Open Access Library Journal
2026, Volume 13, e14915
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Constraint Minimizers of the Gross-Pitaevskii Functional with Logarithmic Convolution and Ringed Shape Potential

Xuanxuan Li

School of Mathematics, Liaoning Normal University, Dalian, China
Email: xxli833@163.com

How to cite this paper: Li, X.X. (2026) Constraint Minimizers of the Gross-Pitaevskii Functional with Logarithmic Convolution and Ringed Shape Potential. *Open Access Library Journal*, **13**: e14915.
https://doi.org/10.4236/oalib.1114915

Received: January 22, 2026
Accepted: February 9, 2026
Published: February 12, 2026

Abstract

We consider a constrained variational problem where the energy functional includes a logarithmic convolution term and an external potential $(|x|-A)^2$. There is a threshold $a^*\in(0,\infty)$ that we establish existence and nonexistence results for constraint minimizers: for $a<a^*$, minimizers exist for any $\gamma\neq 0$; for $a>a^*$ with $\gamma\neq 0$ and $a=a^*$ with $\gamma>0$, no minimizer exists. Furthermore, for $\gamma>0$ and $a\nearrow a^*$, we analyze the limiting behavior of positive minimizers, showing that after suitable scaling, they converge to the standard ground state solution $Q(x)$ of $-\Delta u+u=u^3$ in $\mathbb{R}^2$. We also derive asymptotic estimates for the location of the maximum points of minimizers.

Subject Areas

Functional Analysis, Mathematical Analysis

Keywords

Gross-Pitaevskii Functional, Constraint Minimizers, Logarithmic Convolution, Ringed Shape Potential

1. Introduction and Main Results

The Schrödinger-Poisson system is a basic model in quantum mechanics that describes how charged particles interact through self-consistent electric fields. This system appears in many areas of physics, including semiconductor physics, quantum chemistry, and Bose-Einstein condensates. Since the landmark experiments on Bose-Einstein condensation (BEC) in ultracold atomic gases, a large number of bosonic atoms have been confined in a trapping potential and cooled to extremely low temperatures. Below the critical temperature, a macroscopic number of particles are observed to condense into the same single-particle quantum state. These Bose-Einstein condensates exhibit a variety of intriguing quantum phenomena, such as critical mass collapse and the emergence of quantum vortices in rotating traps. Specifically, if the interatomic interactions within the condensate are attractive, the system will collapse abruptly as soon as the particle number exceeds a critical threshold. We can find detailed information in [1]-[5].

In this paper, we consider the constraint minimizers of the following variational problem

$$e(\gamma,a) := \inf_{u\in S} E_{\gamma,a}(u), \tag{1.1}$$

the energy functional

$$\begin{aligned} E_{\gamma,a}(u) := &\frac{1}{2}\int_{\mathbb{R}^2}\left[\left|\nabla u(x)\right|^2 + \left(|x|-A\right)^2 u^2(x)\right]\mathrm{d}x \\ &+\frac{\gamma}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\,u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y - \frac{a}{4}\int_{\mathbb{R}^2}u^4(x)\,\mathrm{d}x, \end{aligned} \tag{1.2}$$

where $a>0$ describes the attractive interactions. We study a two-dimensional system with the external potential

$$V(x) = \left(|x|-A\right)^2.$$

$A>0$ is a fixed parameter. This potential has a ring shape, with its minimum value on the circle $|x| = A$. In recent years, such ring-shaped potentials have become important, because they can be created in laboratories using optical traps and magnetic fields. Physically, this potential can trap quantum particles to move mainly along a circular path, leading to interesting effects like persistent currents and vortex patterns. We can see [6]-[8].

From a mathematical perspective, the ring potential $V(x) = \left(|x|-A\right)^2$ poses several analytical challenges. First, it is non-convex, with its minimum attained on a closed curve rather than at an isolated point, resulting in a continuum of critical points that complicate standard minimization methods. Second, near its minimum, the potential exhibits a "Mexican hat" geometry, which requires new approaches to study concentration phenomena near critical parameters.

A further difficulty arises from combining the ring potential with the logarithmic convolution term. The broken symmetry of the potential interacts with the long-range nature of the logarithmic term, leading to a nontrivial variational structure. Proving existence of constrained minimizers demands careful estimates in weighted spaces and refined inequalities to balance the potential and nonlocal contributions. Moreover, the geometry of the ring potential complicates the precise location of concentration points, rendering classical symmetry arguments ineffective and calling for new analytical tools.

The logarithmic term derives from the fundamental solution of the Poisson equation in two dimensions and models electrostatic interactions in $\mathbb{R}^2$. Unlike the three-dimensional Coulomb potential, it is singular at both short and long distances, adding further mathematical complexity to the analysis.

The logarithmic term creates several mathematical problems that make this work different from previous studies. First, $E_{\gamma,a}$ is not well-defined on the standard Sobolev space $H^1(\mathbb{R}^2)$, because the logarithmic kernel decays slowly and has singular behavior. To solve the problem, drew inspiration from [9] [10], we denote the subspace

$$\mathcal{W} := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |x|^2 u^2(x)\,\mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|u\| := \left\{ \int_{\mathbb{R}^2} \left[|\nabla u|^2 + \left(1+|x|^2\right) u^2 \right] \mathrm{d}x \right\}^{\frac{1}{2}},\ u \in \mathcal{W},$$

the norm of $H^1(\mathbb{R}^2)$ is denoted by $\|u\|_1 := \left[\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2 \right) \mathrm{d}x \right]^{\frac{1}{2}}$ for any $u \in H^1(\mathbb{R}^2)$, and then define the constraint set

$$S := \left\{ u \in \mathcal{W} : |u|_2 = 1 \right\},$$

where and below we use $|\cdot|_q$ to denote the standard Lebegure norm on $L^q(\mathbb{R}^2)$ for $q \in [1,\infty)$. Recall from ([11], Lemma 3.1) that for any $p \in [2,\infty)$,

$$\mathcal{W} \text{ is compactly embedded into } L^p(\mathbb{R}^2). \tag{1.3}$$

We claim that $\int_{\mathbb{R}^2} (|x|-A)^2 u^2(x)\,\mathrm{d}x < \infty$ for any $u \in \mathcal{W}$. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^2} (|x|-A)^2 u^2(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} |x|^2 u^2(x)\,\mathrm{d}x - 2A\int_{\mathbb{R}^2} |x| u^2(x)\,\mathrm{d}x + A^2 \int_{\mathbb{R}^2} u^2(x)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^2} |x|^2 u^2(x)\,\mathrm{d}x - 2A\left(\int_{\mathbb{R}^2} |x|^2 u^2(x)\,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} u^2(x)\,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad + A^2 \int_{\mathbb{R}^2} u^2(x)\,\mathrm{d}x \\
&< \infty,
\end{aligned}$$

it means that we can work in $\mathcal{W}$.

Second, the logarithmic term is unbounded both at infinity and near the origin, requiring careful decomposition and estimation methods. To overcome these problems, we work in a weighted Sobolev space and split the logarithmic kernel into positive and negative parts. This approach lets us handle the negative part using the Hardy-Littlewood-Sobolev inequality while controlling the positive part through weighted estimates.

Inspired by [9] [10] [12], we perform that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u^2(x) u^2(y)\,\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+|x-y|\right) u^2(x) u^2(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+|x-y|^{-1}\right) u^2(x) u^2(y)\,\mathrm{d}x\mathrm{d}y \\
&:= I_1(u) - I_2(u).
\end{aligned} \tag{1.4}$$

Denote

$$|u|_* := \left(\int_{\mathbb{R}^2} |x|^2 u^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}},\ u \in \mathcal{W}.$$

Similar to [13] we then deduce that

$$I_1(u) \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (|x|+|y|) u^2(x) u^2(y)\,\mathrm{d}x\mathrm{d}y \le 2|u|_2^3 |u|_*,\ u \in \mathcal{W}. \tag{1.5}$$

In virtue of the Hardy-Littlewood-Sobolev inequality (cf. [14]):

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u(x)||v(y)|}{|x-y|}\,\mathrm{d}x\mathrm{d}y \le C|u|_{\frac{4}{3}} |v|_{\frac{4}{3}},\ u,v \in L^{\frac{4}{3}}(\mathbb{R}^2), \tag{1.6}$$

there exists a constant $C > 0$ such that

$$I_2(u) \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u^2(x)u^2(y)}{|x-y|}\,\mathrm{d}x\mathrm{d}y \le C|u|_{\frac{8}{3}}^4,\ u \in L^{\frac{8}{3}}(\mathbb{R}^2). \tag{1.7}$$

It follows from (1.4)-(1.7) that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u^2(x) u^2(y)\,\mathrm{d}x\mathrm{d}y$ is well defined on $\mathcal{W}$, and thus the energy functional $E_{\gamma,a}$ is well defined on $\mathcal{W}$.

If the external potential $V(x) = (|x| - A)^2$ in (1.2) is ignored, it was shown in [9] that $e(\gamma,a)$ possesses minimizers if $\gamma > 0$, $0 < a < a^*$, or $\gamma < 0$, $a \le 0$. Here $a^* := |Q|_2^2$, and $Q = Q(|x|) > 0$ is the unique positive radially symmetric solution of the following scalar field equation (cf. [15] [16]):

$$-\Delta u + u = u^3 \ \text{ in } \mathbb{R}^2,\ u \in H^1(\mathbb{R}^2). \tag{1.8}$$

Furthermore, Q decays exponentially as $|x| \to \infty$ in the sense that

$$|Q(x)|, |\nabla Q(x)| = O\left(|x|^{-\frac{1}{2}} \mathrm{e}^{-|x|}\right) \ \text{as } |x| \to \infty. \tag{1.9}$$

In [17], it has been proved that the above function Q is an achievement function of the equality in the following classical Gagliardo-Nirenberg inequality:

$$\int_{\mathbb{R}^2} u^4\,\mathrm{d}x \le \frac{2}{|Q|_2^2} \int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x \int_{\mathbb{R}^2} u^2\,\mathrm{d}x,\ u \in H^1(\mathbb{R}^2). \tag{1.10}$$

From (1.8) and (1.10) we can get that

$$\int_{\mathbb{R}^2} |\nabla Q|^2\,\mathrm{d}x = \int_{\mathbb{R}^2} Q^2\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^2} Q^4\,\mathrm{d}x. \tag{1.11}$$

By means of these results, we establish the existence and nonexistence of minimizers for (1.1).

Theorem 1.1. *Let* $Q(x) = Q(|x|)$ *be the unique positive solution of* (1.8) *and* $a^* := |Q|_2^2$. [(1)]

1) *If* $a < a^*$, *then there exists at least one minimizer of* $e(\gamma,a)$ *for* $\gamma \ne 0$;

2) *If* $a > a^*$, *then there is no minimizer of* $e(\gamma,a)$ *for* $\gamma \ne 0$;

3) *If* $a = a^*$, *then there is no minimizer of* $e(\gamma,a)$ *for* $\gamma > 0$.

Moreover, $\lim_{a \nearrow a^*} e(\gamma,a) = e(\gamma,a^*) = -\infty$ *for* $\gamma > 0$ *and* $e(\gamma,a) = -\infty$ *for*

$\gamma \neq 0$ *when* $a > a^*$.

The proof of Theorem 1.1 needs the Hardy-Littlewood-Sobolev inequality (1.6) and Gagliardo-Nirenberg inequality (1.10). However, because of $V(x) = (|x| - A)^2$, it is hard to get the lower bound of $E_{\gamma,a}$, furthermore, in the case of $\gamma < 0$, the weak lower semicontinuity of $E_{\gamma,a}$ is difficult to obtain. To solve the above-mentioned difficulties, we decomposed the potential term and bounded the logarithmic convolution term to prove that $E_{\gamma,a}$ is bounded from below. Subsequently, we proved the continuity of the logarithmic term to obtain the weak lower semicontinuity of $E_{\gamma,a}$.

The following result deals with the limiting behavior of minimizers of $e(\gamma, a)$ as $a \nearrow a^*$, where $\gamma > 0$ is fixed. We only consider positive minimizers of $e(\gamma, a)$ in the subsequent analysis.

Theorem 1.2. *Assume that* $u_{\gamma,a} \in S$ *is a positive minimizer of* $e(\gamma, a)$ *for* $\gamma > 0$ *and* $0 < a < a^*$. *Then we have*

$$\lim_{a \nearrow a^*} \left(\frac{8\pi(a^* - a)}{\gamma a^*} \right)^{\frac{1}{2}} u_{\gamma,a} \left(\left(\frac{8\pi(a^* - a)}{\gamma a^*} \right)^{\frac{1}{2}} x + x_{\gamma,a} \right) = \frac{Q(x)}{\sqrt{a^*}} \quad \text{in } \mathcal{W} \cap L^\infty(\mathbb{R}^2),$$

where $x_{\gamma,a}$ *is the unique maximum point of* $u_{\gamma,a}$ *as* $a \nearrow a^*$ *and there exists some constant* C_0 *satisfies*

$$\lim_{a \nearrow a^*} \frac{|x_{\gamma,a}| - A}{(a^* - a)^{\frac{1}{2}}} = C_0.$$

In establishing Theorem 1.2, we confront several additional challenges. First, the presence of the logarithmic convolution term complicates the analysis of the energy functional $E_{\gamma,a}$ given in (1.2). To address this, we employ a key estimate: there exists a constant $C > 0$, independent of $\gamma > 0$ and $0 < a < a^*$, satisfying

$$\int_{\mathbb{R}^2} \ln\left(1 + |x - y|^{-1}\right) v_{\gamma,a}^2(y) \mathrm{d}y \leq C \quad \text{for all } x \in \mathbb{R}^2,$$

where $v_{\gamma,a}$ denotes a rescaled version of the positive minimizer $u_{\gamma,a}$. Second, the scaled minimizer $v_{\gamma,a}$ lacks uniform boundedness as $a \nearrow a^*$ due to the translation-variance of the norm in $\mathcal{W}$. We resolve this by adapting techniques from [18]. Lastly, obtaining precise details about the maximum point $x_{\gamma,a}$ as $a \nearrow a^*$ requires novel approaches, because the term $\int_{\mathbb{R}^2} (|x| - A)^2 u_{\gamma,a}^2 \mathrm{d}x$ changes under translations and $\lim_{a \nearrow a^*} e(\gamma, a) = -\infty$ when $\gamma > 0$.

2. Existence and Nonexistence of Minimizers

In this section, we shall give the proof of Theorem 1.1 on the existence and nonexistence of minimizers for $e(\gamma, a)$.

Proof of Theorem 1.1. 1) We first prove the existence of minimizers for $e(\gamma, a)$ when $\gamma \neq 0$ and $a < a^*$. For any function $u \in H^1(\mathbb{R}^2)$ with $|u|_2 = 1$, we recall the following Gagliardo-Nirenberg inequality from [17]:

$$|u|_q \le \left(\frac{q}{2|Q_q|_2^{q-2}}\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right)^{\frac{q-2}{2q}}, \ q \ge 2, \tag{2.1}$$

where Q_q is the unique positive solution of the following elliptic equation

$$-\frac{q-2}{2}\Delta u + u = u^{q-1}, \ u \in H^1(\mathbb{R}^2).$$

Combining Equation (1.7) with inequality (2.1), we find that there exists a constant $C > 0$ such that

$$I_2(u) \le C\left(\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}, \ \forall u \in S. \tag{2.2}$$

In view of above facts, we now prove Theorem 1.1 (1) by the following four cases.

Case 1: $\gamma > 0$ and $0 < a < a^*$. By Young's Inequality, we have that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^2} |x| u^2(x) \mathrm{d}x \le \varepsilon \int_{\mathbb{R}^2} |x|^2 u^2(x) \mathrm{d}x + \frac{1}{4\varepsilon}\int_{\mathbb{R}^2} u^2(x) \mathrm{d}x, \tag{2.3}$$

taking $\varepsilon = \frac{1}{4A}$, then by (1.2), (1.4), (1.10), (2.2) and (2.3), we get that for $u \in S$,

$$E_{\gamma,a}(u) \ge \frac{1}{2}\left(1 - \frac{a}{a^*}\right)\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2} |x|^2 u^2 \mathrm{d}x - C\left(\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} - \frac{A^2}{2}, \tag{2.4}$$

this shows that $E_{\gamma,a}(u)$ is bounded below. Let $\{u_n\} \subset S$ be a minimizing sequence for $e(\gamma,a)$. From (2.4), we observe that both $\int_{\mathbb{R}^2} |\nabla u_n|^2 \mathrm{d}x$ and $\int_{\mathbb{R}^2} |x|^2 u_n^2 \mathrm{d}x$ is bounded independently of n. Combined with $|u_n|_2 = 1$, we conclude that u_n is uniformly bounded in $\mathcal{W}$. Due to (1.3), there exists a function $u \in \mathcal{W}$ such that

$$u_n \rightharpoonup u \text{ in } \mathcal{W} \text{ and } u_n \to u \text{ in } L^q(\mathbb{R}^2) \text{ for any } q \in [2,\infty) \text{ as } n \to \infty, \tag{2.5}$$

which means that $u \in S$. Moreover, it follows from ([12], Lemma 2.2) that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y \le \liminf_{n\to\infty} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u_n^2(x) u_n^2(y) \mathrm{d}x\mathrm{d}y.$$

By the weak lower semicontinuity, we have

$$\int_{\mathbb{R}^2} \left[|\nabla u|^2 + (|x| - A)^2 u^2\right] \mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathbb{R}^2} \left[|\nabla u_n|^2 + (|x| - A)^2 u_n^2\right] \mathrm{d}x.$$

We conclude from the above that

$$e(\gamma,a) \le E_{\gamma,a}(u) \le \liminf_{n\to\infty} E_{\gamma,a}(u_n) = e(\gamma,a),$$

which implies that $e(\gamma,a) = E_{\gamma,a}(u)$, and thus u is a minimizer of $e(\gamma,a)$.

Case 2: $\gamma > 0$ and $a \le 0$. By (1.2), (1.4), (2.2) and (2.3), we get that for $u \in S$,

$$E_{\gamma,a}(u) \ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2} |x|^2 u^2 \mathrm{d}x - C\left(\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} - \frac{A^2}{2},$$

it shows that $E_{\gamma,a}(u)$ is bounded below. Following the same approach as in Case

1, we find that $e(\gamma,a)$ has at least one minimizer for $\gamma>0$ and $a\le 0$.

Case 3: $\gamma<0$ and $0<a<a^*$. Applying Young's inequality, we deduce from (1.5) that for any $\epsilon>0$,

$$I_1(u)\le 2\epsilon|u|_*^2+\frac{1}{2\epsilon}|u|_2^6,\ u\in\mathcal{W}. \tag{2.6}$$

Taking $\epsilon=-\frac{\pi}{\gamma}$, by (1.2), (1.4), (1.10) and (2.6), we then obtain that for $u\in S$,

$$E_{\gamma,a}(u)\ge\frac{1}{2}\left(1-\frac{a}{a^*}\right)\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^2}(|x|-A)^2u^2\mathrm{d}x-\frac{1}{4}\int_{\mathbb{R}^2}|x|^2u^2\mathrm{d}x-\frac{\gamma^2}{16\pi^2},$$

by (2.3), taking $\varepsilon=\frac{1}{8A}$, then we get

$$E_{\gamma,a}(u)\ge\frac{1}{2}\left(1-\frac{a}{a^*}\right)\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x+\frac{1}{8}\int_{\mathbb{R}^2}|x|^2u^2\mathrm{d}x-\frac{3}{2}A^2-\frac{\gamma^2}{16\pi^2}, \tag{2.7}$$

which yields that $E_{\gamma,a}(u)$ is bounded from below. Similar to Case 1, one can choose a minimizing sequence $\{u_n\}\subset S$ and $\lim_{n\to\infty}E_{\gamma,a}(u_n)=e(\gamma,a)$, it follows that we obtain the equivalent conclusion (2.5).

Using the same argumentation method as in [13], we can obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y. \tag{2.8}$$

Then then following proof is similar to that of Case 1.

Case 4: $\gamma<0$ and $a\le 0$. By (1.2), (1.4) (1.5) and (2.3), we get that for $u\in S$,

$$E_{\gamma,a}(u)\ge\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x+\frac{1}{8}\int_{\mathbb{R}^2}|x|^2u^2\mathrm{d}x-\frac{3}{2}A^2-\frac{\gamma^2}{16\pi^2},$$

which means that $E_{\gamma,a}(u)$ is bounded from below. The following proof is similar to that of Case 3.

2) We next prove the nonexistence of minimizers of $e(\gamma,a)$ for $a>a^*$, $\gamma\ne 0$, or $a=a^*$, $\gamma>0$. Consider the test function

$$u_\tau(x):=\frac{\tau}{|Q|_2}Q(\tau(x-x_0)),\ \tau>0\text{ and }|x_0|=A.$$

Clearly, $u_\tau\in S$ for all $\tau>0$. It follows from (1.10) and (1.11) that for $a>a^*$, $\gamma\ne 0$, or $a=a^*$, $\gamma>0$,

$$\begin{aligned}
e(\gamma,a)&\le E_{\gamma,a}(u_\tau)\\
&=\frac{\tau^2}{2a^*}\int_{\mathbb{R}^2}|\nabla Q|^2\mathrm{d}x-\frac{a\tau^2}{4(a^*)^2}\int_{\mathbb{R}^2}Q^4\mathrm{d}x+\frac{1}{2a^*}\int_{\mathbb{R}^2}\left(\left|\frac{x}{\tau}+x_0\right|-|x_0|\right)^2Q^2\mathrm{d}x\\
&\quad-\frac{\gamma}{8\pi}\ln\tau+\frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\
&\le\frac{1}{2}\left(1-\frac{a}{a^*}\right)\tau^2+\frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2Q^2(x)\mathrm{d}x-\frac{\gamma}{8\pi}\ln\tau\\
&\quad+\frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\
&\to-\infty\ \text{ as }\tau\to\infty,
\end{aligned} \tag{2.9}$$

it means that $e(\gamma,a)$ has no minimizer when $a>a^*$ with $\gamma\neq 0$, or when $a=a^*$ with $\gamma>0$. Moreover, $e(\gamma,a)=-\infty$ for $a>a^*$ with $\gamma\neq 0$, and $e(\gamma,a^*)=-\infty$ for $\gamma>0$.

3) We now examine the limit $\lim_{a\nearrow a^*} e(\gamma,a)$. Assuming $0<a<a^*$, we set $\tau=(a^*-a)^{-\frac{1}{2}}$ in (2.9) and find that $\tau\to\infty$ as $a\nearrow a^*$. It implies that $\lim_{a\nearrow a^*} E_{\gamma,a}(u_\tau)=-\infty$ for $\gamma>0$. Therefore, $\lim_{a\nearrow a^*} e(\gamma,a)=-\infty$ when $\gamma>0$. This establishes the proof of Theorem 1.1. □

3. Limiting Behavior of Minimizers

In this section, we prove Theorem 1.2 on the limiting behavior of positive minimizers for $e(\gamma,a)$ as $a\nearrow a^*$, where $\gamma>0$ is fixed. If $u_{\gamma,a}$ is a minimizer of $e(\gamma,a)$ for $\gamma\neq 0$ and $a<a^*$, then the variational theory shows that $u_{\gamma,a}$ satisfies the following Euler-Lagrange equation:

$$-\Delta u_{\gamma,a}+\left(|x|-A\right)^2 u_{\gamma,a}+\frac{\gamma}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|u_{\gamma,a}^2(y)\mathrm{d}y u_{\gamma,a}=\lambda_{\gamma,a}u_{\gamma,a}+au_{\gamma,a}^3 \text{ in } \mathbb{R}^2, \quad (3.1)$$

where $\lambda_{\gamma,a}\in\mathbb{R}$ is the associated Lagrange multiplier and satisfies that

$$\lambda_{\gamma,a}=2e(\gamma,a)+\frac{\gamma}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_{\gamma,a}^2(x)u_{\gamma,a}^2(y)\mathrm{d}x\mathrm{d}y-\frac{a}{2}\int_{\mathbb{R}^2}u_{\gamma,a}^4\mathrm{d}x. \quad (3.2)$$

We first establish some estimates for positive minimizers.

Lemma 3.1. *Let* $u_{\gamma,a}$ *be a positive minimizer of* $e(\gamma,a)$ *for* $\gamma>0$ *and* $a\in(0,a^*)$, *define*

$$\varepsilon_{\gamma,a}:=\left(\int_{\mathbb{R}^2}|\nabla u_{\gamma,a}|^2\mathrm{d}x\right)^{-\frac{1}{2}} \text{ and } v_{\gamma,a}(x):=\varepsilon_{\gamma,a}u_{\gamma,a}\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right) \text{ in } \mathbb{R}^2, \quad (3.3)$$

where $x_{\gamma,a}$ *is a global maximum point of* $u_{\gamma,a}$. *Then we have*

1) $\varepsilon_{\gamma,a}>0$ *satisfies that for any* $\gamma>0$,

$$\varepsilon_{\gamma,a}\to 0 \text{ and } \lambda_{\gamma,a}\varepsilon_{\gamma,a}^2\to -1 \text{ as } a\nearrow a^*; \quad (3.4)$$

2) *There exists a constant* $\alpha>0$, *independent of* $\gamma>0$ *and* $a\in(0,a^*)$, *such that*

$$\int_{B_2(0)}v_{\gamma,a}^2(x)\mathrm{d}x\geq\alpha>0 \text{ as } a\nearrow a^*; \quad (3.5)$$

3) $v_{\gamma,a}>0$ *satisfies that for any* $\gamma>0$,

$$v_{\gamma,a}(x)\to v_0(x):=\frac{Q(|x|)}{\sqrt{a^*}} \text{ in } H^1(\mathbb{R}^2) \text{ as } a\nearrow a^*, \quad (3.6)$$

where $Q(x)>0$ *is the unique positive solution of* (1.8).

We can draw on the proof method of Lemma 3.1 in [13], and will not elaborate on it in detail here.

Lemma 3.2. *Let* $v_{\gamma,a}(x)$ *be given by* (3.3), *and* $x_{\gamma,a}$ *is a global maximum point of* $u_{\gamma,a}$. *Then we have* [(1)]

1) *There exists a large constant* $R>0$ *such that for any* $\gamma>0$,

$$\left|v_{\gamma,a}(x)\right| \le Ce^{-\frac{2}{3}|x|} \quad \textit{and} \quad \left|\nabla v_{\gamma,a}(x)\right| \le Ce^{-\frac{1}{2}|x|} \quad \textit{uniformly for } |x| \ge R \quad \textit{as } a \nearrow a^*, \tag{3.7}$$

where the constant $C > 0$ *is independent of* $a \in (0, a^*)$;

2) *The global maximum point* $x_{\gamma,a}$ *of* $u_{\gamma,a}$ *is unique and satisfies that for any* $\gamma > 0$, *there exists a constant* $C_0 > 0$ *independent of* $\gamma > 0$ *and* $0 < a < a^*$ *such that*

$$\left|x_{\gamma,a}\right| \to A, \ \frac{\left|x_{\gamma,a}\right| - A}{\left(a^* - a\right)^{\frac{1}{2}}} \to C_0 \text{ as } a \nearrow a^*. \tag{3.8}$$

Proof. Similar to [13], we can obtain (3.7). Now we prove (3.8). Due to (1.2) (1.10) and (3.3), that

$$\begin{aligned} e(\gamma,a) &= E_{\gamma,a}\left(u_{\gamma,a}\right) \\ &\ge \frac{\gamma}{8\pi}\ln\varepsilon_{\gamma,a} + \frac{\gamma}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(x)v_{\gamma,a}^2(y)\mathrm{d}x\mathrm{d}y \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left(\left|\varepsilon_{\gamma,a}x + x_{\gamma,a}\right| - A\right)^2 v_{\gamma,a}^2(x)\mathrm{d}x. \end{aligned} \tag{3.9}$$

We claim that $\left\{\frac{\left|x_{\gamma,a}\right| - A}{\varepsilon_{\gamma,a}}\right\} \subset \mathbb{R}$ is uniformly bounded as $a \nearrow a^*$. Otherwise, there exists a sequence of $\{a\}$, denoted by $\{a_k\}$, such that $\left|\frac{\left|x_{\gamma,a_k}\right| - A}{\varepsilon_{\gamma,a_k}}\right| \to \infty$ as $k \to \infty$, it means that for any constant $C > 0$ and $\xi \in \mathbb{R}^2$,

$$\begin{aligned} &\lim_{k\to\infty}\varepsilon_{\gamma,a_k}^{-2}\int_{\mathbb{R}^2}\left(\left|\varepsilon_{\gamma,a_k}x + x_{\gamma,a_k}\right| - A\right)^2 v_{\gamma,a_k}^2(x)\mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^2}\left(\frac{\left|\varepsilon_{\gamma,a_k}x + x_{\gamma,a_k}\right|}{\varepsilon_{\gamma,a_k}} - \frac{A}{\varepsilon_{\gamma,a_k}}\right)^2 v_{\gamma,a_k}^2(x)\mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^2}\left(\frac{\left|\varepsilon_{\gamma,a_k}x + x_{\gamma,a_k}\right| - \left|x_{\gamma,a_k}\right|}{\varepsilon_{\gamma,a_k}} + \frac{\left|x_{\gamma,a_k}\right| - A}{\varepsilon_{\gamma,a_k}}\right)^2 v_{\gamma,a_k}^2(x)\mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^2}\left(\frac{\varepsilon_{\gamma,a_k}\xi + x_{\gamma,a_k}}{\left|\varepsilon_{\gamma,a_k}\xi + x_{\gamma,a_k}\right|}x + \frac{\left|x_{\gamma,a_k}\right| - A}{\varepsilon_{\gamma,a_k}}\right)^2 v_{\gamma,a_k}^2(x)\mathrm{d}x \ge C, \end{aligned}$$

This estimate and (3.9) imply that

$$e(\gamma,a_k) \ge \frac{\gamma}{8\pi}\ln\varepsilon_{\gamma,a_k} + \frac{\gamma}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a_k}^2(x)v_{\gamma,a_k}^2(y)\mathrm{d}x\mathrm{d}y + C\varepsilon_{\gamma,a_k}^2 \tag{3.10}$$

holds for any $C > 0$, and $e(\gamma,a_k) \to +\infty$ as $C \to +\infty$, which contradicts with Theorem 1.1. So, the above claim is proved. Then, we obtain that there exists a constant $C_0 > 0$ independent of $\gamma > 0$ and $0 < a < a^*$ such that

$$\frac{\left|x_{\gamma,a}\right| - A}{\left(a^* - a\right)^{\frac{1}{2}}} \to C_0 \text{ as } a \nearrow a^*,$$

it further implies that for any $\gamma > 0$, $\left|x_{\gamma,a}\right| \to A$ as $a \nearrow a^*$.

We finally prove the uniqueness of the global maximum point $x_{\gamma,a}$ of $u_{\gamma,a}$ as $a \nearrow a^*$. Due to (3.1) and (3.3), we obtain that $v_{\gamma,a}$ satisfies

$$\begin{aligned}&-\Delta v_{\gamma,a}+\varepsilon_{\gamma,a}^2\left(\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|-A\right)^2 v_{\gamma,a}+\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(y)\mathrm{d}y v_{\gamma,a}\\&=\lambda_{\gamma,a}\varepsilon_{\gamma,a}^2 v_{\gamma,a}+av_{\gamma,a}^3-\frac{\gamma}{2\pi}\varepsilon_{\gamma,a}^2\ln\varepsilon_{\gamma,a}v_{\gamma,a}\ \text{ in } \mathbb{R}^2.\end{aligned} \tag{3.11}$$

Denote

$$\begin{aligned}K_{\gamma,a}(x):=&-\varepsilon_{\gamma,a}^2\left(\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|-A\right)^2 v_{\gamma,a}-\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(y)\mathrm{d}y v_{\gamma,a}\\&+\lambda_{\gamma,a}\varepsilon_{\gamma,a}^2 v_{\gamma,a}+av_{\gamma,a}^3-\frac{\gamma}{2\pi}\varepsilon_{\gamma,a}^2\ln\varepsilon_{\gamma,a}v_{\gamma,a}\ \text{ in } \mathbb{R}^2.\end{aligned}$$

So we get

$$-\Delta v_{\gamma,a}=K_{\gamma,a}(x)\ \text{ in } \mathbb{R}^2. \tag{3.12}$$

We deduce from (3.3) that

$$\left\|v_{\gamma,a}\right\|_1^2=\varepsilon_{\gamma,a}^2\int_{\mathbb{R}^2}\left|\nabla u_{\gamma,a}\right|^2\mathrm{d}x+\int_{\mathbb{R}^2}u_{\gamma,a}^2\mathrm{d}x=2, \tag{3.13}$$

it means that $\{v_{\gamma,a}\}$ is bounded uniformly in $L^q(\mathbb{R}^2)$ for $q\in[2,\infty)$, and $K_{\gamma,a}(x)$ is bounded uniformly in $L_{\text{loc}}^q(\mathbb{R}^2)$. Employing the L^p-estimate (refer to[19], Theorem 9.11) on (3.12), we establish that $\{v_{\gamma,a}\}$ is uniformly bounded in $W_{\text{loc}}^{2,q}(\mathbb{R}^2)$ as $a\nearrow a^*$. Through the standard Sobolev embedding theorem, we further deduce

$$\{v_{\gamma,a}\}\text{ maintains uniform boundedness in } C_{\text{loc}}^{1,\mu}(\mathbb{R}^2)\text{ for certain } \mu\in(0,1) \text{ as } a\nearrow a^*. \tag{3.14}$$

According to ([12], Proposition 2.3), the expression $\varepsilon_{\gamma,a}^2\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(y)\mathrm{d}y$ belongs to $C_{\text{loc}}^{2,\mu}(\mathbb{R}^2)$ when $a\nearrow a^*$. Consequently, $\{K_{\gamma,a}\}$ shows uniform boundedness in $C^\mu(B_r(0))$ for large $r>0$ during this limit. Using the Schauder estimate (see ([19], Theorem 6.2)) on Equation (3.12), we determine that $\{v_{\gamma,a}\}$ is uniformly bounded in $C^{2,\mu}(B_r(0))$ as $a\nearrow a^*$. This implies that there exists $\tilde{v}_0\in C^2(B_r(0))$ such that, up to a subsequence, $v_{\gamma,a}$ converges to $\tilde{v}_0$ in $C^2(B_r(0))$ as $a\nearrow a^*$. From Equation (3.6), we have $v_0=\tilde{v}_0$, which gives

$$v_{\gamma,a}(x)\to v_0(x)=\frac{Q(|x|)}{\sqrt{a^*}}\ \text{ in } C^2(B_r(0))\ \text{ as } a\nearrow a^*, \tag{3.15}$$

for sufficiently large $r>0$.

Since the origin 0 is the unique global maximum point of $Q(x)$, (3.15) demonstrates that all local maxima of $v_{\gamma,a}$ must converge toward 0 and remain confined within $B_\delta(0)$ for some small $\delta>0$ as $a\nearrow a^*$. The condition $Q''(0)<0$ ensures $Q''(t)<0$ holds for $t\in[0,\delta)$. Following ([20], Lemma 4.2), we establish that as $a\nearrow a^*$, each $v_{\gamma,a}$ possesses precisely one maximum point located at 0. Thus, the global maximum $x_{\gamma,a}$ of $u_{\gamma,a}$ becomes unique as $a\nearrow a^*$, thereby concluding the proof of Lemma 3.2. □

Now we prove the refined limiting behavior of positive minimizers of $e(\gamma,a)$ in $\mathcal{H}\cap L^\infty(\mathbb{R}^2)$ for $\gamma>0$ as $a\nearrow a^*$.

Proof of Theorem 1.2. In view of Lemma 3.1 and Lemma 3.2, we only need to prove that

$$v_{\gamma,a}(x)\to\frac{Q(|x|)}{\sqrt{a^*}}\ \text{in } \mathcal{W}\cap L^\infty(\mathbb{R}^2)\ \text{as } a\nearrow a^*, \tag{3.16}$$

and

$$\varepsilon_{\gamma,a}=\left[\frac{8\pi(a^*-a)}{\gamma a^*}\right]^{\frac{1}{2}}(1+o(1)), \tag{3.17}$$

where $v_{\gamma,a}$ is defined by (3.3) and $x_{\gamma,a}$ is the unique maximum point of the positive minimizer $u_{\gamma,a}$.

We can obtain (3.16) by using the same argumentation method as that in [13]. Next, we prove (3.16) as follows. We first give the upper estimate of $e(\gamma,a)$ as $a\nearrow a^*$. Setting $\tau=\left[\frac{\gamma a^*}{8\pi(a^*-a)}\right]^{\frac{1}{2}}$ in (2.9), we obtain that

$$\begin{aligned} e(\gamma,a)\le{}&\frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\ &+\frac{\gamma}{16\pi}\left[\ln\frac{8\pi(a^*-a)}{\gamma a^*}+1\right]+o(1)\ \text{as } a\nearrow a^*. \end{aligned} \tag{3.18}$$

We now give the lower estimate of $e(\gamma,a)$ as $a\nearrow a^*$. From [13] we have that

$$\begin{aligned} &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(x)v_{\gamma,a}^2(y)\mathrm{d}x\mathrm{d}y\\ &\to\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\frac{Q^2(x)}{a^*}\frac{Q^2(y)}{a^*}\mathrm{d}x\mathrm{d}y\ \text{as } a\nearrow a^*. \end{aligned} \tag{3.19}$$

We infer from (1.2), (1.10), (1.11), (3.16) and (3.19) that

$$\begin{aligned} e(\gamma,a)&=E_{\gamma,a}(u_{\gamma,a})\\ &=\frac{\varepsilon_{\gamma,a}^{-2}}{2}\int_{\mathbb{R}^2}|\nabla v_{\gamma,a}|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^2}\left(|\varepsilon_{\gamma,a}x+x_{\gamma,a}|-A\right)^2v_{\gamma,a}^2(x)\mathrm{d}x+\frac{\gamma}{8\pi}\ln\varepsilon_{\gamma,a}\\ &\quad+\frac{\gamma}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(x)v_{\gamma,a}^2(y)\mathrm{d}x\mathrm{d}y-\frac{a\varepsilon_{\gamma,a}^{-2}}{4}\int_{\mathbb{R}^2}v_{\gamma,a}^4\mathrm{d}x\\ &\ge\frac{\gamma}{8\pi}\ln\varepsilon_{\gamma,a}+\frac{a^*-a}{2a^*}(1+o(1))\varepsilon_{\gamma,a}^{-2}\\ &\quad+\frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y+o(1)\\ &\ge\frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\ &\quad+\frac{\gamma}{16\pi}\left[\ln\frac{8\pi(a^*-a)}{\gamma a^*}+1\right]+o(1)\ \text{as } a\nearrow a^*, \end{aligned} \tag{3.20}$$

the equal sign can be obtained when $\varepsilon_{\gamma,a}$ takes the following value,

$$\varepsilon_{\gamma,a}=\left(\frac{8\pi\left(a^{*}-a\right)}{\gamma a^{*}}\right)^{\frac{1}{2}}\left(1+o(1)\right) \text{ as } a\nearrow a^{*}, \tag{3.21}$$

so we get a more precise estimate of $\varepsilon_{\gamma,a}$ as $a\nearrow a^{*}$.

Combining (3.18) with (3.9), we conclude that as $a\nearrow a^{*}$,

$$e(\gamma,a)\approx\frac{\gamma}{16\pi}\left[\ln\frac{8\pi\left(a^{*}-a\right)}{\gamma a^{*}}+1\right]$$
$$+\frac{\gamma}{8\pi\left(a^{*}\right)^{2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y+o(1),$$

and $\varepsilon_{\gamma,a}>0$ satisfies (3.21). Moreover, we derive from (3.16) and (3.21) that

$$\lim_{a\nearrow a^{*}}\left(\frac{8\pi\left(a^{*}-a\right)}{\gamma a^{*}}\right)^{\frac{1}{2}}u_{\gamma,a}\left(\left(\frac{8\pi\left(a^{*}-a\right)}{\gamma a^{*}}\right)^{\frac{1}{2}}x+x_{\gamma,a}\right)=\frac{Q(x)}{\sqrt{a^{*}}} \text{ in } \mathcal{W}\cap L^{\infty}\left(\mathbb{R}^2\right).$$

This completes the proof of Theorem 1.2. □

4. Conclusion

In conclusion, we have completely classified the existence and non-existence of constraint minimizers for a Gross-Pitaevskii functional with ring potential and logarithmic nonlocality. The critical parameter a^{*} is identified, and the blow-up profile of minimizers as $a\nearrow a^{*}$ is rigorously derived. This work extends previous studies on local interaction models to the nonlocal logarithmic case under a ring-shaped constraint, revealing new concentration phenomena and offering a basis for further analytical and numerical investigations.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Anderson, M.H., Ensher, J.R., Matthews, M.R., Wieman, C.E. and Cornell, E.A. (1995) Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor. *Science*, **269**, 198-201. https://doi.org/10.1126/science.269.5221.198

[2] Dalfovo, F., Giorgini, S., Pitaevskii, L.P. and Stringari, S. (1999) Theory of Bose-Einstein Condensation in Trapped Gases. *Reviews of Modern Physics*, **71**, 463-512. https://doi.org/10.1103/revmodphys.71.463

[3] Sackett, C.A., Stoof, H.T.C. and Hulet, R.G. (1998) Growth and Collapse of a Bose-Einstein Condensate with Attractive Interactions. *Physical Review Letters*, **80**, 2031-2034. https://doi.org/10.1103/physrevlett.80.2031

[4] Bradley, C.C., Sackett, C.A. and Hulet, R.G. (1997) Bose-Einstein Condensation of Lithium: Observation of Limited Condensate Number. *Physical Review Letters*, **78**, 985-989. https://doi.org/10.1103/physrevlett.78.985

[5] Huepe, C., Métens, S., Dewel, G., Borckmans, P. and Brachet, M.E. (1999) Decay Rates in Attractive Bose-Einstein Condensates. *Physical Review Letters*, **82**, 1616-1619. https://doi.org/10.1103/physrevlett.82.1616

[6] Rendön, P.L. and Ley-Koo, E. (2015) Complete Sets of Circular, Elliptic and Bipolar Harmonic Vortices on a Plane. *Revista Mexicana de Física*, **61**, 120-129.

[7] Ryu, C., Andersen, M., Cladé, P., Natarajan, V., Helmerson, K. and Phillips, W. (2007) Observation of Persistent Flow of a Bose-Einstein Condensate in a Toroidal Trap. *Physical Review Letters*, **99**, Article ID: 260401. https://doi.org/10.1103/physrevlett.99.260401

[8] Halkyard, P.L. (2010) Dynamics in Cold Atomic Gases: Resonant Behaviour of the Quantum Delta-Kicked Accelerator and Bose-Einstein Condensates in Ring Traps. Ph.D. Thesis, Durham University.

[9] Cingolani, S. and Jeanjean, L. (2019) Stationary Waves with Prescribed L^2-Norm for the Planar Schrödinger—Poisson System. *SIAM Journal on Mathematical Analysis*, **51**, 3533-3568. https://doi.org/10.1137/19m1243907

[10] Stubbe, J. (2008) Bound States of Two-Dimensional Schrödinger-Newton Equations.

[11] Zhang, J. (2000) Stability of Standing Waves for Nonlinear Schrödinger Equations with Unbounded Potentials. *Zeitschrift für angewandte Mathematik und Physik*, **51**, 498-503. https://doi.org/10.1007/pl00001512

[12] Cingolani, S. and Weth, T. (2016) On the Planar Schrödinger-Poisson System. *Annales de l Institut Henri Poincaré C, Analyse non Linéaire*, **33**, 169-197. https://doi.org/10.1016/j.anihpc.2014.09.008

[13] Liu, M. and Zhang, S. (2025) Normalized Solutions for the L^2-Critical Schrödinger-Poisson System in $\mathbb{R}^2$. *Annals of Functional Analysis*, **16**, 33-97. https://doi.org/10.1007/s43034-025-00447-z

[14] Lieb, E.H. and Loss, M. (2001) Analysis, Graduate Studies in Mathematics. Vol. 14, 2nd Edition, American Mathematical Society.

[15] Gidas, B., Ni, W.-M. and Nirenberg, L. (1981) Symmetry of Positive Solutions of Nonlinear Elliptic Equations in $\mathbb{R}^N$. *Advances in Mathematics. Supplementary Studies*, **7**, 369-402.

[16] Kwong, M.K. (1989) Uniqueness of Positive Solutions of $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^N$. *Archive for Rational Mechanics and Analysis*, **105**, 243-266. https://doi.org/10.1007/bf00251502

[17] Weinstein, M.I. (1983) Nonlinear Schrödinger Equations and Sharp Interpolation Estimates. *Communications in Mathematical Physics*, **87**, 567-576. https://doi.org/10.1007/bf01208265

[18] Guo, Y., Luo, Y. and Yang, W. (2020) The Nonexistence of Vortices for Rotating Bose-Einstein Condensates with Attractive Interactions. *Archive for Rational Mechanics and Analysis*, **238**, 1231-1281. https://doi.org/10.1007/s00205-020-01564-w

[19] Gilbarg, D. and Trudinger, N.S. (1997) Elliptic Partial Differential Equations of Second Order. Springer.

[20] Ni, W.-M. and Takagi, I. (1991) On the Shape of Least-Energy Solutions to a Semilinear Neumann Problem. *Communications on Pure and Applied Mathematics*, **44**, 819-851. https://doi.org/10.1002/cpa.3160440705