that

$$\left(v_1^a,v_2^a\right)\to\left(v_1^*,v_2^*\right)\ \text{in}\ H^1\left(\mathbb{R}^2\right)\times H^1\left(\mathbb{R}^2\right)\ \text{as}\ a\nearrow 2a^*.\tag{3.28}$$

We obtain from (3.17) and (3.28) that

$$\int_{\mathbb{R}^2}\left(\left|\nabla v_1^*\right|^2+\left|\nabla v_2^*\right|^2\right)\mathrm{d}x=1.\tag{3.29}$$

Additionally, we also get from (3.26) that

$$\int_{\mathbb{R}^2}\left(\left|\nabla v_1^*\right|^2+\left|\nabla v_2^*\right|^2\right)\mathrm{d}x=\frac{a^*}{2}\int_{\mathbb{R}^2}\left(\left|v_1^*\right|^2+\left|v_2^*\right|^2\right)^2\mathrm{d}x.\tag{3.30}$$

Applying (3.19) and (3.28) we have $\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left(\left|v_1^*\right|^2-\left|v_2^*\right|^2\right)\right)^2\mathrm{d}x=0$. This further implies that $\left|v_1^*\right|=\left|v_2^*\right|$ a.e. on $\mathbb{R}^2$. It then follows from (3.27), (3.29) and (3.30) that for $i=1,2$,

$$\int_{\mathbb{R}^2}\left|v_i^*\right|^2\mathrm{d}x=\frac{1}{2}\ \text{and}\ \frac{1}{2}=\int_{\mathbb{R}^2}\left|\nabla v_i^*\right|^2\mathrm{d}x=a^*\int_{\mathbb{R}^2}\left|v_i^*\right|^4\mathrm{d}x.$$

Obviously, the identity in (1.14) is attained at $\left|v_i^*\right|$ for $i=1,2$. We can derive from the Lagrange multiplier rule that for $i=1,2$,

$$-\Delta|v_i^*|+|v_i^*|=2a^*|v_i^*|^3 \text{ in } \mathbb{R}^2.$$

The uniqueness (up to translations) of positive solutions of (1.2) yields that

$$|v_1^*(x)|=|v_2^*(x)|=\frac{Q(x-x_0)}{\sqrt{2a^*}} \text{ for some } x_0\in\mathbb{R}^2.$$

This completes the proof of Lemma 3.1.

In what follows, we assume that the minimizer (u_1^a,u_2^a) of $e(\gamma,a)$ is non-negative. Following Lemma 3.1, we continue to analyze the refined limiting behavior of non-negative (u_1^a,u_2^a) as $a\nearrow 2a^*$. The exponential decay of non-negative minimizers (u_1^a,u_2^a) at infinity needs to be proved first.

Lemma 3.2 *Let* $(u_1^a,u_2^a)\in\mathcal{M}$ *be a non-negative minimizer of* $e(\gamma,a)$, *where* $\gamma>0$ *is fixed and* $a\in(0,2a^*)$, *together with* $0<k(x)\le 1$ *satisfies* (1.7) *and* (1.8). *Then*

1) There exists a large constant $R>0$, independent of $a\in(0,2a^*)$, such that

$$|v_i^a(x)|\le Ce^{-\frac{2}{3}|x|} \text{ uniformly for } |x|\ge R \text{ as } a\nearrow 2a^*, \tag{3.31}$$

where v_i^a is defined by (3.4) and $i=1,2$;

2) There results

$$\varepsilon_a y_{\varepsilon_a}\to 0 \text{ as } a\nearrow 2a^*. \tag{3.32}$$

Proof. 1) Note from (1.16) that (v_1^a,v_2^a) solves

$$\begin{cases}-\Delta v_1^a+\varepsilon_a^2|\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}|^2 v_1^a+\frac{\gamma\varepsilon_a^2}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|v_1^a(y)v_2^a(y)\mathrm{d}y\,v_2^a\\ =\lambda_a\varepsilon_a^2 v_1^a+ak(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a})|v_2^a|^2 v_1^a-\frac{\gamma\varepsilon_a^2}{2\pi}\ln\varepsilon_a\int_{\mathbb{R}^2}v_1^a(y)v_2^a(y)\mathrm{d}y\,v_2^a \text{ in } \mathbb{R}^2,\\ -\Delta v_2^a+\varepsilon_a^2|\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}|^2 v_2^a+\frac{\gamma\varepsilon_a^2}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|v_1^a(y)v_2^a(y)\mathrm{d}y\,v_1^a\\ =\lambda_a\varepsilon_a^2 v_2^a+ak(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a})|v_1^a|^2 v_2^a-\frac{\gamma\varepsilon_a^2}{2\pi}\ln\varepsilon_a\int_{\mathbb{R}^2}v_1^a(y)v_2^a(y)\mathrm{d}y\,v_1^a \text{ in } \mathbb{R}^2.\end{cases} \tag{3.33}$$

Using the same argument as (2.6), we then obtain from (3.17) that

$$\int_{\mathbb{R}^2}\frac{v_1^a(y)v_2^a(y)}{|x-y|}\mathrm{d}y\le C \text{ uniformly for any } x\in\mathbb{R}^2 \text{ and } a\in(0,2a^*).$$

This further implies that

$$\begin{aligned}&\int_{\mathbb{R}^2}\ln|x-y|v_1^a(y)v_2^a(y)\mathrm{d}y\\ &=\int_{\mathbb{R}^2}\ln(1+|x-y|)v_1^a(y)v_2^a(y)\mathrm{d}y-\int_{\mathbb{R}^2}\ln(1+|x-y|^{-1})v_1^a(y)v_2^a(y)\mathrm{d}y\\ &\ge -C \text{ for any } x\in\mathbb{R}^2 \text{ and } a\in(0,2a^*).\end{aligned}$$

Hence we derive from (3.2) and (3.33) that as $a\nearrow 2a^*$,

$$\begin{cases}-\Delta v_1^a+\frac{2}{3}v_1^a\le 2a^*|v_2^a|^2 v_1^a+o(1)v_2^a \text{ in } \mathbb{R}^2,\\ -\Delta v_2^a+\frac{2}{3}v_2^a\le 2a^*|v_1^a|^2 v_2^a+o(1)v_1^a \text{ in } \mathbb{R}^2.\end{cases} \tag{3.34}$$

Using the De Giorgi-Nash-Moser theory (cf. [13] Theorem 4.1), we then deduce from (3.17) that as $a \nearrow 2a^*$,

$$\max_{B_1(\xi)} v_i^a \le C\left[\left(\int_{B_2(\xi)} \left|v_1^a\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{B_2(\xi)} \left|v_2^a\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}\right] \text{ for } i = 1, 2,$$

where ξ is an arbitrary point in $\mathbb{R}^2$ and $C > 0$ is a constant independent of $a \in \left(0, 2a^*\right)$. We then obtain from (3.5) that

$$v_i^a \in L^\infty\left(\mathbb{R}^2\right) \text{ and } v_i^a(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly for } a \nearrow 2a^* \text{ with } i = 1, 2. \quad (3.35)$$

Therefore, we infer from (3.34) that there exists a large constant $R > 0$ independent of $a \in \left(0, 2a^*\right)$ such that

$$-\Delta\left(v_1^a + v_2^a\right) + \frac{4}{9}\left(v_1^a + v_2^a\right) \le 0 \text{ uniformly for } |x| \ge R \text{ as } a \nearrow 2a^*.$$

Applying the comparision principle, it implies that there is a constant $C > 0$ independent of $a \in \left(0, 2a^*\right)$ such that

$$v_1^a(x) + v_2^a(x) \le Ce^{-\frac{2}{3}|x|} \text{ uniformly for } |x| \ge R \text{ as } a \nearrow 2a^*.$$

Together with the non-negativity of v_i^a for $i = 1, 2$, we conclude that (3.31) holds true.

3) We now prove that

$$\lim_{a \nearrow 2a^*} \varepsilon_a y_{\varepsilon_a} = 0.$$

Otherwise, there is a subsequence of $\varepsilon_a y_{\varepsilon_a}$, still denoted by itself, such that

$$\varepsilon_a y_{\varepsilon_a} \to z_0 \ne 0 \text{ as } a \nearrow 2a^*.$$

Set $x_a := \int_{\mathbb{R}^2} x\left(\left|u_1^a\right|^2 + \left|u_2^a\right|^2\right)\mathrm{d}x$. It follows from (3.4) and (3.5) that as $a \nearrow 2a^*$,

$$x_a = \int_{\mathbb{R}^2} \left(\varepsilon_a x + \varepsilon_a y_{\varepsilon_a}\right)\left(\left|v_1^a\right|^2 + \left|v_2^a\right|^2\right)\mathrm{d}x = \varepsilon_a y_{\varepsilon_a} + \varepsilon_a\left(x_0 + o(1)\right) \to z_0.$$

Define $\bar{u}_i^a(x) := u_i^a(x + x_a)$ for $i = 1, 2$. Then $\left(\bar{u}_1^a, \bar{u}_2^a\right) \in \mathcal{M}$ and

$$\int_{\mathbb{R}^2} \left(\left|\nabla \bar{u}_1^a\right|^2 + \left|\nabla \bar{u}_2^a\right|^2\right)\mathrm{d}x = \int_{\mathbb{R}^2} \left(\left|\nabla u_1^a\right|^2 + \left|\nabla u_2^a\right|^2\right)\mathrm{d}x,$$

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x - y| \bar{u}_1(x)\bar{u}_2(x)\bar{u}_1(y)\bar{u}_2(y)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x - y| u_1(x)u_2(x)u_1(y)u_2(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

In addition, we deduce from (3.4) and (3.31) that as $a \nearrow 2a^*$,

$$\begin{aligned}\int_{\mathbb{R}^2} |x|^2\left(\left|\bar{u}_1^a\right|^2 + \left|\bar{u}_2^a\right|^2\right)\mathrm{d}x &= \int_{\mathbb{R}^2} \left|x - x_a\right|^2\left(\left|u_1^a\right|^2 + \left|u_2^a\right|^2\right)\mathrm{d}x \\ &= \varepsilon_a^2 \int_{\mathbb{R}^2} \left|x - \left(x_0 + o(1)\right)\right|^2\left(\left|v_1^a\right|^2 + \left|v_2^a\right|^2\right)\mathrm{d}x \\ &= O\left(\varepsilon_a^2\right),\end{aligned}$$

$$\int_{\mathbb{R}^2} |x|^2\left(\left|u_1^a\right|^2 + \left|u_2^a\right|^2\right)\mathrm{d}x = \int_{\mathbb{R}^2} \left|\varepsilon_a x + \varepsilon_a y_{\varepsilon_a}\right|^2\left(\left|v_1^a\right|^2 + \left|v_2^a\right|^2\right)\mathrm{d}x \to \left|z_0\right|^2.$$

Using (1.7), (3.4) and (3.31), we derive that

$$
\begin{aligned}
a\int_{\mathbb{R}^2} k(x)|\bar{u}_1|^2|\bar{u}_2|^2\,\mathrm{d}x &= a\int_{\mathbb{R}^2} k(x-x_a)|u_1^a|^2|u_2^a|^2\,\mathrm{d}x \\
&= a\varepsilon_a^{-2}\int_{\mathbb{R}^2} k\big(\varepsilon_a x-\varepsilon_a(x_0+o(1))\big)|v_1|^2|v_2|^2\,\mathrm{d}x \\
&= \varepsilon_a^{-2}\left[1+o(1)\right] \text{ as } a\nearrow 2a^*,
\end{aligned}
$$

$$
\begin{aligned}
a\int_{\mathbb{R}^2} k(x)|u_1|^2|u_2|^2\,\mathrm{d}x &= a\varepsilon_a^{-2}\int_{\mathbb{R}^2} k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)|v_1|^2|v_2|^2\,\mathrm{d}x \\
&= \varepsilon_a^{-2}\left[k(z_0)+o(1)\right] \text{ as } a\nearrow 2a^*.
\end{aligned}
$$

Note that $k(z_0)\le 1$, then $E_{\gamma,a}\left(\bar{u}_1^a,\bar{u}_2^a\right)<E_{\gamma,a}\left(u_1^a,u_2^a\right)$ as $a\nearrow 2a^*$, which contradicts with the fact that $\left(u_1^a,u_2^a\right)$ is the minimizer of $e(\gamma,a)$, which ensures that (3.32) is true. The proof of Lemma 3.2 is thus complete.

Next, we shall apply Lemma 3.2 to prove the convergence behavior of non-negative minimizers $\left(u_1^a,u_2^a\right)$ as $a\nearrow 2a^*$ for given $\gamma>0$.

Lemma 3.3 *Let* $\left(u_1^a,u_2^a\right)\in\mathcal{M}$ *be a non-negative minimizer of* $e(\gamma,a)$, *where* $\gamma>0$ *is fixed and* $a\in\left(0,2a^*\right)$, *together with* $0<k(x)\le 1$ *satisfies* (1.7) *and* (1.8) *and* $k(x)\in C^{0,\alpha}\left(\mathbb{R}^2\right)$ *for some* $\alpha\in(0,1)$. *Define*

$$
\tilde{v}_i^a(x):=\varepsilon_a u_i^a\left(\varepsilon_a x+x_i^a\right),\ i=1,2, \tag{3.36}
$$

where ε_a is given by (3.1) and x_i^a is a global maximum point of u_i^a. Then for $i=1,2$, x_i^a is unique and satisfies

$$
x_i^a\to 0 \text{ and } \frac{\left|x_1^a-x_2^a\right|}{\varepsilon_a}\to 0 \text{ as } a\nearrow 2a^*. \tag{3.37}
$$

Moreover, there results

$$
\tilde{v}_i^a(x)\to\frac{Q(x)}{\sqrt{2a^*}} \text{ in } H\cap L^\infty\left(\mathbb{R}^2\right) \text{ as } a\nearrow 2a^* \text{ with } i=1,2. \tag{3.38}
$$

Proof. We first prove (3.37). For any given $a\in\left(0,2a^*\right)$, each u_i^a has a global maximum point x_i^a for $i=1,2$ by means of (3.3) and (3.35). Then v_i^a achieves its global maximum point at $\varepsilon_a^{-1}\left(x_i^a-\varepsilon_a y_{\varepsilon_a}\right)$ for $i=1,2$. Using (3.3) and (3.35) again, we obtain that $\varepsilon_a^{-1}\left(x_i^a-\varepsilon_a y_{\varepsilon_a}\right)$ is bounded uniformly as $a\nearrow 2a^*$ for $i=1,2$. Together with (3.2) and (3.32), we have

$$
x_i^a\to 0 \text{ as } a\nearrow 2a^* \text{ for } i=1,2. \tag{3.39}
$$

Observe from (3.4) and (3.36) that

$$
\tilde{v}_i^a(x):=v_i^a\left(x+\frac{x_i^a-\varepsilon_a y_{\varepsilon_a}}{\varepsilon_a}\right) \text{ in } \mathbb{R}^2 \text{ for } i=1,2. \tag{3.40}
$$

We deduce from (3.33) that $\left(\tilde{v}_1^a,\tilde{v}_2^a\right)$ solves

$$
\begin{cases}
-\Delta\tilde{v}_1^a=G_1^a \text{ in } \mathbb{R}^2, \\
-\Delta\tilde{v}_2^a=G_2^a \text{ in } \mathbb{R}^2,
\end{cases} \tag{3.41}
$$

where

$$G_1^a := -\varepsilon_a^2 \left|\varepsilon_a x + x_1^a\right|^2 \tilde{v}_1^a - \frac{\gamma \varepsilon_a^2}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| \tilde{v}_1^a(y) \tilde{v}_2^a\left(y + \frac{x_1^a - x_2^a}{\varepsilon_a}\right) \mathrm{d}y \tilde{v}_2^a\left(\cdot + \frac{x_1^a - x_2^a}{\varepsilon_a}\right)$$
$$+ \lambda_a \varepsilon_a^2 \tilde{v}_1^a + ak\left(x + \frac{x_1^a - \varepsilon_a y_{\varepsilon_a}}{\varepsilon_a}\right) \left|\tilde{v}_2^a\left(\cdot + \frac{x_1^a - x_2^a}{\varepsilon_a}\right)\right|^2 \tilde{v}_1^a$$
$$- \frac{\gamma \varepsilon_a^2}{2\pi} \ln \varepsilon_a \left|\tilde{v}_1^a \tilde{v}_2^a\right|_1 \tilde{v}_2^a\left(\cdot + \frac{x_1^a - x_2^a}{\varepsilon_a}\right),$$

$$G_2^a := -\varepsilon_a^2 \left|\varepsilon_a x + x_2^a\right|^2 \tilde{v}_2^a - \frac{\gamma \varepsilon_a^2}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| \tilde{v}_1^a\left(y + \frac{x_2^a - x_1^a}{\varepsilon_a}\right) \tilde{v}_2^a(y) \mathrm{d}y \tilde{v}_1^a\left(\cdot + \frac{x_2^a - x_1^a}{\varepsilon_a}\right)$$
$$+ \lambda_a \varepsilon_a^2 \tilde{v}_2^a + ak\left(x + \frac{x_2^a - \varepsilon_a y_{\varepsilon_a}}{\varepsilon_a}\right) \left|\tilde{v}_1^a\left(\cdot + \frac{x_2^a - x_1^a}{\varepsilon_a}\right)\right|^2 \tilde{v}_2^a$$
$$- \frac{\gamma \varepsilon_a^2}{2\pi} \ln \varepsilon_a \left|\tilde{v}_1^a \tilde{v}_2^a\right|_1 \tilde{v}_1^a\left(\cdot + \frac{x_2^a - x_1^a}{\varepsilon_a}\right).$$

It follows from (3.5) and (3.40) that

$$\tilde{v}_i^a(x) \to \frac{1}{\sqrt{2a^*}} Q(x + y_i - x_0) \text{ in } H^1(\mathbb{R}^2) \text{ as } a \nearrow 2a^*, \tag{3.42}$$

where $y_i := \lim_{a \nearrow 2a^*} \varepsilon_a^{-1}\left(x_i^a - \varepsilon_a y_{\varepsilon_a}\right)$ for $i = 1,2$. Obviously, $\left\{\tilde{v}_i^a\right\}$ is bounded uniformly as $a \nearrow 2a^*$ in $H^1(\mathbb{R}^2)$ and in $L^q(\mathbb{R}^2)$ for all $q \in [2,\infty)$ with $i = 1,2$. Notice that

$$\left|\int_{\mathbb{R}^2} \ln|x-y| \tilde{v}_1^a(y) \tilde{v}_2^a\left(y + \frac{x_1^a - x_2^a}{\varepsilon_a}\right) \mathrm{d}y\right| \le |x| + C, \tag{3.43}$$

$$\left|\int_{\mathbb{R}^2} \ln|x-y| \tilde{v}_1^a\left(y + \frac{x_2^a - x_1^a}{\varepsilon_a}\right) \tilde{v}_2^a(y) \mathrm{d}y\right| \le |x| + C, \tag{3.44}$$

where $C > 0$ is a constant independent of $a \in (0, 2a^*)$. Applying the L^p theory (cf. ([14] Theorem 9.11)) to (3.41), we then deduce from (3.31) and (3.39) that $\left\{\tilde{v}_i^a\right\}$ is bounded uniformly as $a \nearrow 2a^*$ in $W_{loc}^{2,q}(\mathbb{R}^2)$ for all $q \in [2,\infty)$ with $i = 1,2$. The standard Sobolev embedding theorem implies that $\left\{\tilde{v}_i^a\right\}$ is bounded uniformly as $a \nearrow 2a^*$ in $C_{loc}^{1,\mu}(\mathbb{R}^2)$ for some $\mu \in (0,1)$ with $i = 1,2$. Similar to the proof of [8, Proposition 2.3], we get that as $a \nearrow 2a^*$,

$$\int_{\mathbb{R}^2} \ln|x-y| \tilde{v}_1^a(y) \tilde{v}_2^a\left(y + \frac{x_1^a - x_2^a}{\varepsilon_a}\right) \mathrm{d}y \in C_{loc}^{3,\mu}(\mathbb{R}^2),$$

$$\int_{\mathbb{R}^2} \ln|x-y| \tilde{v}_1^a\left(y + \frac{x_2^a - x_1^a}{\varepsilon_a}\right) \tilde{v}_2^a(y) \mathrm{d}y \in C_{loc}^{3,\mu}(\mathbb{R}^2).$$

Note that $\varepsilon_a^2 \left|\varepsilon_a x + x_i^a\right|^2$ is locally Lipschitz continuous in $\mathbb{R}^2$ for $i = 1,2$. Using the Schauder estimate (cf. ([14] Theorem 6.2)) to (3.41), we further obtain that $\left\{\tilde{v}_i^a\right\}$ is bounded uniformly as $a \nearrow 2a^*$ in $C_{loc}^{2,\mu}(\mathbb{R}^2)$ for some $\mu \in (0,1)$ with $i = 1,2$. Passing to a subsequence, there is a function $\tilde{v}_i^* \in C_{loc}^2(\mathbb{R}^2)$ such that

$$\tilde{v}_i^a \to \tilde{v}_i^* \text{ in } C_{loc}^2(\mathbb{R}^2) \text{ as } a \nearrow 2a^* \text{ for } i = 1,2.$$

Then we have $\tilde{v}_i^* = \frac{1}{\sqrt{2a^*}} Q(. + y_i - x_0)$ for $i = 1,2$ in view of (3.42). Since the origin is a global maximum point of $\tilde{v}_i^a$, it is also a global maximum point of $\tilde{v}_i^*$ for $i = 1,2$. The radially symmetric and decreasing property of Q yields that

$$y_1 = y_2 = x_0. \tag{3.45}$$

Hence we derive that

$$\frac{|x_1^a - x_2^a|}{\varepsilon_a} \le \frac{|x_1^a - \varepsilon_a y_{\varepsilon_a}|}{\varepsilon_a} + \frac{|x_2^a - \varepsilon_a y_{\varepsilon_a}|}{\varepsilon_a} \to |y_1| - |y_2| = 0 \text{ as } a \nearrow 2a^*. \tag{3.46}$$

We next prove (3.38). By (3.42) and (3.45), we get that

$$\tilde{v}_i^a(x) \to \frac{Q(x)}{\sqrt{2a^*}} \text{ in } H^1(\mathbb{R}^2) \text{ as } a \nearrow 2a^* \text{ with } i = 1,2. \tag{3.47}$$

It follows from (3.31) and (3.40) that there is a large constant $\tilde{R} \ge \max\{R + M, 2M\}$, where $M > 0$ is the uniform upper bound of $\varepsilon_a^{-1}|x_i^a - \varepsilon_a y_{\varepsilon_a}|$ as $a \nearrow 2a^*$ for $i = 1,2$, such that

$$|\tilde{v}_i^a(x)| \le C e^{-\frac{1}{3}|x|} \text{ uniformly for } |x| \ge \tilde{R} \text{ as } a \nearrow 2a^* \text{ with } i = 1,2. \tag{3.48}$$

Combining (1.13) with (3.47) and (3.48), we deduce that

$$\int_{\mathbb{R}^2} |x|^2 \left|\tilde{v}_i^a(x) - \frac{Q(x)}{\sqrt{2a^*}}\right| \mathrm{d}x \to 0 \text{ as } a \nearrow 2a^* \text{ with } i = 1,2. \tag{3.49}$$

We then obtain from (3.47) and (3.49) that

$$\tilde{v}_i^a(x) \to \frac{Q(x)}{\sqrt{2a^*}} \text{ in } H \text{ as } a \nearrow 2a^* \text{ for } i = 1,2.$$

Now we show that

$$\tilde{v}_i^a(x) \to \frac{Q(x)}{\sqrt{2a^*}} \text{ in } L^\infty(\mathbb{R}^2) \text{ as } a \nearrow 2a^* \text{ for } i = 1,2.$$

By means of (1.13) and (3.48), we only need to prove the L^∞-uniform convergence of $\{\tilde{v}_i^a\}$ as $a \nearrow 2a^*$ for $i = 1,2$ on any compact subset of $\mathbb{R}^2$. Using (3.17), (3.39), (3.43), (3.44) and (3.48), we see that $\{G_i^a\}$ is bounded uniformly as $a \nearrow 2a^*$ in $L^2(\mathbb{R}^2)$ for $i = 1,2$. For any $r > 0$, it follows from [14], Theorem 8.8] that there is a constant $C > 0$ independent of $a \in (0, 2a^*)$ and $r > 0$ such that for $i = 1,2$,

$$\|\tilde{v}_i^a\|_{H^2(B_r(0))} \le C\left(\|\tilde{v}_i^a\|_{H^1(B_{r+1}(0))} + \|G_i^a\|_{L^2(B_{r+1}(0))}\right) \text{ as } a \nearrow 2a^*.$$

Hence $\{v_i^a\}$ is bounded uniformly as $a \nearrow 2a^*$ in $H^2_{loc}(\mathbb{R}^2)$ for $i = 1,2$. Then we derive from the compact embedding $H^2(B_r(0)) \hookrightarrow L^\infty(B_r(0))$ (cf. ([14] Theorem 7.26)) that there is a subsequence of $\{\tilde{v}_i^a\}$, still denoted by itself, such that

$$\tilde{v}_i^a(x) \to \frac{Q(x)}{\sqrt{2a^*}} \text{ in } L^\infty(B_r(0)) \text{ as } a \nearrow 2a^* \text{ for } i=1,2.$$

Note that the above convergence is independent of the choice of the subsequence and $r>0$ is arbitrary. We conclude that the convergence holds for the whole sequence in $L^\infty_{loc}(\mathbb{R}^2)$ as $a \nearrow 2a^*$ for $i=1,2$.

Finally, we prove the uniqueness of the global maximum point x_i^a of u_i^a for $i=1,2$. Since

$$\tilde{v}_i^a(x) \to \frac{Q(x)}{\sqrt{2a^*}} \text{ in } C^2_{loc}(\mathbb{R}^2) \text{ as } a \nearrow 2a^* \text{ for } i=1,2,$$

and the origin is the unique global maximum point of Q, we see that all local maximum points of $\tilde{v}_i^a$ must approach the origin and thus stay in a small ball $B_\epsilon(0)$ as $a \nearrow 2a^*$ for some small constant $\epsilon>0$ with $i=1,2$. Due to $Q''(r)<0$, we can take $\epsilon>0$ small enough such that $Q''(0)<0$ for $r\in[0,\epsilon]$. It follows from ([15] Lemma 4.2) that $\tilde{v}_i^a$ has no local maximum points other than the origin as $a \nearrow 2a^*$ for $i=1,2$. Hence the global maximum point of u_i^a is unique as $a \nearrow 2a^*$ for $i=1,2$. This completes the proof of Lemma 3.3.

In view of above lemmas, we are now ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. According to Lemma 3.1 and Lemma 3.3, it suffices to prove that

$$\varepsilon_a = \left[\frac{8\pi(2a^*-a)}{\gamma a^*}\right]^{\frac{1}{2}}(1+o(1)) \text{ and } \lim_{a\nearrow 2a^*}\frac{x_i^a}{\varepsilon_a}=0 \text{ for } i=1,2.$$

We first estimate $e(\gamma,a)$ to get the explicit blowing-up rate of (u_1^a,u_2^a) as $a \nearrow 2a^*$. Taking $\theta=\frac{1}{2}(1+o(1))$ and $\tau=\left[\frac{\gamma a^*}{8\pi(2a^*-a)}\right]^{\frac{1}{2}}$ in (2.10), we then obtain that

$$\begin{aligned} e(\gamma,\alpha) &\le C\frac{\tau^{2-b}}{4a^*}\int_{\mathbb{R}^2}|x|^b Q^4(x)\,\mathrm{d}x+\frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2Q^2(x)\,\mathrm{d}x-\frac{\gamma\theta(1-\theta)}{4\pi}\ln\tau \\ &\quad+\frac{\gamma\theta(1-\theta)}{4\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\,\mathrm{d}x\mathrm{d}y \\ &\le \frac{\gamma}{32\pi}\ln\frac{8\pi(2a^*-a)}{\gamma a^*} \\ &\quad+\frac{\gamma}{16\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\,\mathrm{d}x\mathrm{d}y+C \text{ as } a\nearrow 2a^*. \end{aligned} \tag{3.50}$$

The same argument as (2.5) together with (3.5) yields that

$$\begin{aligned} &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_1^a(x)v_2^a(x)v_1^a(y)v_2^a(y)\,\mathrm{d}x\mathrm{d}y \\ &\to\frac{1}{(2a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\,\mathrm{d}x\mathrm{d}y \text{ as } a\nearrow 2a^*. \end{aligned} \tag{3.51}$$

Following (1.12) and (1.15), we then obtain from (3.5), (3.7) and (3.51) that

$$
\begin{aligned}
e(\gamma,a) &= \frac{1}{2\varepsilon_a^2}\int_{\mathbb{R}^2}\left(\left|\nabla v_1^a\right|^2+\left|\nabla v_2^a\right|^2\right)\mathrm{d}x \\
&\quad -\frac{a^*}{4\varepsilon_a^2}\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left(\left|v_1^a\right|^2+\left|v_2^a\right|^2\right)\right)^2\mathrm{d}x \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^2}\left|\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right|^2\left(\left|v_1^a\right|^2+\left|v_2^a\right|^2\right)\mathrm{d}x \\
&\quad +\frac{\gamma}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_1^a(x)v_2^a(x)v_1^a(y)v_2^a(y)\mathrm{d}x\mathrm{d}y \\
&\quad +\frac{\gamma}{4\pi}\ln\varepsilon_a\int_{\mathbb{R}^2}v_1^a(x)v_2^a(x)\mathrm{d}x\int_{\mathbb{R}^2}v_1^a(y)v_2^a(y)\mathrm{d}y \\
&\quad +\frac{a^*}{4\varepsilon_a^2}\int_{\mathbb{R}^2}\left(\sqrt{k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)}\left(\left|v_1^a\right|^2-\left|v_2^a\right|^2\right)\right)^2\mathrm{d}x \\
&\quad +\frac{2a^*-a}{2\varepsilon_a^2}\int_{\mathbb{R}^2}k\left(\varepsilon_a x+\varepsilon_a y_{\varepsilon_a}\right)\left|v_1^a\right|^2\left|v_2^a\right|^2\mathrm{d}x \\
&\geq \frac{\gamma}{16\pi\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y+o(1) \\
&\quad +\frac{\gamma}{16\pi}\left(1+o(1)\right)\ln\varepsilon_a+\frac{2a^*-a}{4a^*\varepsilon_a^2}\left(1+o(1)\right) \\
&\geq \frac{\gamma}{16\pi\left(a^*\right)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y+o(1) \\
&\quad +\frac{\gamma}{32\pi}\left[1+\ln\frac{8\pi\left(2a^*-a\right)}{\gamma a^*}\right]\left(1+o(1)\right),
\end{aligned}
\tag{3.52}
$$

where the identity in the last inequality is achieved at

$$
\varepsilon_a=\left[\frac{8\pi\left(2a^*-a\right)}{\gamma a^*}\right]^{\frac{1}{2}}\left(1+o(1)\right)\ \text{as } a\nearrow 2a^*. \tag{3.53}
$$

Hence it follows from (3.50) and (3.52) that

$$
e(\gamma,a)\approx\frac{\gamma}{32\pi}\ln\frac{8\pi\left(2a^*-a\right)}{\gamma a^*}+C\ \text{ as } a\nearrow 2a^*,
$$

and ε_a satisfies (3.53). We conclude from (3.38) and (3.53) that for $i=1,2$ and as $a\nearrow 2a^*$,

$$
\sqrt{\frac{8\pi\left(2a^*-a\right)}{\gamma a^*}}\,u_i^a\left(\sqrt{\frac{8\pi\left(2a^*-a\right)}{\gamma a^*}}x+x_i^a\right)\to\frac{Q(x)}{\sqrt{2a^*}}\ \text{ in } H\cap L^\infty\left(\mathbb{R}^2\right).
$$

This completes the proof of Theorem 1.2.

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Conflicts of Interest

The author declares no conflicts of interest.

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