Open Access Library Journal
2026, Volume 13, e14782
ISSN Online: 2333-9721
ISSN Print: 2333-9705

The de Rham Cohomology of Compact Lie Groups

Yingling Liu, Yu Wang

College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, China
Email: 18982859249@163.com

How to cite this paper: Liu, Y.L. and Wang, Y. (2026) The de Rham Cohomology of Compact Lie Groups. *Open Access Library Journal*, **13**: e14782.
https://doi.org/10.4236/oalib.1114782

Received: December 19, 2025
Accepted: January 25, 2026
Published: January 28, 2026

Abstract

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. We prove that the de Rham cohomology $H^{\bullet}_{\mathrm{dR}}(G)$ is isomorphic to the Lie algebra cohomology $H^{\bullet}(\mathfrak{g},\mathbb{R})$, and that both are isomorphic to the space $\left(\left(\Lambda^{\bullet}\mathfrak{g}\right)^{*}\right)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant forms on the exterior algebra. In other words,

$$H^{\bullet}(\mathfrak{g},\mathbb{R}) \cong H^{\bullet}_{\mathrm{dR}}(G) \cong \left(\left(\Lambda^{\bullet}\mathfrak{g}\right)^{*}\right)^{\mathfrak{g}}.$$

This result reduces the geometric problem of computing $H^{\bullet}_{\mathrm{dR}}(G)$ to the algebraic problem. We use the above isomorphism to calculate the de Rham cohomology of several classical groups $SO(3)$, $SO(4)$, $SO(5)$, $SO(6)$, and $SU(3)$. We provide detailed computations for $SO(3)$. For the higher-dimensional cases $SO(4)$, $SO(5)$, $SO(6)$, and $SU(3)$, we utilize MATLAB to compute the de Rham cohomology groups efficiently.

Subject Areas

Lie Groups and Lie Algebras, Homological Algebra

Keywords

Lie Group, Lie Algebra, De Rham Cohomology, Lie Algebra Cohomology, Invariant Differential Forms

1. Introduction

The cohomology of Lie groups and Lie algebras constitutes a fundamental topic in differential geometry and algebraic topology, with profound connections to representation theory and mathematical physics. For a compact connected Lie group G with Lie algebra $\mathfrak{g}$, the cohomology of G is defined as the de Rham cohomology arising from the complex of differential forms on G. We build on the work of Chevalley and Eilenberg [1], who introduced Lie algebra cohomology to compute de Rham cohomology, thereby translating a geometric problem into an algebraic one. Specifically, when $\mathfrak{g}$ acts trivially on $\mathbb{R}$, there exists an isomorphism:

$$H^{\bullet}(\mathfrak{g},\mathbb{R}) \cong H^{\bullet}_{\mathrm{dR}}(G) \cong \left(\left(\Lambda^{\bullet}\mathfrak{g}\right)^{*}\right)^{\mathfrak{g}},$$

where $\left(\left(\Lambda^{\bullet}\mathfrak{g}\right)^{*}\right)^{\mathfrak{g}}$ denotes the space of $\mathfrak{g}$-invariant elements. This isomorphism provides a powerful tool for computing the de Rham cohomology of G by leveraging the algebraic structure of $\mathfrak{g}$. While the theory is well-known, computations become increasingly complex as the dimension grows for Lie groups. Existing literature often focuses on lower-dimensional cases (e.g., $SO(3)$), but systematic computations remain challenge for classical groups of higher dimensions. This paper provides a rigorous exposition of the isomorphism and presents detailed cohomology calculations for specific classical Lie groups, namely $SO(4)$, $SO(5)$, $SO(6)$, and $SU(3)$.

This paper is organized as follows: In Sect. 2, we recall basic concepts about the de Rham cohomology and the Lie algebra cohomology. In Sect. 3, we provide a detailed proof of the isomorphism. In Sect. 4, we compute specific examples.

2. Preliminary Knowledge

This section recalls the definitions and fundamental properties of the de Rham cohomology and the Lie algebra cohomology.

Definition 1 [2] *Let* K *be a field. A Lie algebra* $\mathfrak{g}$ *is a vector space over* K *with a bilinear bracket* $[-,-]$:

$$\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},$$

satisfying the following axioms for all $X,Y,Z\in\mathfrak{g}$ and $\lambda_1,\lambda_2\in K$:

1) Bilinearity: $[\lambda_1 X+\lambda_2 Y,Z]=\lambda_1[X,Z]+\lambda_2[Y,Z]$;

2) Antisymmetry: $[X,X]=0$;

3) Jacobi identity: $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$.

Definition 2 [2] *Let* K *be a field and* A *be a unital ring. If the additive group of* A *forms a* K*-vector space, and*

$$\lambda(ab)=(\lambda a)b=a(\lambda b)$$

for all $\lambda\in K$, and $a,b\in A$, then A is called an associative algebra over K, or simply a K-algebra.

Let A be an associative algebra over a field K. The commutator bracket $[a,b]=ab-ba$ defines a Lie algebra structure on the underlying K-vector space of A.

Definition 3 [3] *Let* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be Lie algebras over a field* K. *A* K*-linear map* $\mathscr{A}:\mathfrak{g}_1\to\mathfrak{g}_2$ *is a Lie algebra homomorphism if for all* $X,Y\in\mathfrak{g}_1$

$$[\mathscr{A}(X),\mathscr{A}(Y)]=\mathscr{A}([X,Y]).$$

A canonical example is a homomorphism from a Lie algebra $\mathfrak{g}$ to $\mathrm{gl}(V)$, the general linear Lie algebra on a vector space V.

Definition 4 [3] *A representation of a Lie algebra* $\mathfrak{g}$ *over a field* K *is a pair* (V,ρ), *where* V *is a* K*-vector space and* $\rho:\mathfrak{g}\to gl(V)$ *is a Lie algebra homomorphism.*

Definition 5 [3] *For any Lie algebra* $\mathfrak{g}$, *the adjoint representation is the homomorphism* $ad:\mathfrak{g}\to gl(\mathfrak{g})$, *defined by* $ad(X)(Y)=[X,Y]$.

The adjoint representation is fundamental for studying the structure of Lie algebra. To connect this algebraic framework to the differential geometry of Lie groups, we recall basic concepts from smooth manifold theory.

Definition 6 [4] *Let* M *be an* n*-dimensional topological manifold. If a smooth structure* Σ *is specified on* M, *then* (M,Σ) *is called an* n*-dimensional smooth manifold.*

Definition 7 [4] *Let* M *be a smooth manifold. A function* $f:M\to\mathbb{R}$ *is called smooth if it is smooth with respect to the smooth structure of* M. *The set of all smooth functions on* M *is denoted by* $C^\infty(M)$.

Definition 8 [5] *Let* M *be a smooth manifold and* $p\in M$. *Denote by* C_p^∞ *the algebra of germs of smooth functions at* p. *A tangent vector at* p *is a linear map* $v:C_p^\infty\to\mathbb{R}$ *satisfying the following axioms: for all* $f,g\in C_p^\infty$ *and* $\lambda\in\mathbb{R}$,

1) $v(f+\lambda g)=v(f)+\lambda v(g)$;
2) $v(f\cdot g)=v(f)g(p)+f(p)v(g)$.

The tangent space at p, denoted T_pM, is the vector space of all tangent vectors.

Definition 9 [5] *A cotangent vector at* $p\in M$ *is a linear functional* $\alpha:T_pM\to\mathbb{R}$. *The cotangent space* T_p^*M *is the dual space of* T_pM.

Definition 10 [5] *Let* M *be a smooth manifold. The tangent bundle of* M *is defined as* $TM=\bigcup_{p\in M}T_pM$, *equipped with a natural smooth structure that makes it a smooth manifold.*

Definition 11 [6] *A smooth vector field on a smooth manifold* M *is a smooth map* $X:M\to TM$ *such that* $\pi\circ X=id_M$, *where* $\pi:TM\to M$ *is the canonical projection. In other words,* X *is a smooth section of the tangent bundle.*

The set of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

Definition 12 [7] *Let* M *be a smooth manifold. A differential* k*-form on* M *is a smooth section of the* k*-th exterior power of the cotangent bundle, i.e., a smooth map*

$$\omega:M\to\Lambda^kT^*M=\bigcup_{p\in M}\Lambda^k\left(T_p^*M\right),$$

such that $\omega(p)\in\Lambda^k\left(T_p^*M\right)$ for each $p\in M$. The set of all differential k-forms on M is denoted by $\Omega^k(M)$.

Property 1 [6] *Let* M *be a smooth manifold. There exists a unique operator* $d:\Omega^r(M)\to\Omega^{r+1}(M)(r\geq 0)$, *called the exterior derivative, satisfying the following properties:*

1) $d:\Omega^r(M)\to\Omega^{r+1}(M)$ is a linear map.

2) $d(\varphi\wedge\psi)=d\varphi\wedge\psi+(-1)^r\varphi\wedge d\psi$, $\varphi\in\Omega^r(M)$, $\psi\in\Omega^s(M)$.

3) For $f\in C^\infty(M)=\Omega^0(M)$, df is the ordinary differential of f.

4) $d\circ d=0$.

Proposition 1 [6] *The space* $\Omega^k(M)$ *of differential* k*-forms on a smooth manifold* M *is isomorphic as a* $C^\infty(M)$*-module to the space of alternating* $C^\infty(M)$*-multilinear maps* $\mathfrak{X}(M)^k\to C^\infty(M)$.

Proposition 2 [8] (*Invariant formula*) *For any* $\omega\in\Omega^k(M)$ *and* $X_0,\cdots,X_k\in\mathfrak{X}(M)$,

$$d\omega(X_0,\cdots,X_k)=\sum_{i=0}^{k}(-1)^i X_i\left(\omega\left(X_0,\cdots,\hat{X}_i,\cdots,X_k\right)\right)+\sum_{0\le i<j\le k}(-1)^{i+j}\omega\left(\left[X_i,X_j\right],X_0,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_k\right),$$

where $\hat{X}_i$ denotes that the element X_i is omitted.

Definition 13 [6] *Let* M *and* N *be smooth manifolds and* $f:M\to N$ *a smooth map. For each* $p\in M$, *the pushforward of* f *at* p *is the linear map* $f_{*p}:T_pM\to T_{f(p)}N$ *defined by*

$$\left(f_{*p}(X)\right)(g)=X(g\circ f)$$

for all $X\in T_pM$ and $g\in C^\infty_{f(p)}$.

Definition 14 [6] *Let* $f:M\to N$ *be a smooth map. The pullback induced by* f *is the map* $f^*:\Omega^k(N)\to\Omega^k(M)$ *defined by*

$$f^*(\omega)(X_1,\cdots,X_k)=\omega\left(f_*(X_1),\cdots,f_*(X_k)\right)$$

for all $\omega\in\Omega^k(N)$ and $X_1,\cdots,X_k\in\mathfrak{X}(M)$.

Property 2 [6] *Let* $f:M\to N$ *be a smooth map. The pullback* $f^*:\Omega^k(N)\to\Omega^k(M)$ *satisfies the following properties:*

1) f^* is a linear map.

2) For all $\omega\in\Omega^r(N)$ and $\eta\in\Omega^s(N)$, $f^*(\omega\wedge\eta)=f^*\omega\wedge f^*\eta$.

3) $f^*(d\omega)=d\left(f^*\omega\right)$.

Definition 15 [9] *A Lie group is a group* G *that is also a smooth manifold such that the group operations* $\varphi:G\times G\to G,(g,h)\mapsto gh$ *and inversion* $\tau:G\to G,g\mapsto g^{-1}$ *are smooth maps.*

For any fixed $g\in G$, the maps

$$L_g:G\to G,\ L_g(h)=gh$$

and

$$R_g:G\to G,\ R_g(h)=hg^{-1}$$

are smooth diffeomorphisms, called left multiplication and right multiplication, respectively.

The group $G\times G$ acts smoothly on G via

$$(g,h)\cdot x = R_h L_g(x) = gxh^{-1},\ \ g,h,x\in G.$$

If $g=h$, this action gives the conjugation by g, denoted $c_g = R_g L_g$.

Definition 16 [7] *Let a Lie group* G *acts smoothly on a smooth manifold* M *via a map* $G\times M\to M$. *For each* $g\in G$, *denote by* g *the smooth map* $M\to M$ *given by the action. A differential form* $\omega\in\Omega^k(M)$ *is called* G*-invariant if* $g^*\omega=\omega$, for all $g\in G$.

The space of all G-invariant k-forms on M is denoted by $\Omega^k(M)^G$.

Definition 17 [9] *Let* G *be a Lie group. A vector field* $X\in\mathfrak{X}(G)$ *is left-invariant if for all* $g,h\in G$,

$$(L_g)_{*h}(X(h)) = X(gh).$$

The set of all left-invariant vector fields on G forms a Lie algebra under the Lie bracket of vector fields. This Lie algebra is denoted by $\mathfrak{g}$ and is isomorphic to the tangent space T_eG at the identity element $e\in G$. It is called the Lie algebra of G.

Definition 18 [10] *A chain complex* $\mathbf{C}$ *of* R*-modules is a family* $\{C_n\}_{n\in Z}$ *of* R*-modules together with* R*-modules map* $d_n: C_n\to C_{n-1}$ *such that the sequence*

$$\cdots\to C_{n+1}\xrightarrow{d_{n+1}} C_n\xrightarrow{d_n} C_{n-1}\to\cdots$$

satisfies $d_n\circ d_{n+1}=0$ for all $n\in Z$.

Definition 19 [10] *Let* $(\mathbf{C}_\bullet, d_\bullet)$ *and* $(\mathbf{C}'_\bullet, d'_\bullet)$ *be chain complexes. A chain map*

$$f = f_\bullet : (\mathbf{C}_\bullet, d_\bullet)\to(\mathbf{C}'_\bullet, d'_\bullet)$$

is a family of morphisms $\{f_n: C_n\to C'_n\}_{n\in\mathbb{Z}}$ such that diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow[d'_{n+1}]{} & C'_n & \xrightarrow[d'_n]{} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

commutes, *i.e.*, $f_{n-1}\circ d_n = d'_n\circ f_n$ for all $n\in Z$.

Definition 20 [10] *Let* $\mathbf{C}:\cdots\to C_{n+1}\xrightarrow{d_{n+1}} C_n\xrightarrow{d_n} C_{n-1}\to\cdots$ *be a chain complex. Its* n*-th homology is defined as the quotient module*

$$H_n(\mathbf{C}) = \frac{\ker d_n}{\operatorname{Im} d_{n+1}}.$$

Definition 21 [10] *A chain map* $f:(A,d)\to(C,\delta)$ *is called a quasi-isomorphism if for every integer* n *the induced map* $f_*: H_n(A)\to H_n(C)$ *are an isomorphism.*

Definition 22 [11] *Let* K *be a feld,* $\mathfrak{g}$ *a Lie algebra over* K, *and* Γ *a* $\mathfrak{g}$*-*

module. Define

$$C^n(\mathfrak{g},\Gamma) := \mathrm{Hom}_K(\Lambda^n \mathfrak{g},\Gamma),\ n>0,\ C^0(\mathfrak{g},\Gamma) := \Gamma.$$

The space $C^n(\mathfrak{g},\Gamma)$ can be identified with the space of alternating n-linear maps $\mathfrak{g}^n \to \Gamma$. For $c \in C^n(\mathfrak{g},\Gamma)$, define $dc \in C^{n+1}(\mathfrak{g},\Gamma)$ by

$$dc(X_1,\cdots,X_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1} X_i\left(c\left(X_1,\cdots,\hat{X}_i,\cdots,X_{n+1}\right)\right) + \sum_{1\le i<j\le n+1}(-1)^{i+j} c\left(\left[X_i,X_j\right],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{n+1}\right)$$

for all $X_1,\cdots,X_{n+1} \in \mathfrak{g}$.

One can verify that $d \circ d = 0$, so we obtain a cochain complex

$$\cdots \to C^{n-1}(\mathfrak{g},\Gamma) \xrightarrow{d^{n-1}} C^n(\mathfrak{g},\Gamma) \xrightarrow{d^n} C^{n+1}(\mathfrak{g},\Gamma) \to \cdots.$$

The Lie algebra cohomology of $\mathfrak{g}$ with coefficients in Γ is

$$H^n(\mathfrak{g},\Gamma) := H^n\left(\left(C^*(\mathfrak{g},\Gamma),d\right)\right) = \frac{\ker d^n}{\operatorname{Im} d^{n-1}}.$$

3. Isomorphism between de Rham Cohomology and Lie Algebra Cohomology

This section constructs explicit chain complexe isomorphisms that induce isomorphisms in cohomology. As a consequence, we prove that for a compact connected Lie group G with Lie algebra $\mathfrak{g}$, the de Rham cohomology $H^\bullet_{\mathrm{dR}}(G)$ is isomorphic to the Lie algebra cohomology $H^\bullet(\mathfrak{g},\mathbb{R})$.

Proposition 3 [11] *Let M be a smooth manifold and G a connected compact Lie group acting smoothly on M via $\alpha: G\times M \to M$. Then the inclusion map*

$$\varphi: \Omega^*(M)^G \hookrightarrow \Omega^*(M)$$

is a quasi-isomorphism.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let V be a vector space and $\pi: G \to \mathrm{Aut}(V)$ a representation of G. Its derivative at the identity gives the induced Lie algebra representation

$$\rho = D_e\pi : \mathfrak{g} \to \mathrm{End}(V).$$

Recall that for any $X \in \mathfrak{g}$, the exponential map $\exp: \mathfrak{g} = T_eG \to G$ satisfies

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX) = X,$$

where exp is defined by $\exp(X) = \theta_X(1)$, with θ_X being the maximal integral curve of the left-invariant vector field determined by X and satisfying $\theta_X(0) = e$. Applying the chain rule to $\pi \circ \exp$ yields

$$\rho(X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(\exp(tX)).$$

Definition 23 [12] *Let G be a Lie group acting linearly on a vector space V*

via a representation $\pi: G \to \mathrm{GL}(V)$, *and let* $\rho: \mathfrak{g} \to \mathrm{End}(V)$ *be the corresponding Lie algebra representation. A vector* $v \in V$ *is called* G*-invariant if* $\pi(g)(v) = v$, for all $g \in G$.

The subspace of G-invariant vectors is denoted by V^G. A vector $v \in V$ is called $\mathfrak{g}$-invariant if $\rho(X)(v) = 0$, for all $X \in \mathfrak{g}$.

The subspace of $\mathfrak{g}$-invariant vectors is denoted by $V^{\mathfrak{g}}$.

We shall now prove that these two subspaces coincide.

Proposition 4 $V^G = V^{\mathfrak{g}}$.

Proof. We prove the equality by showing two inclusions.

1) $V^G \subseteq V^{\mathfrak{g}}$.

Let $v \in V^G$, so that $\pi(g)(v) = v$ for every $g \in G$. For any $X \in \mathfrak{g}$,

$$\rho(X)(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(\exp(tX))(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} v = 0,$$

hence $v \in V^{\mathfrak{g}}$.

2) $V^{\mathfrak{g}} \subseteq V^G$.

Let $v \in V^{\mathfrak{g}}$, *i.e.*, $\rho(X)(v) = 0$ for every $X \in \mathfrak{g}$. Define the evaluation map $\mathrm{ev}_v : \mathrm{Aut}(V) \to V$ by $\mathrm{ev}_v(A) = A(v)$. Then

$$D_e(\mathrm{ev}_v \circ \pi)(X) = (\mathrm{ev}_v \circ D_e\pi)(X) = \mathrm{ev}_v(\rho(X)) = \rho(X)(v) = 0.$$

Since G is connected, the map $\mathrm{ev}_v \circ \pi$ is constant. As $\mathrm{ev}_v \circ \pi(e) = \pi(e)(v) = v$, we obtain $\mathrm{ev}_v \circ \pi(g) = \pi(g)(v) = v$, where e is the identity of G. Thus $v \in V^G$. Combining (1) and (2) we conclude $V^G = V^{\mathfrak{g}}$.

Building on Proposition 4, we now establish an isomorphism between the complex of invariant differential forms and the cochain complex of the Lie algebra, thereby connecting the de Rham cohomology of a Lie group to its Lie algebra cohomology.

Proposition 5 *The evaluation map at the identity* $e \in G$, $\varepsilon: \Omega^k(G)^G \to C^k(\mathfrak{g}, \mathbb{R})$, $\omega \mapsto \omega_e$, *defines an isomorphism of cochain complexes.*

Proof. First, we verify that ε is well-defined. Identifying $\mathfrak{g}$ with the tangent space T_eG, we have

$$\varepsilon(\omega) = \omega_e \in \mathrm{Hom}_{\mathbb{R}}(\Lambda^k(T_eG), \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\Lambda^k \mathfrak{g}, \mathbb{R}) = C^k(\mathfrak{g}, \mathbb{R}).$$

Hence ε is well-defined.

Consider the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega^{k-1}(G)^G & \xrightarrow{d^{k-1}} & \Omega^k(G)^G & \xrightarrow{d^k} & \Omega^{k+1}(G)^G & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varepsilon^{k-1}} & & \downarrow{\scriptstyle \varepsilon^k} & & \downarrow{\scriptstyle \varepsilon^{k+1}} & & \\
\cdots & \longrightarrow & C^{k-1}(\mathfrak{g}, \mathbb{R}) & \xrightarrow[d'^{k-1}]{} & C^k(\mathfrak{g}, \mathbb{R}) & \xrightarrow[d'^k]{} & C^{k+1}(\mathfrak{g}, \mathbb{R}) & \longrightarrow & \cdots
\end{array}$$

Let $\omega \in \Omega^k(G)^G$ and $g \in G$. Since ω is G-invariant, $g^*\omega = \omega$. We have $g^*(d^k\omega) = d^k(g^*\omega) = d^k\omega$, so $d^k\omega$ is also G-invariant, *i.e.*, $d^k\omega \in \Omega^{k+1}(G)^G$.

Next, we show that ε is a chain map, *i.e.*, $\varepsilon^{k+1} \circ d^k = d'^k \circ \varepsilon^k$. Let $\omega \in \Omega^k(G)^G$ and $v_1, \cdots, v_{k+1} \in T_e G$. Let X_i be the left-invariant vector field on G with $X_i(e) = v_i$. Since ω and each X_i are left-invariant, we have

$$(L_g)_{*h}(X_i(h)) = X_i(gh),$$

$$(L_g)^* \omega = \omega$$

for all $g, h \in G$. The invariance of ω means that for any tangent vectors $u_1, \cdots, u_k \in T_h G$,

$$w_h(u_1, \cdots, u_k) = \left((L_g)^* \omega\right)_h (u_1, \cdots, u_k) = w_{gh}\left((L_g)_{*h}(u_1), \cdots, (L_g)_{*h}(u_k)\right).$$

Define the function $f : G \to \mathbb{R}$ by $f(h) = \omega(X_1, \cdots, X_k)_h = \omega_h(X_1(h), \cdots, X_k(h))$. Then

$$\begin{aligned} f(gh) &= \omega(X_1, \cdots, X_k)_{gh} \\ &= \omega_{gh}(X_1(gh), \cdots, X_k(gh)) \\ &= \omega_{gh}\left((L_g)_{*h}(X_1(h)), \cdots, (L_g)_{*h}(X_k(h))\right) \\ &= \left((L_g)^* \omega\right)_h (X_1(h), \cdots, X_k(h)) \\ &= \omega_h(X_1(h), \cdots, X_k(h)) \\ &= f(h). \end{aligned}$$

Thus, f is left-invariant. Consequently, $\omega(X_1, \cdots, X_k)$ is left-invariant and therefore constant.

Since $\mathfrak{g}$ acts trivially on $\mathbb{R}$, we have

$$\begin{aligned} &\varepsilon^{k+1}(d^k \omega)(v_1, \cdots, v_{k+1}) \\ &= (d^k \omega)_e (v_1, \cdots, v_{k+1}) = d^k \omega(X_1, \cdots, X_{k+1})(e) \\ &= \sum_{i=1}^{k+1} (-1)^{i+1} X_i\left(\omega\left(X_1, \cdots, \hat{X}_i, \cdots, X_{k+1}\right)\right)(e) \\ &\quad + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\left(\left[X_i, X_j\right], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{k+1}\right)(e) \\ &= \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\left(\left[X_i, X_j\right], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{k+1}\right)(e) \\ &= \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega_e\left(\left[v_i, v_j\right], v_1, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_{k+1}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} d'^k(\varepsilon^k \omega)(v_1, \cdots, v_{k+1}) &= d'^k(\omega_e)(v_1, \cdots, v_{k+1}) \\ &= \sum_{i=1}^{k+1} (-1)^{i+1} v_i\left(\omega_e(v_1, \cdots, \hat{v}_i, \cdots, v_{k+1})\right) \\ &\quad + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega_e\left(\left[v_i, v_j\right], v_1, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_{k+1}\right) \\ &= \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega_e\left(\left[v_i, v_j\right], v_1, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_{k+1}\right). \end{aligned}$$

This implies that $\varepsilon^{k+1}\circ d^k = d'^k \circ \varepsilon^k$, so ε is a chain map.

Next we prove that ε is an isomorphism. Let $\omega \in \Omega^k(G)^G$, $g \in G$ and $u'_1,\cdots,u'_k \in T_g G$. Then

$$\omega_g(u'_1,\cdots,u'_k) = \left(L^*_{g^{-1}}\omega\right)_g(u'_1,\cdots,u'_k) = \omega_e\left(D_g L_{g^{-1}}(u'_1),\cdots,D_g L_{g^{-1}}(u'_k)\right).$$

Hence $\ker\varepsilon = \left\{\omega \in \Omega^k(G)^G \mid \varepsilon(\omega) = \omega_e = 0\right\} = 0$. It follows that ε is injective.

Let $c \in C^k(\mathfrak{g},\mathbb{R})$, there exists $\omega \in \Omega^k(G)$ such that

$$\omega_g(u'_1,\cdots,u'_k) = c\left(D_g L_{g^{-1}}(u'_1),\cdots,D_g L_{g^{-1}}(u'_k)\right).$$

For any $g,h \in G$,

$$\begin{aligned}
\left((L_h)^*\omega\right)_g(u'_1,\cdots,u'_k) &= \omega_{hg}\left((L_h)_{*g}(u'_1),\cdots,(L_h)_{*g}(u'_k)\right)\\
&= \omega_{hg}\left(D_g L_h(u'_1),\cdots,D_g L_h(u'_k)\right)\\
&= c\left(D_{hg}L_{(hg)^{-1}}\left(D_g L_h(u'_1)\right),\cdots,D_{hg}L_{(hg)^{-1}}\left(D_g L_h(u'_k)\right)\right)\\
&= c\left(D_g L_{g^{-1}}(u'_1),\cdots,D_g L_{g^{-1}}(u'_k)\right)\\
&= \omega_g(u'_1,\cdots,u'_k).
\end{aligned}$$

Thus, $\omega \in \Omega^k(G)^G$. Additionally, since

$$\begin{aligned}
\varepsilon(\omega)(v_1,\cdots,v_k) &= \omega_e(v_1,\cdots,v_k)\\
&= c\left(D_e L_{e^{-1}}(v_1),\cdots,D_e L_{e^{-1}}(v_k)\right)\\
&= c\left(\left(L_{e^{-1}}\right)_{*e}(X_1(e)),\cdots,\left(L_{e^{-1}}\right)_{*e}(X_k(e))\right)\\
&= c\left(X_1\left(e^{-1}\cdot e\right),\cdots,X_k\left(e^{-1}\cdot e\right)\right)\\
&= c\left(X_1(e),\cdots,X_k(e)\right)\\
&= c(v_1,\cdots,v_k),
\end{aligned}$$

it follows that $\varepsilon(\omega) = c$. Thus, ε is surjective. We conclude that ε is an isomorphism.

Next, we construct a representation of the Lie group G with Lie algebra $\mathfrak{g}$ on the cochain complex $C^*(\mathfrak{g},\mathbb{R})$, and show that the subspace of G-invariant cochains coincides with the subspace of $\mathfrak{g}$-invariant cochains.

Proposition 6 $\left(C^*(\mathfrak{g},\mathbb{R})^G, d\right) = \left(C^*(\mathfrak{g},\mathbb{R})^{\mathfrak{g}}, d\right)$.

Proof. First, we construct the action $\pi : G \to \mathrm{Aut}\left(C^k(\mathfrak{g},\mathbb{R})\right)$.

For $g,h \in G$ and $X, X_1,\cdots,X_k \in \mathfrak{g}$, recall that G acts on $\mathfrak{g}$ via the adjoint action

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}),\ \mathrm{Ad}(g)(X) = T_e c_g(X),$$

where $c_g(h) = ghg^{-1}$ is conjugation by g. This action extends naturally to the exterior power $\Lambda^k\mathfrak{g}$, which we also denote by $\mathrm{Ad} : G \to \mathrm{Aut}\left(\Lambda^k\mathfrak{g}\right)$, satisfying

$$\mathrm{Ad}(g)(X_1\wedge\cdots\wedge X_k) := \mathrm{Ad}(g)(X_1)\wedge\cdots\wedge\mathrm{Ad}(g)(X_k).$$

Dualising gives an action π of G on $C^k(\mathfrak{g},\mathbb{R})$, $\pi : G \to \mathrm{Aut}\left(C^k(\mathfrak{g},\mathbb{R})\right)$, $g \mapsto \mathrm{Ad}(g)^*$, $\left(\mathrm{Ad}(g)^* c\right)(X_1,\cdots,X_k) := c\left(\mathrm{Ad}\left(g^{-1}\right)(X_1),\cdots,\mathrm{Ad}\left(g^{-1}\right)(X_k)\right)$.

Next we verify that π is a homomorphism. For $g,h \in G$ and $c \in C^k(\mathfrak{g},\mathbb{R})$,

$$\begin{aligned}(\pi(gh)c)(X_1,\cdots,X_k) &= \mathrm{Ad}(gh)^* c(X_1,\cdots,X_k)\\ &= c\left(\mathrm{Ad}\left((gh)^{-1}\right)(X_1),\cdots,\mathrm{Ad}\left((gh)^{-1}\right)(X_k)\right)\\ &= c\left(T_e c_{(gh)^{-1}}(X_1),\cdots,T_e c_{(gh)^{-1}}(X_k)\right),\end{aligned}$$

$$\begin{aligned}(\pi(g)\circ\pi(h)c)(X_1,\cdots,X_k) &= \left((\mathrm{Ad}g)^* \circ (\mathrm{Ad}h)^*\right)c(X_1,\cdots,X_k)\\ &= (\mathrm{Ad}g)^*\left((\mathrm{Ad}h)^* c\right)(X_1,\cdots,X_k)\\ &= (\mathrm{Ad}h)^* c\left(T_e c_{g^{-1}}(X_1),\cdots,T_e c_{g^{-1}}(X_k)\right)\\ &= c\left(T_e c_{h^{-1}}\left(T_e c_{g^{-1}}(X_1)\right),\cdots,T_e c_{h^{-1}}\left(T_e c_{g^{-1}}(X_k)\right)\right)\\ &= c\left(T_e c_{(gh)^{-1}}(X_1),\cdots,T_e c_{(gh)^{-1}}(X_k)\right),\end{aligned}$$

it follows that $\pi(gh) = \pi(g)\circ\pi(h)$. Thus, π is a representation of G.

Now we construct the corresponding Lie algebra action $\rho : \mathfrak{g} \to \mathrm{End}\left(C^k(\mathfrak{g},\mathbb{R})\right)$. The adjoint action of $\mathfrak{g}$ on itself is

$$\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g}),\ \mathrm{ad}(X) = [X,Y],$$

which we can extend to an action of $\mathfrak{g}$ on $\Lambda^k(\mathfrak{g})$ by

$$\mathrm{ad}(X)(X_1 \wedge \cdots \wedge X_k) = \sum_{i=1}^{k} X_1 \wedge \cdots \wedge [X,X_i] \wedge \cdots \wedge X_k.$$

Again this dualises to an action on $C^k(\mathfrak{g},\mathbb{R})$,

$$\rho : \mathfrak{g} \to \mathrm{End}\left(C^k(\mathfrak{g},\mathbb{R})\right),\ X \mapsto \mathrm{ad}(X)^*,$$

satisfying

$$\left(\mathrm{ad}(X)^* c\right)(X_1,\cdots,X_k) = \sum_{i=1}^{k} c(X_1,\cdots,[X_i,X],\cdots,X_k).$$

Next we show that ρ is a Lie algebra homomorphism, *i.e.*, $\rho([X,Y]) = [\rho(X),\rho(Y)]$ for all $X,Y \in \mathfrak{g}$. Recall that $[\rho(X),\rho(Y)] = \rho(X)\circ\rho(Y) - \rho(Y)\circ\rho(X)$. For $c \in C^k(\mathfrak{g},\mathbb{R})$ and $X_1,\cdots,X_k \in \mathfrak{g}$,

$$\begin{aligned}(\rho(X)\circ\rho(Y)c)(X_1,\cdots,X_k) &= \sum_{i=1}^{k} \rho(Y)c(X_1,\cdots,[X_i,X],\cdots,X_k)\\ &= \sum_{1\le j<i\le k} c\left(X_1,\cdots,[X_j,Y],\cdots,[X_i,X],\cdots,X_k\right)\\ &\quad + \sum_{1\le j=i\le k} c\left(X_1,\cdots,[[X_i,X],Y],\cdots,X_k\right)\\ &\quad + \sum_{1\le i<j\le k} c\left(X_1,\cdots,[X_i,X],\cdots,[X_j,Y],\cdots,X_k\right).\end{aligned}$$

Similarly,

$$
\begin{aligned}
\left(\rho(Y)\circ\rho(X)c\right)(X_1,\cdots,X_k) &= \sum_{j=1}^{k}\rho(X)c\left(X_1,\cdots,\left[X_j,Y\right],\cdots,X_k\right) \\
&= \sum_{1\le j<i\le k} c\left(X_1,\cdots,\left[X_j,Y\right],\cdots,\left[X_i,X\right],\cdots,X_k\right) \\
&\quad + \sum_{1\le j=i\le k} c\left(X_1,\cdots,\left[\left[X_j,Y\right],X\right],\cdots,X_k\right) \\
&\quad + \sum_{1\le i<j\le k} c\left(X_1,\cdots,\left[X_i,X\right],\cdots,\left[X_j,Y\right],\cdots,X_k\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(\left[\rho(X),\rho(Y)\right]c\right)(X_1,\cdots,X_k) \\
&= \left(\rho(X)\circ\rho(Y)-\rho(Y)\circ\rho(X)\right)c(X_1,\cdots,X_k) \\
&= \sum_{1\le j=i\le k} c\left(X_1,\cdots,\left[\left[X_i,X\right],Y\right]-\left[\left[X_j,Y\right],X\right],\cdots,X_k\right) \\
&= \sum_{i=1}^{k} c\left(X_1,\cdots,\left[\left[X_i,X\right],Y\right]-\left[\left[X_i,Y\right],X\right],\cdots,X_k\right) \\
&= \sum_{i=1}^{k} c\left(X_1,\cdots,\left[X_i,\left[X,Y\right]\right],\cdots,X_k\right) \\
&= \left(\rho(\left[X,Y\right])c\right)(X_1,\cdots,X_k),
\end{aligned}
$$

it follows that $\rho(\left[X,Y\right])=\left[\rho(X),\rho(Y)\right]$. Thus, ρ is a representation of $\mathfrak{g}$.

Next we prove that $\rho(X)=D_e\pi(X)$ for every $X\in\mathfrak{g}$. Since

$$
\begin{aligned}
\left(D_e\pi(X)c\right)(X_1,\cdots,X_k) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\pi(\exp(tX))c\right)(X_1,\cdots,X_k) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathrm{Ad}(\exp(tX))^{*}c\right)(X_1,\cdots,X_k) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} c\left(\mathrm{Ad}(\exp(tX))^{-1}(X_1),\cdots,\mathrm{Ad}(\exp(tX))^{-1}(X_k)\right) \\
&= \sum_{i=1}^{k} c\left(X_1,\cdots,\mathrm{ad}(-X)(X_i),\cdots,X_k\right) \\
&= \sum_{i=1}^{k} c\left(X_1,\cdots,\left[X_i,X\right],\cdots,X_k\right) \\
&= \left((\mathrm{ad}\,X)^{*}c\right)(X_1,\cdots,X_k) \\
&= \left(\rho(X)c\right)(X_1,\cdots,X_k),
\end{aligned}
$$

it follows that $\rho=D_e\pi$. According to Proposition 4, we conclude $C^k(\mathfrak{g},\mathbb{R})^G=C^k(\mathfrak{g},\mathbb{R})^{\mathfrak{g}}$.

Next we prove that for every $c\in C^k(\mathfrak{g},\mathbb{R})^G$, its differential satisfies $dc\in C^{k+1}(\mathfrak{g},\mathbb{R})^G$. Let $g\in G$ and $X_1,\cdots,X_{k+1}\in\mathfrak{g}$. Because c is G-invariant,

$$
\pi(g)c(X_1,\cdots,X_k)=\left(\mathrm{Ad}(g)\right)^{*}c(X_1,\cdots,X_k)=c(X_1,\cdots,X_k).
$$

Now compute $\left(\mathrm{Ad}(g)^* dc\right)(X_1,\cdots,X_{k+1})$:

$$\begin{aligned}
&\left(\mathrm{Ad}(g)^* dc\right)(X_1,\cdots,X_{k+1})\\
&= dc\left(\mathrm{Ad}\left(g^{-1}\right)(X_1),\cdots,\mathrm{Ad}\left(g^{-1}\right)(X_{k+1})\right)\\
&= \sum_{i=1}^{k+1}(-1)^{i+1}\mathrm{Ad}\left(g^{-1}\right)(X_i)\left(c\left(\mathrm{Ad}\left(g^{-1}\right)(X_1),\cdots,\mathrm{Ad}\left(\widehat{g^{-1}}\right)(X_i),\cdots,\mathrm{Ad}\left(g^{-1}\right)(X_{k+1})\right)\right)\\
&\quad+\sum_{i<j}(-1)^{i+j}c\left(\left[\mathrm{Ad}\left(g^{-1}\right)(X_i),\mathrm{Ad}\left(g^{-1}\right)(X_j)\right],\mathrm{Ad}\left(g^{-1}\right)(X_1),\cdots,\right.\\
&\quad\left.\mathrm{Ad}\left(\widehat{g^{-1}}\right)(X_i),\cdots,\mathrm{Ad}\left(\widehat{g^{-1}}\right)(X_j),\cdots,\mathrm{Ad}\left(g^{-1}\right)(X_{k+1})\right)\\
&= \sum_{i<j}(-1)^{i+j}c\left(\mathrm{Ad}\left(g^{-1}\right)\left(\left[X_i,X_j\right]\right),\mathrm{Ad}\left(g^{-1}\right)(X_1),\cdots,\mathrm{Ad}\left(\widehat{g^{-1}}\right)(X_i),\cdots,\right.\\
&\quad\left.\mathrm{Ad}\left(\widehat{g^{-1}}\right)(X_j),\cdots,\mathrm{Ad}\left(g^{-1}\right)(X_{k+1})\right)\\
&= \sum_{i<j}(-1)^{i+j}\left(\mathrm{Ad}(g)^* c\right)\left(\left[X_i,X_j\right],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}\right)\\
&= \sum_{i<j}(-1)^{i+j}c\left(\left[X_i,X_j\right],X_i,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}\right)\\
&= dc(X_1,\cdots,X_{k+1}).
\end{aligned}$$

It follows that $dc\in C^{k+1}(\mathfrak{g},\mathbb{R})^G$. We finally obtain $\left(C^*(\mathfrak{g},\mathbb{R})^G,d\right)=\left(C^*(\mathfrak{g},\mathbb{R})^{\mathfrak{g}},d\right)$.

We now consider the action of $G\times G$ on G and establish an isomorphism of cochain complexes between the space of $G\times G$-invariant differential forms on G and the G-invariant supspace of the Lie algebra cochain complex.

Proposition 7 *Evaluation at* $e\in G$, $\tau:\Omega^k(G)^{G\times G}\to C^k(\mathfrak{g},\mathbb{R})^G$, $\omega\mapsto\omega_e$ *defines an isomorphism of chain complexes.*

Proof. Let $\omega\in\Omega^k(G)^{G\times G}$, we show that $\omega_e\in C^k(\mathfrak{g},\mathbb{R})^G$, that is, ω_e is G-invariant. For any $g\in G$ and $X_1,\cdots,X_k\in\mathfrak{g}$, the invariance of ω gives $c_g^*\omega=\omega$. Hence

$$\begin{aligned}
\omega_e(X_1,\cdots,X_k)&=\left(c_{g^{-1}}^*\omega\right)_e(X_1,\cdots,X_k)\\
&=\omega_e\left(D_e c_{g^{-1}}(X_1),\cdots,D_e c_{g^{-1}}(X_k)\right)\\
&=\omega_e\left(\mathrm{Ad}\left(g^{-1}\right)(X_1),\cdots,\mathrm{Ad}\left(g^{-1}\right)(X_k)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\pi(g)\omega_e\right)(X_1,\cdots,X_k)&=\left((Adg)^*\omega_e\right)(X_1,\cdots,X_k)\\
&=\omega_e\left(Ad\left(g^{-1}\right)(X_1),\cdots,Ad\left(g^{-1}\right)(X_k)\right)\\
&=\omega_e(X_1,\cdots,X_k).
\end{aligned}$$

It follows that $\pi(g)\omega_e=\omega_e$, which means that ω_e is G-invariant. Consequently, τ is well-defined.

Consider the following diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & \Omega^{k-1}(G)^{G\times G} & \xrightarrow{d'^{k-1}} & \Omega^{k}(G)^{G\times G} & \xrightarrow{d'^{k}} & \Omega^{k+1}(G)^{G\times G} & \longrightarrow \\
 & \downarrow \tau^{k-1} & & \downarrow \tau^{k} & & \downarrow \tau^{k+1} & \\
\longrightarrow & C^{k-1}(\mathfrak{g},\mathbb{R})^{G} & \xrightarrow[d^{k-1}]{} & C^{k}(\mathfrak{g},\mathbb{R})^{G} & \xrightarrow[d^{k}]{} & C^{k+1}(\mathfrak{g},\mathbb{R})^{G} & \longrightarrow
\end{array}$$

Let $\omega \in \Omega^k(G)^{G\times G}$ and $g,h \in G$. Since ω is $G\times G$-invariant, we have

$$\left(R_g L_h\right)^*\left(d'^k\omega\right) = L_h^*\left(R_g^*\left(d'^k\omega\right)\right) = L_h^* d'^k\left(R_g^*\omega\right) = d'^k\left(L_h^* R_g^*\omega\right) = d'^k\omega,$$

so $d'^k\omega \in \Omega^{k+1}(G)^{G\times G}$.

Next we show that τ is a chain map, *i.e.*, $\tau^{k+1}\circ d'^k = d^k\circ\tau^k$. Let $\omega \in \Omega^k(G)^{G\times G}$ and $v_1,\cdots,v_{k+1}\in T_eG$. Let X_i be the left-invariant vector field on G with $X_i(e)=v_i$. Because ω and each X_i are left-invariant, the function $\omega(X_1,\cdots,X_k)$ is left-invariant and therefore is constant. Moreover, $\mathfrak{g}$ acts trivially on $\mathbb{R}$. Consequently,

$$\begin{aligned}
&\left(\tau^{k+1}\left(d'^k\omega\right)\right)(v_1,\cdots,v_{k+1})\\
&=\left(d'^k\omega\right)_e(v_1,\cdots,v_{k+1}) = d'^k\omega(X_1,\cdots,X_k)(e)\\
&=\sum_{i=1}^{k+1}(-1)^{i+1}X_i\left(\omega\left(X_1,\cdots,\hat{X}_i,\cdots,X_{k+1}\right)\right)(e)\\
&\quad+\sum_{1\le i<j\le k+1}(-1)^{i+j}\omega\left(\left[X_i,X_j\right],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}\right)(e)\\
&=\sum_{1\le i<j\le k+1}(-1)^{i+j}\omega\left(\left[X_i,X_j\right],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}\right)(e)\\
&=\sum_{1\le i<j\le k+1}(-1)^{i+j}\omega_e\left(\left[v_i,v_j\right],v_1,\cdots,\hat{v}_i,\cdots,\hat{v}_j,\cdots,v_{k+1}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left(d^k\left(\tau^k\omega\right)\right)(v_1,\cdots,v_{k+1}) &= d^k(\omega_e)(v_1,\cdots,v_{k+1})\\
&=\sum_{i=1}^{k+1}(-1)^{i+1}v_i\left(\omega_e\left(v_1,\cdots,\hat{v}_i,\cdots,v_{k+1}\right)\right)\\
&\quad+\sum_{1\le i<j\le k+1}^{k}(-1)^{i+j}\omega_e\left(\left[v_i,v_j\right],v_1,\cdots,\hat{v}_i,\cdots,\hat{v}_j,\cdots,v_{k+1}\right)\\
&=\sum_{1\le i<j\le k+1}^{k}(-1)^{i+j}\omega_e\left(\left[v_i,v_j\right],v_1,\cdots,\hat{v}_i,\cdots,\hat{v}_j,\cdots,v_{k+1}\right).
\end{aligned}$$

This implies that $\tau^{k+1}\circ d'^k = d^k\circ\tau^k$. Thus, τ is a chain map.

Next, we prove that τ is an isomorphism. Let $\omega\in\Omega^k(G)^{G\times G}$, $g\in G$ and $u_1,\cdots,u_k\in T_gG$. Because a $G\times G$-invariant form is in particular left-invariant, we have

$$\omega_g(u_1,\cdots,u_k) = \left(L_{g^{-1}}^*\omega\right)_g(u_1,\cdots,u_k) = \omega_e\left(D_gL_{g^{-1}}(u_1),\cdots,D_gL_{g^{-1}}(u_k)\right).$$

This gives $\ker\tau = \left\{\omega \in \Omega^k(G)^{G\times G} \mid \tau(\omega) = \omega_e = 0\right\} = 0$. Thus, τ is injective. Let $c \in C^k(\mathfrak{g}, \mathbb{R})^G$, there exists $\omega \in \Omega^k(G)^G$ such that

$$\omega_g(u_1, \cdots, u_k) = c\left(D_g L_{g^{-1}}(u_1), \cdots, D_g L_{g^{-1}}(u_k)\right).$$

For any $x, y, g \in G$,

$$\begin{aligned}
&\left(\left(R_y L_x\right)^* \omega\right)_g (u_1, \cdots, u_k) = \omega_{xgy^{-1}}\left(D_g\left(R_y L_x\right)(u_1), \cdots, D_g\left(R_y L_x\right)(u_k)\right) \\
&= c\left(D_{xgy^{-1}} L_{yg^{-1}x^{-1}}\left(D_g\left(R_y L_x\right)(u_1)\right), \cdots, D_{xgy^{-1}} L_{yg^{-1}x^{-1}}\left(D_g\left(R_y L_x\right)(u_k)\right)\right) \\
&= c\left(D_g\left(c_y \circ L_{g^{-1}}\right)(u_1), \cdots, D_g\left(c_y \circ L_{g^{-1}}\right)(u_k)\right) \\
&= c\left(D_e c_y \circ D_g L_{g^{-1}}(u_1), \cdots, D_e c_y \circ D_g L_{g^{-1}}(u_k)\right) \\
&= c\left(\mathrm{Ad}(y)\left(D_g L_{g^{-1}}(u_1)\right), \cdots, \mathrm{Ad}(y)\left(D_g L_{g^{-1}}(u_k)\right)\right) \\
&= c\left(D_g L_{g^{-1}}(u_1), \cdots, D_g L_{g^{-1}}(u_k)\right) \\
&= \omega_g(u_1, \cdots, u_k).
\end{aligned}$$

Thus, $\omega \in \Omega^k(G)^{G\times G}$. Additionally, since

$$\begin{aligned}
\tau(\omega)(v_1, \cdots, v_k) &= \omega_e(v_1, \cdots, v_k) \\
&= c\left(D_e L_{e^{-1}}(v_1), \cdots, D_e L_{e^{-1}}(v_k)\right) \\
&= c\left(\left(L_{e^{-1}}\right)_{*e}\left(X_1(e)\right), \cdots, \left(L_{e^{-1}}\right)_{*e}\left(X_k(e)\right)\right) \\
&= c\left(X_1\left(e^{-1}\cdot e\right), \cdots, X_k\left(e^{-1}\cdot e\right)\right) \\
&= c\left(X_1(e), \cdots, X_k(e)\right) \\
&= c(v_1, \cdots, v_k),
\end{aligned}$$

it follows that $\tau(\omega) = c$. Thus, τ is surjective. We conclude that τ is an isomorphism.

The isomorphisms of chain complexes established in Propositions 5 and 7, together with the quasi-isomorphism in Proposition 3 and the equality in Proposition 6, induce isomorphisms in cohomology.

Theorem 1 *If* G *is a compact connected Lie group with Lie algebra* $\mathfrak{g}$, *then*

$$H^\bullet(\mathfrak{g}, \mathbb{R}) \cong H^\bullet_{dR}(G) \cong \left(\left(\Lambda^\bullet \mathfrak{g}\right)^*\right)^{\mathfrak{g}},$$

where $\mathfrak{g}$ acts trivially on $\mathbb{R}$.

Proof. According to Proposition 3, we have $H^\bullet\left(\Omega^k(G)^G\right) \cong H^\bullet_{dR}(G)$. Moreover, Proposition 5 also gives an isomorphism of complexes $\Omega^k(G)^G \cong C^k(g, R)^G$. Consequently,

$$\begin{aligned}
H^k\left(\Omega^k(G)^G\right) &\cong H^k(\mathfrak{g}, \mathbb{R}) \\
&\cong H^k\left(C^k(\mathfrak{g}, R)^G\right) \\
&= H^k\left(C^k(\mathfrak{g}, R)^{\mathfrak{g}}\right) \\
&= H^k\left(\left(\left(\wedge^k \mathfrak{g}\right)^*\right)^{\mathfrak{g}}\right).
\end{aligned}$$

We obtain $H^\bullet(\mathfrak{g},\mathbb{R}) \cong H^\bullet_{dR}(G) \cong H\left(\left((\Lambda^\bullet \mathfrak{g})^*\right)^{\mathfrak{g}}\right)$.

Now we prove that $d=0$ of the chain complex

$$\left(C^\bullet(\mathfrak{g},\mathbb{R})^G, d\right) = \left(C^\bullet(\mathfrak{g},\mathbb{R})^{\mathfrak{g}}, d\right) = \left(\left((\Lambda^\bullet \mathfrak{g})^*\right)^{\mathfrak{g}}, d\right).$$

For any $c \in C^k(\mathfrak{g},\mathbb{R})^{\mathfrak{g}}$ and $X_1,\cdots,X_{k+1} \in \mathfrak{g}$. Because $\mathfrak{g}$ acts trivially on $\mathbb{R}$ and c is $\mathfrak{g}$-invariant, we have

$$\begin{aligned}
2dc(X_1,\cdots,X_{k+1}) &= 2\sum_{i=1}^{k+1}(-1)^{i+1} X_i\left(c\left(X_1,\cdots,\hat{X}_i,\cdots,X_{k+1}\right)\right) \\
&\quad + 2\sum_{i<j}(-1)^{i+j} c\left(\left[X_i,X_j\right],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}\right) \\
&= \sum_{i<j}(-1)^{i+j} c\left(\left[X_i,X_j\right],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}\right) \\
&\quad + \sum_{i<j}(-1)^{i+j} c\left(\left[X_i,X_j\right],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1}\right) \\
&= \sum_{i<j}(-1)^{j} c\left(X_1,\cdots,\left[X_i,X_j\right],\cdots,\hat{X}_j,\cdots,X_{k+1}\right) \\
&\quad + \sum_{j<i}(-1)^{j} c\left(X_1,\cdots,\hat{X}_j,\cdots,\left[X_i,X_j\right],\cdots,X_{k+1}\right) \\
&= \sum_{j=1}^{k+1}(-1)^{j}\left(\operatorname{ad}X_j\right)^* c\left(X_1,\cdots,\hat{X}_j,\cdots,X_{k+1}\right) \\
&= 0.
\end{aligned}$$

This gives $H\left(\left((\wedge^k \mathfrak{g})^*\right)^{\mathfrak{g}}\right) = \left((\wedge^k \mathfrak{g})^*\right)^{\mathfrak{g}}$. Combining all isomorphisms we finally obtain

$$H^\bullet(\mathfrak{g},\mathbb{R}) \cong H^\bullet_{dR}(G) \cong \left((\Lambda^\bullet \mathfrak{g})^*\right)^{\mathfrak{g}}.$$

4. Example

This section provides explicit cohomology computations for several examples, with a detailed derivation for $SO(3)$. For higher-dimensional cases ($n \geq 4$), the rapid growth in combinatorial complexity and computational demands necessitates algorithmic approaches. We therefore compute these results programmatically using MATLAB.

Example 1 *Calculate the de Rham cohomology of* $SO(3)$ *and the Lie algebra cohomology of its Lie algebra* $\mathfrak{so}(3)$.

The Lie group $SO(3)$ is defined as

$$SO(3) = \left\{A \in M_3(\mathbb{R}) \mid A^t A = id, \det A = 1\right\},$$

with Lie algebra

$$\mathfrak{so}(3) = \left\{X \in M_3(\mathbb{R}) \mid X^t = -X\right\}.$$

By Theorem 1, we have isomorphisms

$$H^{\bullet}\left(\mathfrak{so}(3),\mathbb{R}\right) \cong H_{dR}^{\bullet}\left(SO(3)\right) \cong \left(\left(\Lambda^{\bullet}\mathfrak{so}(3)\right)^{*}\right)^{\mathfrak{so}(3)}.$$

The Lie algebra $\mathfrak{so}(3)$ is spanned by the basis matrices

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

whose Lie brackets satisfy

$$[E_1,E_2] = -E_3,\ [E_1,E_3] = E_2,\ [E_2,E_3] = -E_1.$$

Next, we compute $H^1\left(\mathfrak{so}(3),\mathbb{R}\right) \cong H_{dR}^1\left(SO(3)\right) \cong \left(\left(\mathfrak{so}(3)\right)^{*}\right)^{\mathfrak{so}(3)}$. For any $c \in \left(\left(\mathfrak{so}(3)\right)^{*}\right)^{\mathfrak{so}(3)}$, $c:\mathfrak{so}(3)\to\mathbb{R}$ is an $\mathfrak{so}(3)$-invariant linear map, *i.e.*, $\rho:\mathfrak{so}(3)\to \mathrm{End}\left(c\left(\mathfrak{so}(3),\mathbb{R}\right)\right)$ satisfying

$$\rho(E_m)c(E_i) = (\mathrm{ad}E_m)^{*}c(E_i) = c([E_i,E_m]) = 0,\ 1 \le m,i \le 3.$$

Write $c = \sum_{i=1}^{3}\lambda_i E_i^{*}$, with $E_i^{*}(E_j) = \delta_i^j$, we have

$$(\mathrm{ad}E_1)^{*}c(E_1) = c([E_1,E_1]) = c(0) = 0,$$

$$(\mathrm{ad}E_1)^{*}c(E_2) = c([E_2,E_1]) = c(E_3) = \lambda_3 = 0,$$

$$(\mathrm{ad}E_1)^{*}c(E_3) = c([E_3,E_1]) = -c(E_2) = -\lambda_2 = 0,$$

$$(\mathrm{ad}E_2)^{*}c(E_1) = c([E_1,E_2]) = -c(E_3) = -\lambda_3 = 0,$$

$$(\mathrm{ad}E_2)^{*}c(E_2) = c([E_2,E_2]) = c(0) = 0,$$

$$(\mathrm{ad}E_2)^{*}c(E_3) = c([E_3,E_2]) = c(E_1) = \lambda_1 = 0,$$

$$(\mathrm{ad}E_3)^{*}c(E_1) = c([E_1,E_3]) = c(E_2) = \lambda_2 = 0,$$

$$(\mathrm{ad}E_3)^{*}c(E_2) = c([E_2,E_3]) = -c(E_1) = -\lambda_1 = 0,$$

$$(\mathrm{ad}E_3)^{*}c(E_3) = c([E_3,E_3]) = c(0) = 0.$$

Hence $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Consequently,

$$H^1\left(\mathfrak{so}(3),\mathbb{R}\right) \cong H_{dR}^1\left(SO(3)\right) \cong \left(\left(\mathfrak{so}(3)\right)^{*}\right)^{\mathfrak{so}(3)} = 0.$$

Next, we compute $H^2\left(\mathfrak{so}(3),\mathbb{R}\right) \cong H_{dR}^2\left(SO(3)\right) \cong \left(\left(\Lambda^2\mathfrak{so}(3)\right)^{*}\right)^{\mathfrak{so}(3)}$. For any $c \in \left(\left(\Lambda^2\mathfrak{so}(3)\right)^{*}\right)^{\mathfrak{so}(3)}$, $c:\mathfrak{so}(3)^2\to\mathbb{R}$ is a bilinear, alternating, $\mathfrak{so}(3)$-invariant map, *i.e.*,

$$\rho(E_m)c(E_i,E_j) = (\mathrm{ad}E_m)^{*}c(E_i,E_j) = c([E_i,E_m],E_j) + c(E_i,[E_j,E_m]) = 0,$$

for $1 \le m \le 3$, $1 \le i < j \le 3$. Write

$$c=\sum_{1\le i<j\le 3}\lambda_{ij}\left(E_i^*\wedge E_j^*\right),\ \left(E_i^*\wedge E_j^*\right)\left(E_k,E_l\right)=\delta_i^k\delta_j^l\left(1\le k<l\le 3\right),$$

it follows that

$$\left(\operatorname{ad}E_1\right)^*c\left(E_1,E_2\right)=c\left(\left[E_1,E_1\right],E_2\right)+c\left(E_1,\left[E_2,E_1\right]\right)=c\left(\left[E_1,E_3\right]\right)=\lambda_{13}=0,$$

$$\left(\operatorname{ad}E_1\right)^*c\left(E_1,E_3\right)=c\left(\left[E_1,E_1\right],E_2\right)+c\left(E_1,\left[E_3,E_1\right]\right)=-c\left(\left[E_1,E_2\right]\right)=-\lambda_{12}=0,$$

$$\left(\operatorname{ad}E_1\right)^*c\left(E_2,E_3\right)=c\left(\left[E_2,E_1\right],E_3\right)+c\left(E_2,\left[E_3,E_1\right]\right)=c\left(\left[E_3,E_3\right]\right)-c\left(E_2,E_2\right)=0,$$

$$\left(\operatorname{ad}E_2\right)^*c\left(E_1,E_2\right)=c\left(\left[E_1,E_2\right],E_2\right)+c\left(E_1,\left[E_2,E_2\right]\right)=-c\left(\left[E_3,E_2\right]\right)=\lambda_{23}=0.$$

This implies that $\lambda_{12}=\lambda_{13}=\lambda_{23}=0$. Thus,

$$H^2\left(\mathfrak{so}(3),\mathbb{R}\right)\cong H_{dR}^2\left(SO(3)\right)\cong((\Lambda^2\mathfrak{so}(3))^*)^{\mathfrak{so}(3)}=0.$$

Next, we compute $H^3\left(\mathfrak{so}(3),\mathbb{R}\right)\cong H_{dR}^3\left(SO(3)\right)\cong\left(\left(\Lambda^3\mathfrak{so}(3)\right)^*\right)^{\mathfrak{so}(3)}$. For any $c\in\left(\left(\Lambda^3\mathfrak{so}(3)\right)^*\right)^{\mathfrak{so}(3)}$, $c:\mathfrak{so}(3)^3\to\mathbb{R}$ is a 3-linear, alternating, $\mathfrak{so}(3)$- invariant map, *i.e.*,

$$\begin{aligned}\rho\left(E_m\right)c\left(E_i,E_j,E_k\right)&=\left(\operatorname{ad}E_m\right)^*c\left(E_i,E_j,E_k\right)\\&=c\left(\left[E_i,E_m\right],E_j,E_k\right)+c\left(E_i,\left[E_j,E_m\right],E_k\right)\\&\quad+c\left(E_i,E_j,\left[E_k,E_m\right]\right)\\&=0,\end{aligned}$$

for $1\le m\le 3$ and $1\le i<j<k\le 3$. According to $c=\sum_{1\le i<j<k\le 3}\lambda_{ijk}\left(E_i^*\wedge E_j^*\wedge E_k^*\right)$ and $\left(E_i^*\wedge E_j^*\wedge E_k^*\right)\left(E_s,E_t,E_l\right)=\delta_s^i\delta_t^j\delta_l^k\quad\left(1\le s<t<l\le 3\right)$, we obtain

$$\begin{aligned}&\left(\operatorname{ad}E_1\right)^*c\left(E_1,E_2,E_3\right)\\&=c\left(\left[E_1,E_1\right],E_2,E_3\right)+c\left(E_1,\left[E_2,E_1\right],E_3\right)+c\left(E_1,E_2,\left[E_3,E_1\right]\right)\\&=c\left(E_1,E_3,E_3\right)-c\left(E_1,E_2,E_2\right)\equiv 0,\end{aligned}$$

$$\begin{aligned}&\left(\operatorname{ad}E_2\right)^*c\left(E_1,E_2,E_3\right)\\&=c\left(\left[E_1,E_2\right],E_2,E_3\right)+c\left(E_1,\left[E_2,E_2\right],E_3\right)+c\left(E_1,E_2,\left[E_3,E_2\right]\right)\\&=-c\left(E_3,E_2,E_3\right)+c\left(E_1,E_2,E_1\right)\equiv 0,\end{aligned}$$

$$\begin{aligned}&\left(\operatorname{ad}E_3\right)^*c\left(E_1,E_2,E_3\right)\\&=c\left(\left[E_1,E_3\right],E_2,E_3\right)+c\left(E_1,\left[E_2,E_3\right],E_3\right)+c\left(E_1,E_2,\left[E_3,E_3\right]\right)\\&=-c\left(E_2,E_2,E_3\right)-c\left(E_1,E_1,E_3\right)\equiv 0.\end{aligned}$$

All three equations are automatically satisfied for any λ_{123}. This implies that λ_{123} is free. Consequently,

$$H^3\left(\mathfrak{so}(3),\mathbb{R}\right)\cong H_{dR}^3\left(SO(3)\right)\cong\left(\left(\Lambda^3\mathfrak{so}(3)\right)^*\right)^{\mathfrak{so}(3)}\cong\mathbb{R}.$$

Summarising the results for $k=1,2,3$:

$$H^k\left(\mathfrak{so}(3),\mathbb{R}\right)\cong H_{dR}^k\left(SO(3)\right)\cong\begin{cases}\mathbb{R} & k=3\\ 0 & k=1,2\end{cases}.$$

The Lie algebra cohomology of $SO(4)$, $SO(5)$ and $SO(6)$ is computed below using a method analogous to that employed for $SO(3)$. However, the increased dimensionality leads to substantial growth in computational complexity, rendering manual calculations impractical. Consequently, the results presented below were obtained via MATLAB.

Example 2 *Calculate the de Rham cohomology of* $SO(4)$ *and the Lie algebra cohomology of its Lie algebra* $\mathfrak{so}(4)$.

The Lie group $SO(4)$ is defined as

$$SO(4)=\left\{A\in M_4\left(\mathbb{R}\right)\mid A^tA=id,\det A=1\right\},$$

with Lie algebra

$$\mathfrak{so}(4)=\left\{X\in M_4\left(\mathbb{R}\right)\mid X^t=-X\right\}.$$

By Theorem 1 we have the isomorphisms

$$H^\bullet\left(\mathfrak{so}(4),\mathbb{R}\right)\cong H_{dR}^\bullet\left(SO(4)\right)\cong\left(\left(\Lambda^\bullet\mathfrak{so}(4)\right)^*\right)^{\mathfrak{so}(4)}.$$

A basis of $\mathfrak{so}(4)$ is given by the matrices

$$e_1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},e_2=\begin{pmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{pmatrix},e_3=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{pmatrix},$$

$$e_4=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix},e_5=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix},e_6=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},$$

whose Lie brackets satisfy

$$[e_1,e_2]=-e_4,\ [e_1,e_3]=-e_5,\ [e_1,e_4]=e_2,\ [e_1,e_5]=e_3,\ [e_1,e_6]=0,$$

$$[e_2,e_3]=-e_6,\ [e_2,e_4]=-e_1,\ [e_2,e_5]=0,\ [e_2,e_6]=e_3,\ [e_3,e_4]=0,$$

$$[e_3,e_5]=-e_1,\ [e_3,e_6]=-e_2,\ [e_4,e_5]=-e_6,\ [e_4,e_6]=e_5,\ [e_5,e_6]=-e_4.$$

We calculated the results below using MATLAB.

For H^1, all coefficients $\lambda_i=0(1\le i\le 6)$.

For H^2, all coefficients $\lambda_{ij}=0(1\le i<j\le 6)$.

For H^3, the coefficients satisfy

$$\lambda_{123}=\lambda_{145}=\lambda_{246}=\lambda_{356},$$
$$\lambda_{124}=\lambda_{135}=\lambda_{236}=\lambda_{456},$$

with all other $\lambda_{ijk} = 0(1 \le i < j < k \le 6)$.

For H^4, all coefficients $\lambda_{ijkl} = 0(1 \le i < j < k < l \le 6)$.

For H^5, all coefficients $\lambda_{ijkls} = 0(1 \le i < j < k < l < s \le 6)$.

For H^6, the coefficient λ_{123456} is a free parameter in $\mathbb{R}$.

Hence,

$$H^k(\mathfrak{so}(4), \mathbb{R}) \cong H^k_{dR}(SO(4)) \cong \begin{cases} 0 & k = 1,2,4,5 \\ \mathbb{R} & k = 6 \\ \mathbb{R}^2 & k = 3 \end{cases}.$$

Example 3 *Calculate the de Rham cohomology of* $SO(5)$ *and the Lie algebra cohomology of the Lie algebra* $\mathfrak{so}(5)$.

The Lie group $SO(5)$ is defined as

$$SO(5) = \{A \in M_5(\mathbb{R}) \mid A^t A = id, \det A = 1\},$$

with Lie algebra

$$\mathfrak{so}(5) = \{X \in M_5(\mathbb{R}) \mid X^t = -X\}.$$

By Theorem 1 we have the isomorphisms

$$H^\bullet(\mathfrak{so}(5), \mathbb{R}) \cong H^\bullet_{dR}(SO(5)) \cong \left(\left(\Lambda^\bullet \mathfrak{so}(5)\right)^*\right)^{\mathfrak{so}(5)}.$$

The Lie algebra $\mathfrak{so}(5)$ is spanned by the basis matrices

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, e_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$e_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

whose Lie brackets satisfy

$$[e_1, e_2] = -e_5,\ [e_1, e_3] = -e_6,\ [e_1, e_4] = -e_7,\ [e_1, e_5] = e_2,\ [e_1, e_6] = e_3,$$

$$[e_1,e_7]=e_4,\ [e_1,e_8]=0,\ [e_1,e_9]=0,\ [e_1,e_{10}]=0,\ [e_2,e_3]=-e_8,$$
$$[e_2,e_4]=-e_9,\ [e_2,e_5]=-e_1,\ [e_2,e_6]=0,\ [e_2,e_7]=0,\ [e_2,e_8]=e_3$$
$$[e_2,e_9]=e_4,\ [e_2,e_{10}]=0,\ [e_3,e_4]=-e_{10},\ [e_3,e_5]=0,\ [e_3,e_6]=-e_1,$$
$$[e_3,e_7]=0,\ [e_3,e_8]=-e_2,\ [e_3,e_9]=0,\ [e_3,e_{10}]=e_4,\ [e_4,e_5]=0,$$
$$[e_4,e_6]=0,\ [e_4,e_7]=-e_1,\ [e_4,e_8]=0,\ [e_4,e_9]=-e_2,\ [e_4,e_{10}]=-e_3,$$
$$[e_5,e_6]=-e_8,\ [e_5,e_7]=-e_9,\ [e_5,e_8]=e_6,\ [e_5,e_9]=e_7,\ [e_5,e_{10}]=0,$$
$$[e_6,e_7]=-e_{10},\ [e_6,e_8]=-e_5,\ [e_6,e_9]=0,\ [e_6,e_{10}]=e_7,\ [e_7,e_8]=0,$$
$$[e_7,e_9]=-e_5,\ [e_7,e_{10}]=-e_6,\ [e_8,e_9]=-e_{10},\ [e_8,e_{10}]=e_9,\ [e_9,e_{10}]=-e_8.$$

We calculated the results below using MATLAB.

For H^1, all coefficients $\lambda_i=0(1\le i\le 10)$.

For H^2, all coefficients $\lambda_{ij}=0(1\le i<j\le 10)$.

For H^3, the coefficients satisfy

$$\begin{aligned}\lambda_{125}&=\lambda_{136}=\lambda_{147}=\lambda_{238}\\&=\lambda_{249}=\lambda_{34,10}=\lambda_{568}\\&=\lambda_{579}=\lambda_{67,10}=\lambda_{89,10},\end{aligned}$$

with all other other $\lambda_{ijk}=0(1\le i<j<k\le 10)$.

For H^4, all coefficients $\lambda_{ijkl}=0(1\le i<j<k<l\le 10)$.

For H^5, all coefficients $\lambda_{ijkls}=0(1\le i<j<k<l<s\le 10)$.

For H^6, all coefficients $\lambda_{ijklst}=0(1\le i<j<k<l<s<t\le 10)$.

For H^7, the coefficients satisfy

$$\begin{aligned}\lambda_{1234567}&=\lambda_{1234589}=\lambda_{123468,10}=\lambda_{123479,10}\\&=\lambda_{1256789}=\lambda_{135678,10}=\lambda_{145679,10}\\&=\lambda_{235689,10}=\lambda_{245789,10}=\lambda_{346789,10},\end{aligned}$$

with all other $\lambda_{ijklstx}=0(1\le i<j<k<l<s<t<x\le 10)$.

For H^8, all coefficients $\lambda_{ijklstxp}=0(1\le i<j<k<l<s<t<x<p\le 10)$.

For H^9, all coefficients $\lambda_{ijklstxpz}=0(1\le i<j<k<l<s<t<x<p<z\le 10)$.

For H^{10}, the coefficient $\lambda_{123456789,10}$ is a free parameter in $\mathbb{R}$.

Hence,

$$H^k(\mathfrak{so}(5),\mathbb{R})\cong H^k_{dR}(SO(5))\cong\begin{cases}0 & k=1,2,4,5,6,8,9\\ \mathbb{R} & k=3,7,10\end{cases}.$$

Example 4 *Calculate the de Rham cohomology of* $SO(6)$ *and the Lie algebra cohomology of the Lie algebra* $\mathfrak{so}(6)$.

The Lie group $SO(6)$ is defined as

$$SO(6)=\{A\in M_6(\mathbb{R})\mid A^tA=id,\det A=1\},$$

with Lie algebra

$$\mathfrak{so}(6)=\left\{X\in M_6(\mathbb{R})\mid X^t=-X\right\}.$$

By Theorem 1 we have the isomorphisms

$$H^\bullet(\mathfrak{so}(6),\mathbb{R})\cong H^\bullet_{dR}(SO(6))\cong\left(\left(\Lambda^\bullet\mathfrak{so}(6)\right)^*\right)^{\mathfrak{so}(6)}.$$

The Lie algebra $\mathfrak{so}(6)$ is spanned by the basis matrices

$$e_1=\begin{pmatrix}0&1&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_2=\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_3=\begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$e_4=\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_5=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\end{pmatrix},e_6=\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$e_7=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_8=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_9=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\end{pmatrix},$$

$$e_{10}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_{11}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_{12}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\end{pmatrix},$$

$$e_{13}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&-1&0&0\\0&0&0&0&0&0\end{pmatrix},e_{14}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&-1&0&0\end{pmatrix},e_{15}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&-1&0\end{pmatrix},$$

whose Lie brackets satisfy

$$[e_1,e_2]=-e_6,\ [e_1,e_3]=-e_7,\ [e_1,e_4]=-e_8,\ [e_1,e_5]=-e_9,\ [e_1,e_6]=e_2,$$

$$[e_1,e_7]=e_3,\ [e_1,e_8]=e_4,\ [e_1,e_9]=e_5,\ [e_1,e_{10}]=0,\ [e_1,e_{11}]=0,$$

$$[e_1,e_{12}]=0,\ [e_1,e_{13}]=0,\ [e_1,e_{14}]=0,\ [e_1,e_{15}]=0,\ [e_2,e_3]=-e_{10},$$

$$[e_2,e_4]=-e_{11},\ [e_2,e_5]=-e_{12},\ [e_2,e_6]=-e_1,\ [e_2,e_7]=0,\ [e_2,e_8]=0,$$

$$[e_2,e_9]=0,\ [e_2,e_{10}]=e_3,\ [e_2,e_{11}]=e_4,\ [e_2,e_{12}]=e_5,\ [e_2,e_{13}]=0,$$
$$[e_2,e_{14}]=0,\ [e_2,e_{15}]=0,\ [e_3,e_4]=-e_{13},\ [e_3,e_5]=-e_{14},\ [e_3,e_6]=0,$$
$$[e_3,e_7]=-e_1,[e_3,e_8]=0,[e_3,e_9]=0,[e_3,e_{10}]=-e_2,[e_3,e_{11}]=0,$$
$$[e_3,e_{12}]=0,\ [e_3,e_{13}]=e_4,\ [e_3,e_{14}]=e_5,\ [e_3,e_{15}]=0,\ [e_4,e_5]=-e_{15},$$
$$[e_4,e_6]=0,\ [e_4,e_7]=0,\ [e_4,e_8]=-e_1,[e_4,e_9]=0,\ [e_4,e_{10}]=0,$$
$$[e_4,e_{11}]=-e_2,\ [e_4,e_{12}]=0,\ [e_4,e_{13}]=-e_3,\ [e_4,e_{14}]=0,\ [e_4,e_{15}]=e_5,$$
$$[e_5,e_6]=0,\ [e_5,e_7]=0,\ [e_5,e_8]=0,\ [e_5,e_9]=-e_1,\ [e_5,e_{10}]=0,$$
$$[e_5,e_{11}]=0,\ [e_5,e_{12}]=-e_2,\ [e_5,e_{13}]=0,\ [e_5,e_{14}]=-e_3,\ [e_5,e_{15}]=-e_4,$$
$$[e_6,e_7]=-e_{10},\ [e_6,e_8]=-e_{11},\ [e_6,e_9]=-e_{12},\ [e_6,e_{10}]=e_7,\ [e_6,e_{11}]=e_8,$$
$$[e_6,e_{12}]=e_9,\ [e_6,e_{13}]=0,\ [e_6,e_{14}]=0,\ [e_6,e_{15}]=0,\ [e_7,e_8]=-e_{13},$$
$$[e_7,e_9]=-e_{14},\ [e_7,e_{10}]=-e_6,\ [e_7,e_{11}]=0,\ [e_7,e_{12}]=0,\ [e_7,e_{13}]=e_8,$$
$$[e_7,e_{14}]=e_9,\ [e_7,e_{15}]=0,\ [e_8,e_9]=-e_{15},\ [e_8,e_{10}]=0,\ [e_8,e_{11}]=-e_6,$$
$$[e_8,e_{12}]=0,\ [e_8,e_{13}]=-e_7,\ [e_8,e_{14}]=0,\ [e_8,e_{15}]=e_9,\ [e_9,e_{10}]=0,$$
$$[e_9,e_{11}]=0,\ [e_9,e_{12}]=-e_6,\ [e_9,e_{13}]=0,\ [e_9,e_{14}]=-e_7,\ [e_9,e_{15}]=-e_8,$$
$$[e_{10},e_{11}]=-e_{13},\ [e_{10},e_{12}]=-e_{14},\ [e_{10},e_{13}]=e_{11},\ [e_{10},e_{14}]=e_{12},\ [e_{10},e_{15}]=0,$$
$$[e_{11},e_{12}]=-e_{15},\ [e_{11},e_{13}]=-e_{10},\ [e_{11},e_{14}]=0,\ [e_{11},e_{15}]=e_{12},\ [e_{12},e_{13}]=0,$$
$$[e_{12},e_{14}]=-e_{10},\ [e_{12},e_{15}]=-e_{11},\ [e_{13},e_{14}]=-e_{15},\ [e_{13},e_{15}]=e_{14},\ [e_{14},e_{15}]=-e_{13}.$$

We calculated the results below using MATLAB.

For H^1, all coefficients $\lambda_i=0(1\le i\le 15)$.

For H^2, all coefficients $\lambda_{ij}=0(1\le i<j\le 15)$.

For H^3, the coefficients satisfy

$$\begin{aligned}\lambda_{126}&=\lambda_{137}=\lambda_{148}=\lambda_{159}=\lambda_{23,10}=\lambda_{24,11}=\lambda_{25,12}=\lambda_{34,13}\\&=\lambda_{35,14}=\lambda_{45,15}=\lambda_{67,10}=\lambda_{68,11}=\lambda_{69,12}=\lambda_{78,13}\\&=\lambda_{79,14}=\lambda_{89,15}=\lambda_{10,11,13}=\lambda_{10,12,14}=\lambda_{11,12,15}=\lambda_{13,14,15},\end{aligned}$$

with all other $\lambda_{ijk}=0(1\le i<j<k\le 15)$.

For H^4, all coefficients $\lambda_{ijkl}=0(1\le i<j<k<l\le 15)$.

For H^5, the coefficients satisfy

$$\begin{aligned}\lambda_{12345}&=\lambda_{12389}=\lambda_{123,11,12}=\lambda_{123,13,14}=\lambda_{12479}=\lambda_{124,10,12}\\&=\lambda_{124,13,15}=\lambda_{12578}=\lambda_{125,10,11}=\lambda_{125,14,15}=\lambda_{13469}\\&=\lambda_{134,10,14}=\lambda_{134,11,15}=\lambda_{13568}=\lambda_{135,10,13}=\lambda_{135,12,15}\\&=\lambda_{14567}=\lambda_{145,11,13}=\lambda_{145,12,14}=\lambda_{16789}=\lambda_{167,11,12}\\&=\lambda_{167,13,14}=\lambda_{168,10,12}=\lambda_{168,13,15}=\lambda_{169,10,11}=\lambda_{169,14,15}\\&=\lambda_{178,10,14}=\lambda_{178,11,15}=\lambda_{179,10,13}=\lambda_{179,12,15}=\lambda_{189,11,13}\\&=\lambda_{189,12,14}=\lambda_{2346,12}=\lambda_{2347,14}=\lambda_{2348,15}=\lambda_{2356,11}\end{aligned}$$

$$
\begin{aligned}
&= \lambda_{2357,13} = \lambda_{2359,15} = \lambda_{2456,10} = \lambda_{2458,13} = \lambda_{2459,14} \\
&= \lambda_{2678,12} = \lambda_{2679,11} = \lambda_{2689,10} = \lambda_{26,10,11,12} = \lambda_{26,10,13,14} \\
&= \lambda_{26,11,13,15} = \lambda_{26,12,14,15} = \lambda_{27,10,11,14} = \lambda_{27,10,12,13} = \lambda_{28,10,11,15} \\
&= \lambda_{28,11,12,13} = \lambda_{29,10,12,15} = \lambda_{29,11,12,14} = \lambda_{3457,10} = \lambda_{3458,11} \\
&= \lambda_{3459,12} = \lambda_{3478,14} = \lambda_{3679,13} = \lambda_{36,10,11,14} = \lambda_{36,10,12,13} \\
&= \lambda_{3789,10} = \lambda_{37,10,11,12} = \lambda_{37,10,13,14} = \lambda_{37,11,13,15} = \lambda_{37,12,14,15} \\
&= \lambda_{38,10,13,15} = \lambda_{38,11,13,14} = \lambda_{39,10,14,15} = \lambda_{39,12,13,14} = \lambda_{4678,15} \\
&= \lambda_{4689,13} = \lambda_{46,10,11,15} = \lambda_{46,11,12,13} = \lambda_{4789,11} = \lambda_{47,10,13,15} \\
&= \lambda_{47,11,13,14} = \lambda_{48,10,11,12} = \lambda_{48,10,13,14} = \lambda_{48,11,13,15} = \lambda_{48,12,14,15} \\
&= \lambda_{49,11,14,15} = \lambda_{49,12,13,15} = \lambda_{5679,15} = \lambda_{5689,14} = \lambda_{56,10,12,15} \\
&= \lambda_{56,11,12,14} = \lambda_{5689,12} = \lambda_{57,10,14,15} = \lambda_{57,12,13,14} = \lambda_{58,11,14,15} \\
&= \lambda_{58,12,13,15} = \lambda_{59,10,11,12} = \lambda_{59,10,13,14} = \lambda_{59,11,13,15} = \lambda_{59,12,14,15},
\end{aligned}
$$

with all other $\lambda_{ijkls} = 0\left(1 \le i < j < k < l < s \le 15\right)$.

For H^6, all coefficients $\lambda_{ijklst} = 0, 1 \le i < j < k < l < s < t \le 15$.

For H^7, the coefficients satisfy

$$
\begin{aligned}
\lambda_{1234678} &= \lambda_{12346,10,11} = \lambda_{12346,14,15} = \lambda_{12347,10,13} = \lambda_{12347,12,15} = \lambda_{12348,11,13} \\
&= \lambda_{12348,12,14} = \lambda_{12349,10,15} = \lambda_{12349,11,14} = \lambda_{12349,12,13} = \lambda_{1235679} \\
&= \lambda_{12356,10,12} = \lambda_{12356,13,15} = \lambda_{12357,10,14} = \lambda_{12357,11,15} = \lambda_{12358,10,15} \\
&= \lambda_{12358,11,14} = \lambda_{12358,12,13} = \lambda_{12359,11,13} = \lambda_{12359,12,14} = \lambda_{1245689} \\
&= \lambda_{12456,11,12} = \lambda_{12456,13,14} = \lambda_{12457,10,15} = \lambda_{12457,11,14} = \lambda_{12457,12,13} \\
&= \lambda_{12458,10,14} = \lambda_{12458,11,15} = \lambda_{12459,10,13} = \lambda_{12459,12,15} = \lambda_{12678,10,11} \\
&= \lambda_{12678,14,15} = \lambda_{12679,10,12} = \lambda_{12679,13,15} = \lambda_{12689,11,12} = \lambda_{12689,13,14} \\
&= \lambda_{126,10,11,14,15} = \lambda_{126,10,12,13,15} = \lambda_{126,11,12,13,14} = \lambda_{12789,10,15} = \lambda_{12789,11,14} \\
&= \lambda_{12789,12,13} = \lambda_{127,10,11,12,15} = \lambda_{127,10,13,14,15} = \lambda_{128,10,11,12,14} = \lambda_{128,11,13,14,15} \\
&= \lambda_{129,10,11,12,13} = \lambda_{129,12,13,14,15} = \lambda_{13456,10,15} = \lambda_{13456,11,14} = \lambda_{13456,12,13} \\
&= \lambda_{1345789} = \lambda_{13457,11,12} = \lambda_{13457,13,14} = \lambda_{13458,10,12} = \lambda_{13458,13,15} \\
&= \lambda_{13459,10,11} = \lambda_{13459,14,15} = \lambda_{13678,10,13} = \lambda_{13678,12,15} = \lambda_{13679,10,14} \\
&= \lambda_{13679,11,15} = \lambda_{13689,10,15} = \lambda_{13689,11,14} = \lambda_{13689,12,13} = \lambda_{136,10,11,12,15} \\
&= \lambda_{136,10,13,14,15} = \lambda_{13789,11,12} = \lambda_{13789,13,14} = \lambda_{137,10,11,14,15} = \lambda_{137,10,12,13,15} \\
&= \lambda_{137,11,12,13,14} = \lambda_{138,10,12,13,14} = \lambda_{138,11,12,13,15} = \lambda_{139,10,11,13,14} = \lambda_{139,11,12,14,15} \\
&= \lambda_{14678,11,13} = \lambda_{14678,12,14} = \lambda_{14679,10,15} = \lambda_{14679,11,14} = \lambda_{14679,12,13} \\
&= \lambda_{14689,10,14} = \lambda_{14689,11,15} = \lambda_{146,10,11,12,14} = \lambda_{146,11,13,14,15} = \lambda_{14789,10,12} \\
&= \lambda_{14789,13,15} = \lambda_{147,10,12,13,14} = \lambda_{147,11,12,13,15} = \lambda_{148,10,11,14,15} = \lambda_{148,10,12,13,15} \\
&= \lambda_{148,11,12,13,14} = \lambda_{149,10,11,13,15} = \lambda_{149,10,12,14,15} = \lambda_{15678,10,15} = \lambda_{15678,11,14} \\
&= \lambda_{15678,12,13} = \lambda_{15679,11,13} = \lambda_{15679,12,14} = \lambda_{15689,10,13} = \lambda_{15689,12,15} \\
&= \lambda_{156,10,11,12,13} = \lambda_{156,12,13,14,15} = \lambda_{15789,10,11} = \lambda_{15789,14,15} = \lambda_{157,10,11,13,14} \\
&= \lambda_{157,11,12,14,15} = \lambda_{158,10,11,13,15} = \lambda_{158,10,12,14,15} = \lambda_{159,10,11,14,15} = \lambda_{159,10,12,13,15} \\
&= \lambda_{159,11,12,13,14} = \lambda_{234567,15} = \lambda_{234568,14} = \lambda_{234569,13} = \lambda_{234578,12} \\
&= \lambda_{234579,11} = \lambda_{234589,10} = \lambda_{2345,10,11,12} = \lambda_{2345,10,1314} = \lambda_{2345,11,13,15} \\
&= \lambda_{2345,12,14,15} = \lambda_{236789,15} = \lambda_{2367,10,11,13} = \lambda_{2367,10,12,14} = \lambda_{2367,11,12,15}
\end{aligned}
$$

$$= \lambda_{2367,13,14,15} = \lambda_{2368,10,12,15} = \lambda_{2368,11,12,14} = \lambda_{2369,10,11,15} = \lambda_{2369,11,12,13}$$
$$= \lambda_{2378,10,14,15} = \lambda_{2378,12,13,14} = \lambda_{2379,10,13,15} = \lambda_{2379,11,13,14} = \lambda_{2389,10,11,12}$$
$$= \lambda_{238910,13,14} = \lambda_{2389,11,13,15} = \lambda_{2389,12,14,15} = \lambda_{23,10,11,12,13,14} = \lambda_{246789,14}$$
$$= \lambda_{2467,10,12,15} = \lambda_{2467,11,12,14} = \lambda_{2468,10,11,13} = \lambda_{2468,10,12,14} = \lambda_{2468,11,12,15}$$
$$= \lambda_{2468,13,14,15} = \lambda_{246910,11,14} = \lambda_{246910,12,13} = \lambda_{2478,11,14,15} = \lambda_{2478,12,13,15}$$
$$= \lambda_{2479,10,11,12} = \lambda_{247910,13,14} = \lambda_{2479,11,13,15} = \lambda_{2479,12,14,15} = \lambda_{248910,13,15}$$
$$= \lambda_{2489,11,13,14} = \lambda_{24,10,11,12,13,15} = \lambda_{256789,13} = \lambda_{2567,10,11,15} = \lambda_{2567,11,12,13}$$
$$= \lambda_{2568,10,11,14} = \lambda_{2568,10,12,13} = \lambda_{2569,10,11,13} = \lambda_{2569,10,12,14} = \lambda_{2569,11,12,15}$$
$$= \lambda_{2569,13,14,15} = \lambda_{2578,10,11,12} = \lambda_{2578,10,13,14} = \lambda_{2578,11,13,15} = \lambda_{2578,12,14,15}$$
$$= \lambda_{2579,11,14,15} = \lambda_{2579,12,13,15} = \lambda_{2589,10,14,15} = \lambda_{2589,12,13,14} = \lambda_{25,10,11,12,14,15}$$
$$= \lambda_{346789,12} = \lambda_{3467,10,14,15} = \lambda_{3467,12,13,14} = \lambda_{3468,11,14,15} = \lambda_{3468,12,13,15}$$
$$= \lambda_{3469,10,11,12} = \lambda_{3469,10,13,14} \lambda_{3469,11,13,15} = \lambda_{3469,12,14,15} = \lambda_{3478,10,11,13}$$
$$= \lambda_{3478,10,12,14} = \lambda_{3478,11,12,15} = \lambda_{3478,13,14,15} = \lambda_{347910,11,14} = \lambda_{3479,10,12,13}$$
$$= \lambda_{3489,10,11,15} = \lambda_{3489,11,12,13} = \lambda_{34,10,11,13,14,15} = \lambda_{356789,11} = \lambda_{3567,10,13,15}$$
$$= \lambda_{3567,11,13,14} = \lambda_{3568,10,11,12} = \lambda_{3568,10,13,14} = \lambda_{3568,11,13,15} = \lambda_{3568,12,14,15}$$
$$= \lambda_{3569,11,14,15} = \lambda_{3569,12,13,15} = \lambda_{3578,10,11,14} = \lambda_{3578,10,12,13} = \lambda_{3579,10,11,13}$$
$$= \lambda_{3579,10,12,14} = \lambda_{3579,11,12,15} = \lambda_{3579,13,14,15} = \lambda_{3589,10,12,15} = \lambda_{3589,11,12,14}$$
$$= \lambda_{35,10,12,13,14,15} = \lambda_{456789,10} = \lambda_{4567,10,11,12} = \lambda_{4567,10,13,14} = \lambda_{4567,11,13,15}$$
$$= \lambda_{4567,12,14,15} = \lambda_{4568,10,13,15} = \lambda_{4568,11,13,14} = \lambda_{4569,10,14,15} = \lambda_{4569,12,13,14}$$
$$= \lambda_{4578,10,11,15} = \lambda_{4578,11,12,13} = \lambda_{4579,10,12,15} = \lambda_{457,9,11,12,14} = \lambda_{4589,10,11,13}$$
$$= \lambda_{4589,10,12,14} = \lambda_{4589,11,12,15} = \lambda_{4589,13,14,15} = \lambda_{45,11,12,13,14,15} = \lambda_{6789,10,11,12}$$
$$= \lambda_{6789,10,13,14} = \lambda_{6789,11,13,15} = \lambda_{6789,12,14,15} = \lambda_{67,10,11,12,13,14} = \lambda_{68,10,11,12,13,15}$$
$$= \lambda_{69,10,11,12,14,15} = \lambda_{78,10,11,13,14,15} = \lambda_{79,10,12,13,14,15} = \lambda_{89,11,12,13,14,15},$$

with all other $\lambda_{ijklstx} = 0 \left(1 \le i < j < k < l < s < t < x \le 15\right)$.

For H^8, the coefficients satisfy

$$\lambda_{1234567,10} = \lambda_{1234568,11} = \lambda_{1234569,12} = \lambda_{1234578,13} = \lambda_{1234579,14}$$
$$= \lambda_{1234589,15} = \lambda_{12345,10,11,13} = \lambda_{12345,10,12,14} = \lambda_{12345,11,12,15}$$
$$= \lambda_{12345,13,14,15} = \lambda_{1236789,10} = \lambda_{12367,10,11,12} = \lambda_{12367,10,13,14}$$
$$= \lambda_{12367,11,13,15} = \lambda_{12367,12,14,15} = \lambda_{12368,10,13,15} = \lambda_{12368,11,13,14}$$
$$= \lambda_{12369,10,14,15} = \lambda_{12369,12,13,14} = \lambda_{12378,10,11,15} = \lambda_{12378,11,12,13}$$
$$= \lambda_{12379,10,12,15} = \lambda_{12379,11,12,14} = \lambda_{12389,10,11,13} = \lambda_{12389,10,12,14}$$
$$= \lambda_{12389,11,12,15} = \lambda_{12389,13,14,15} = \lambda_{123,11,12,13,14,15} = \lambda_{1246789,11}$$
$$= \lambda_{12467,10,13,15} = \lambda_{12467,11,13,14} = \lambda_{12468,10,11,12} = \lambda_{12468,10,13,14}$$
$$= \lambda_{12468,11,13,15} = \lambda_{12468,12,14,15} = \lambda_{12469,11,14,15} = \lambda_{12469,12,13,15}$$
$$= \lambda_{12478,10,11,14} = \lambda_{12478,10,12,13} = \lambda_{12479,10,11,13} = \lambda_{12479,10,12,14}$$
$$= \lambda_{12479,11,12,15} = \lambda_{12479,13,14,15} = \lambda_{12489,10,12,15} = \lambda_{12489,11,12,14}$$
$$= \lambda_{124,10,12,13,14,15} = \lambda_{1256789,12} = \lambda_{12567,10,14,15} = \lambda_{12567,12,13,14}$$
$$= \lambda_{12568,11,14,15} = \lambda_{12568,12,13,15} = \lambda_{12569,10,11,12} = \lambda_{12569,10,13,14}$$
$$= \lambda_{12569,11,13,15} = \lambda_{12569,12,14,15} = \lambda_{12578,10,11,13} = \lambda_{12578,10,12,14}$$
$$= \lambda_{12578,11,12,15} = \lambda_{12578,13,14,15} = \lambda_{12579,10,11,14} = \lambda_{12579,10,12,13}$$

$$= \lambda_{12589,10,11,15} = \lambda_{12589,11,12,13} = \lambda_{125,10,11,13,14,15} = \lambda_{1346789,13}$$
$$= \lambda_{13467,10,11,15} = \lambda_{13467,11,12,13} = \lambda_{13468,10,11,14} = \lambda_{13468,10,12,13}$$
$$= \lambda_{13469,10,11,13} = \lambda_{13469,10,12,14} = \lambda_{13469,11,12,15} = \lambda_{13469,13,14,15}$$
$$= \lambda_{13478,10,11,12} = \lambda_{13478,10,13,14} = \lambda_{13478,11,13,15} = \lambda_{13478,12,14,15}$$
$$= \lambda_{13479,11,14,15} = \lambda_{13479,12,13,15} = \lambda_{13489,10,14,15} = \lambda_{13489,12,13,14}$$
$$= \lambda_{134,10,11,12,14,15} = \lambda_{1356789,14} = \lambda_{13567,10,12,15} = \lambda_{13567,11,12,14}$$
$$= \lambda_{13568,10,11,13} = \lambda_{13568,10,12,14} = \lambda_{13568,11,12,15} = \lambda_{13568,13,14,15}$$
$$= \lambda_{13569,10,11,14} = \lambda_{13569,10,12,13} = \lambda_{13578,11,14,15} = \lambda_{13578,12,13,15}$$
$$= \lambda_{13579,10,11,12} = \lambda_{13579,10,13,14} = \lambda_{13579,11,13,15} = \lambda_{13579,12,14,15}$$
$$= \lambda_{13589,10,13,15} = \lambda_{13589,11,13,14} = \lambda_{135,10,11,12,13,15} = \lambda_{1456789,15}$$
$$= \lambda_{14567,10,11,13} = \lambda_{14567,10,12,14} = \lambda_{14567,11,12,15} = \lambda_{14567,13,14,15}$$
$$= \lambda_{14568,10,12,15} = \lambda_{14568,11,12,14} = \lambda_{14569,10,11,15} = \lambda_{14569,11,12,13}$$
$$= \lambda_{14578,10,14,15} = \lambda_{14578,12,13,14} = \lambda_{14579,10,13,15} = \lambda_{14579,11,13,14}$$
$$= \lambda_{14589,10,11,12} = \lambda_{14589,10,13,14} = \lambda_{14589,11,13,15} = \lambda_{14589,12,14,15}$$
$$= \lambda_{145,10,11,12,13,14} = \lambda_{16789,10,11,13} = \lambda_{16789,10,12,14} = \lambda_{16789,11,12,15}$$
$$= \lambda_{16789,13,14,15} = \lambda_{167,11,12,13,14,15} = \lambda_{168,10,12,13,14,15} = \lambda_{169,10,11,13,14,15}$$
$$= \lambda_{178,10,11,12,14,15} = \lambda_{179,10,11,12,13,15} = \lambda_{189,10,11,12,13,14} = \lambda_{234678,10,15}$$
$$= \lambda_{234678,11,14} = \lambda_{234678,12,13} = \lambda_{234679,11,13} = \lambda_{234679,12,14}$$
$$= \lambda_{234689,10,13} = \lambda_{234689,12,15} = \lambda_{2346,10,11,12,13} = \lambda_{2346,12,13,14,15}$$
$$= \lambda_{234789,10,11} = \lambda_{234789,14,15} = \lambda_{2347,10,11,13,14} = \lambda_{2347,11,12,14,15}$$
$$= \lambda_{2348,10,11,13,15} = \lambda_{2348,10,12,14,15} = \lambda_{2349,10,11,14,15} = \lambda_{2349,10,12,13,15}$$
$$= \lambda_{2349,11,12,13,14} = \lambda_{235678,11,13} = \lambda_{235678,12,14} = \lambda_{235679,10,15}$$
$$= \lambda_{235679,11,14} = \lambda_{235679,12,13} = \lambda_{235689,10,14} = \lambda_{235689,11,15}$$
$$= \lambda_{2356,10,11,12,14} = \lambda_{2356,11,13,14,15} = \lambda_{235789,10,12} = \lambda_{235789,13,15}$$
$$= \lambda_{2357,10,12,13,14} = \lambda_{2357,11,12,13,15} = \lambda_{2358,10,11,14,15} = \lambda_{2358,10,12,13,15}$$
$$= \lambda_{2358,11,12,13,14} = \lambda_{2359,10,11,13,15} = \lambda_{2359,10,12,14,15} = \lambda_{245678,10,13}$$
$$= \lambda_{245678,12,15} = \lambda_{245679,10,14} = \lambda_{245679,11,15} = \lambda_{245689,10,15}$$
$$= \lambda_{245689,11,14} = \lambda_{245689,12,13} = \lambda_{2456,10,11,12,15} = \lambda_{2456,10,13,14,15}$$
$$= \lambda_{245789,11,12} = \lambda_{245789,13,14} = \lambda_{2457,10,11,14,15} = \lambda_{2457,10,12,13,15}$$
$$= \lambda_{2457,11,12,13,14} = \lambda_{2458,10,12,13,14} = \lambda_{2458,11,12,13,15} = \lambda_{2459,10,11,13,14}$$
$$= \lambda_{2459,11,12,14,15} = \lambda_{2678,10,11,12,13} = \lambda_{2678,12,13,14,15} = \lambda_{2679,10,11,12,14}$$
$$= \lambda_{2679,11,13,14,15} = \lambda_{2689,10,11,12,15} = \lambda_{2689,10,13,14,15} = \lambda_{26,10,11,12,13,14,15}$$
$$= \lambda_{2789,10,11,14,15} = \lambda_{2789,10,12,13,15} = \lambda_{2789,11,12,13,14} = \lambda_{345678,10,11}$$
$$= \lambda_{345678,14,15} = \lambda_{345679,10,12} = \lambda_{345679,13,15} = \lambda_{345689,11,12}$$
$$= \lambda_{345689,13,14} = \lambda_{3456,10,11,14,15} = \lambda_{3456,10,12,13,15} = \lambda_{3456,11,12,13,14}$$
$$= \lambda_{345789,10,15} = \lambda_{345789,11,14} = \lambda_{345789,12,13} = \lambda_{3457,10,11,12,15}$$
$$= \lambda_{3457,10,13,14,15} = \lambda_{3458,10,11,12,14} = \lambda_{3458,11,13,14,15} = \lambda_{3459,10,11,12,13}$$
$$= \lambda_{3459,12,13,14,15} = \lambda_{3678,10,11,13,14} = \lambda_{3678,11,12,14,15} = \lambda_{3679,10,12,13,14}$$
$$= \lambda_{3679,11,12,13,15} = \lambda_{3689,10,11,14,15} = \lambda_{3689,10,12,13,15} = \lambda_{3689,11,12,13,14}$$
$$= \lambda_{3789,10,11,12,15} = \lambda_{3789,10,13,14,15} = \lambda_{37,10,11,12,13,14,15} = \lambda_{4678,10,11,13,15}$$

$$\begin{aligned}
&= \lambda_{4678,10,12,14,15} = \lambda_{4679,10,11,14,15} = \lambda_{4679,10,12,13,15} = \lambda_{4679,11,12,13,14} \\
&= \lambda_{4689,10,12,13,14} = \lambda_{4689,11,12,13,15} = \lambda_{4789,10,11,12,14} = \lambda_{4789,11,13,14,15} \\
&= \lambda_{48,10,11,12,13,14,15} = \lambda_{5678,10,11,14,15} = \lambda_{5678,10,12,13,15} = \lambda_{5678,11,12,13,14} \\
&= \lambda_{5679,10,11,13,15} = \lambda_{5679,10,12,14,15} = \lambda_{5689,10,11,13,14} = \lambda_{5689,11,12,14,15} \\
&= \lambda_{5789,10,11,12,13} = \lambda_{5789,12,13,14,15} = \lambda_{59,10,11,12,13,14,15},
\end{aligned}$$

with all other $\lambda_{ijklstxp} = 0\left(1 \le i < j < k < l < s < t < x < p \le 15\right)$.

For H^9, all coefficients $\lambda_{ijklstxpz} = 0\left(1 \le i < j < k < l < s < t < x < p < z \le 15\right)$.

For H^{10}, the coefficients satisfy

$$\begin{aligned}
\lambda_{1234678,10,11,13} &= \lambda_{1234678,10,12,14} = \lambda_{1234678,11,12,15} = \lambda_{1234678,13,14,15} \\
&= \lambda_{1234679,10,11,14} = \lambda_{1234679,10,12,13} = \lambda_{1234689,10,11,15} \\
&= \lambda_{1234689,11,12,13} = \lambda_{12346,10,11,13,14,15} = \lambda_{1234789,10,13,15} \\
&= \lambda_{1234789,11,13,14} = \lambda_{12347,10,11,12,13,15} = \lambda_{12348,10,11,12,13,14} \\
&= \lambda_{1235678,10,11,14} = \lambda_{1235678,10,12,13} = \lambda_{1235679,10,11,13} \\
&= \lambda_{1235679,10,12,14} = \lambda_{1235679,11,12,15} = \lambda_{1235679,13,14,15} \\
&= \lambda_{1235689,10,12,15} = \lambda_{1235689,11,12,14} = \lambda_{12356,10,12,13,14,15} \\
&= \lambda_{1235789,10,14,15} = \lambda_{1235789,12,13,14} = \lambda_{12357,10,11,12,14,15} \\
&= \lambda_{12359,10,11,12,13,14} = \lambda_{1245678,10,11,15} = \lambda_{1245678,11,12,13} \\
&= \lambda_{1245679,10,12,15} = \lambda_{1245679,11,12,14} = \lambda_{1245689,10,11,13} \\
&= \lambda_{1245689,10,12,14} = \lambda_{1245689,11,12,15} = \lambda_{1245689,13,14,15} \\
&= \lambda_{12456,11,12,13,14,15} = \lambda_{1245789,11,14,15} = \lambda_{1245789,12,13,15} \\
&= \lambda_{12458,10,11,12,14,15} = \lambda_{12459,10,11,12,13,15} = \lambda_{12678,10,11,13,14,15} \\
&= \lambda_{12679,10,12,13,14,15} = \lambda_{12689,11,12,13,14,15} = \lambda_{1345678,10,13,15} \\
&= \lambda_{1345678,11,13,14} = \lambda_{1345679,10,14,15} = \lambda_{1345679,12,13,14} \\
&= \lambda_{1345689,11,14,15} = \lambda_{1345689,12,13,15} = \lambda_{1345789,10,11,13} \\
&= \lambda_{1345789,10,12,14} = \lambda_{1345789,11,12,15} = \lambda_{1345789,13,14,15} \\
&= \lambda_{13457,11,12,13,14,15} = \lambda_{13458,10,12,13,14,15} = \lambda_{13459,10,11,13,14,15} \\
&= \lambda_{13678,10,11,12,13,15} = \lambda_{13679,10,11,12,14,15} = \lambda_{13789,11,12,13,14,15} \\
&= \lambda_{14678,10,11,12,13,14} = \lambda_{14689,10,11,12,14,15} = \lambda_{14789,10,12,13,1415} \\
&= \lambda_{15679,10,11,12,13,14} = \lambda_{15689,10,11,12,13,15} = \lambda_{15789,10,11,13,14,15} \\
&= \lambda_{234567,10,11,13,15} = \lambda_{234567,10,12,14,15} = \lambda_{234568,10,11,13,14} \\
&= \lambda_{234568,11,12,14,15} = \lambda_{234569,10,12,13,14} = \lambda_{234569,11,12,13,15} \\
&= \lambda_{234578,10,11,12,13} = \lambda_{234578,12,13,14,15} = \lambda_{234579,10,11,12,14} \\
&= \lambda_{234579,11,13,14,15} = \lambda_{234589,10,11,12,15} = \lambda_{234589,10,13,14,15} \\
&= \lambda_{2345,10,11,12,13,14,15} = \lambda_{236789,10,11,13,15} = \lambda_{236789,10,12,14,15} \\
&= \lambda_{2389,10,11,12,13,14,15} = \lambda_{246789,10,11,13,14} = \lambda_{246789,11,12,14,15} \\
&= \lambda_{2479,10,11,12,13,14,15} = \lambda_{256789,10,12,13,14} = \lambda_{256789,11,12,13,15} \\
&= \lambda_{2578,10,11,12,13,14,15} = \lambda_{346789,10,11,12,13} = \lambda_{346789,12,13,14,15} \\
&= \lambda_{3469,10,11,12,13,14,15} = \lambda_{356789,10,11,12,14} = \lambda_{356789,11,13,14,15} \\
&= \lambda_{3568,10,11,12,13,14,15} = \lambda_{456789,10,11,12,15} = \lambda_{456789,10,13,14,15} \\
&= \lambda_{4567,10,11,12,13,14,15} = \lambda_{6789,10,11,12,13,14,15},
\end{aligned}$$

with all other $\lambda_{ijklstxpzd} = 0(1 \le i < j < k < l < s < t < x < p < z < d \le 15)$.

For H^{11}, all coefficients $\lambda_{ijklstxpzdq} = 0(1 \le i < j < k < l < s < t < x < p < z < d < q \le 15)$.

For H^{12}, the coefficients satisfy

$$\begin{aligned}\lambda_{123456789,10,11,12} &= \lambda_{123456789,10,13,14} = \lambda_{123456789,11,13,15} = \lambda_{123456789,12,14,15} \\ &= \lambda_{1234567,10,11,12,13,14} = \lambda_{1234568,10,11,12,13,15} = \lambda_{1234569,10,11,12,14,15} \\ &= \lambda_{1234578,10,11,13,14,15} = \lambda_{1234579,10,12,13,14,15} = \lambda_{1234589,11,12,13,14,15} \\ &= \lambda_{1236789,10,11,12,13,14} = \lambda_{1246789,10,11,12,13,15} = \lambda_{1256789,10,11,12,14,15} \\ &= \lambda_{1346789,10,11,13,14,15} = \lambda_{1356789,10,12,13,14,15} = \lambda_{1456789,11,12,13,14,15} \\ &= \lambda_{234678,10,11,12,13,14,15} = \lambda_{235679,10,11,12,13,14,15} = \lambda_{245689,10,11,12,13,14,15} \\ &= \lambda_{345789,10,11,12,13,14,15},\end{aligned}$$

with all other $\lambda_{ijklstxpzdqb} = 0(1 \le i < j < k < l < s < t < x < p < z < d < q < b \le 15)$.

For H^{13}, all coefficients $\lambda_{ijklstxpzdqbc} = 0(1 \le i < j < k < l < s < t < x < p < z < d < q < b < c \le 15)$.

For H^{14}, all coefficients $\lambda_{ijklstxpzdqbcg} = 0(1 \le i < j < k < l < s < t < x < p < z < d < q < b < c < g \le 15)$.

For H^{15}, the coefficient $\lambda_{123456789,10,11,12,13,14,15}$ is a free parameter $\mathbb{R}$.

Hence,

$$H^k(\mathfrak{so}(6), \mathbb{R}) \cong H^k_{dR}(SO(6)) \cong \begin{cases} 0 & k = 1,2,4,6,9,11,13,14 \\ \mathbb{R} & k = 3,5,7,8,10,12,15 \end{cases}.$$

Example 5 *Calculate the de Rham cohomology of* $SU(3)$ *and the Lie algebra cohomology of the Lie algebra* $\mathfrak{su}(3)$.

The Lie group $SU(3)$ is defined as

$$SU(3) = \{U \in \mathbb{C}^{3\times3} \mid U^H U = id, \det U = 1\},$$

with Lie algebra

$$\mathfrak{so}(3) = \{X \in \mathbb{C}^{3\times3} \mid X^H = -X\}.$$

By Theorem 1 we have isomorphisms

$$H^\bullet(\mathfrak{su}(3), \mathbb{R}) \cong H^\bullet_{dR}(SU(3)) \cong \left((\Lambda^\bullet \mathfrak{su}(3))^*\right)^{\mathfrak{su}(3)}.$$

The Lie algebra $\mathfrak{su}(3)$ is spanned by the basis matrices

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ e_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\ e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ e_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

whose Lie brackets satisfy

$$[e_1,e_2] = 2ie_3,\ [e_1,e_3] = -2ie_2,\ [e_1,e_4] = ie_7,\ [e_1,e_5] = -ie_6,$$
$$[e_1,e_6] = ie_5,\ [e_1,e_7] = -ie_4,\ [e_1,e_8] = 0,\ [e_2,e_3] = 2ie_1,$$
$$[e_2,e_4] = ie_6,\ [e_2,e_5] = ie_7,\ [e_2,e_6] = -ie_4,\ [e_2,e_7] = -ie_5,$$
$$[e_2,e_8] = 0,\ [e_3,e_4] = ie_5,\ [e_3,e_5] = -ie_4,\ [e_3,e_6] = -ie_7,$$
$$[e_3,e_7] = ie_6,\ [e_3,e_8] = 0,\ [e_4,e_5] = ie_3 + \sqrt{3}ie_8,\ [e_4,e_6] = ie_2,$$
$$[e_4,e_7] = ie_1,\ [e_4,e_8] = -\sqrt{3}ie_5,\ [e_5,e_6] = -ie_1,\ [e_5,e_7] = ie_2,$$
$$[e_5,e_8] = \sqrt{3}ie_4,\ [e_6,e_7] = -ie_3 + \sqrt{3}ie_8,\ [e_6,e_8] = -\sqrt{3}ie_7,\ [e_7,e_8] = \sqrt{3}ie_6,$$

We calculated the results below using MATLAB.

For H^1, all coefficients $\lambda_i = 0\,(1 \le i \le 8)$.

For H^2, all coefficients $\lambda_{ij} = 0\,(1 \le i < j \le 8)$.

For H^3, the coefficients satisfy

$$\begin{aligned}\lambda_{123} &= \lambda_{147} = \lambda_{156} = \lambda_{246} = \lambda_{257} \\ &= \lambda_{345} = \lambda_{367} = \lambda_{458} = \lambda_{678},\end{aligned}$$

with all other $\lambda_{ijk} = 0\,(1 \le i < j < k \le 8)$.

For H^4, all coefficients $\lambda_{ijkl} = 0\,(1 \le i < j < k < l \le 8)$.

For H^5, the coefficients satisfy

$$\begin{aligned}\lambda_{12345} &= \lambda_{12367} = \lambda_{12458} = \lambda_{12678} = \lambda_{13468} \\ &= \lambda_{13578} = \lambda_{23478} = \lambda_{23568} = \lambda_{45678},\end{aligned}$$

with all other $\lambda_{ijkls} = 0\,(1 \le i < j < k < l < s \le 8)$.

For H^6, all coefficients $\lambda_{ijklst} = 0\,(1 \le i < j < k < l < s < t \le 8)$.

For H^7, all coefficients $\lambda_{ijklstx} = 0\,(1 \le i < j < k < l < s < t < x \le 8)$.

For H^8, the coefficient $\lambda_{12345678}$ is a free parameter in $\mathbb{R}$.

Hence,

$$H^k(\mathfrak{su}(3),\mathbb{R}) \cong H^k_{dR}(SU(3)) \cong \begin{cases} 0 & k = 1,2,4,6,7 \\ \mathbb{R} & k = 3,5,8 \end{cases}.$$

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Chevalley, C. and Eilenberg, S. (1948) Cohomology Theory of Lie Groups and Lie

Algebras. *Transactions of the American Mathematical Society*, **63**, 85-124. https://doi.org/10.1090/s0002-9947-1948-0024908-8

[2] Iachello, F. (2006) Lie Algebras and Applications. Springer.

[3] Jacobson, N. (2013) Lie Algebras. Courier Corporation.

[4] Looijenga, E. (2010) Smooth Manifolds. Informal Notes.

[5] Christensen, J.D. and Wu, E. (2014) Tangent Spaces and Tangent Bundles for Ideological Spaces. arXiv:1411.5425.

[6] Guggenheimer, H.W. (2012) Differential Geometry. Courier Corporation.

[7] Vaisman I. (2016) Cohomotopy and Differential Forms. Courier Dover Publications.

[8] Tu, L.W. (2011) Manifolds. In: Tu, L.W., Ed., *Universitext*, Springer, 47-83. https://doi.org/10.1007/978-1-4419-7400-6_3

[9] Hall, B.C. (2015) Lie Groups, Lie Algebras, and Representations. Springer. https://doi.org/10.1007/978-3-319-13467-3

[10] Rotman, J.J. (2009) An Introduction to Homological Algebra. Springer. https://doi.org/10.1007/b98977

[11] Larsen, M.J. and Lunts, V.A. (2021) A Note on Lie Algebra Cohomology. *Algebra & Number Theory*, **15**, 773-783. https://doi.org/10.2140/ant.2021.15.773

[12] Chevalley, C. (2018) Theory of Lie Groups. Courier Dover Publications.