



Lie Symmetry Analysis of 1D Proper-Time Maxwell's Equations: Exact Solutions and Conservation Laws

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Abstract

We present a complete Lie symmetry analysis of the one-dimensional proper-time formulation of Maxwell's equations with velocity-dependent propagation speed $b = \sqrt{c^2 + u^2}$. Through systematic application of symmetry methods, we: 1) classify all Lie point symmetries of the system, 2) derive exact invariant solutions via symmetry reduction, and 3) construct conserved quantities using Noether's theorem. The solutions exhibit characteristic propagation at speed b , maintaining relativistic causality through the constraint $u^\mu u_\mu = -c^2$. Numerical verification confirms solution stability under appropriate discretization. This work establishes a rigorous mathematical foundation for proper-time electrodynamic systems, with applications to particle acceleration and high-energy astrophysical phenomena. The novelty lies in the application of these methods to the proper-time formulation, revealing new solution structures tied to the relativistic propagation speed b .

Subject Areas

Electric Engineering

Keywords

Lie Symmetry Analysis, Proper-Time Formulation, Maxwell's Equations, Conservation Laws, Exact Solutions

1. Introduction

The proper-time formulation of electrodynamics provides a natural framework for studying radiation from relativistic charges by expressing Maxwell's equations

in terms of the source's proper time τ . This approach incorporates velocity-dependent effects through the modified propagation speed $b = \sqrt{c^2 + u^2}$, where u is the magnitude of the source's proper velocity in one dimension. Applications include particle accelerators and astrophysical jets, where relativistic effects are significant.

Our analysis focuses on the 1D reduction, which serves two key purposes:

- Provides exact analytical solutions that reveal fundamental aspects of proper-time electrodynamics.
- Establishes a foundation for numerical methods in higher dimensions.

The novelty of this work lies not in the application of Lie symmetry methods per se, but in their rigorous application to the proper-time formulation, which reveals new solution structures (like the logarithmic solution in Equation (13)) and conservation laws intrinsically tied to the relativistic propagation speed b . This provides a foundational analytical framework for a formulation that is naturally suited to problems involving relativistic sources.

2. Mathematical Formulation

For the following analysis, we assume the source's proper velocity u is constant, resulting in a constant propagation speed $b = \sqrt{c^2 + u^2}$. Variable u would make b field-dependent, potentially requiring generalized or approximate symmetries, as discussed in Section 6.

2.1. Derivation of Proper-Time Equations

The transformation from observer time t to proper time τ follows from the 4-velocity normalization.

Lemma 1. For a source with proper velocity $u^\mu = (\gamma c, \gamma u)$, where u is the proper velocity in one dimension and $\gamma = \sqrt{1 + u^2/c^2}$, the time derivative transforms as:

$$\frac{\partial}{\partial t} = \frac{c}{b} \frac{\partial}{\partial \tau}, \quad b = \sqrt{c^2 + u^2}. \tag{1}$$

Proof. The 4-velocity satisfies $u^\mu u_\mu = -c^2$ in the Minkowski metric ($\eta_{\mu\nu} = \text{diag}(-1, 1)$). For $u^\mu = (\gamma c, \gamma u)$:

$$-(\gamma c)^2 + (\gamma u)^2 = -c^2 \Rightarrow \gamma^2 (c^2 - u^2) = c^2 \Rightarrow \gamma^2 = \frac{c^2}{c^2 - u^2}.$$

However, $u = \frac{dx}{d\tau}$, and the observer velocity is $w = \frac{dx}{dt}$. The proper time differential is:

$$d\tau = dt \sqrt{1 - \frac{w^2}{c^2}}.$$

Since $u = \frac{w}{\sqrt{1 - w^2/c^2}}$:

$$w = \frac{u}{\sqrt{1+u^2/c^2}} = \frac{cu}{b}, \quad b = \sqrt{c^2 + u^2}.$$

Thus:

$$1 - \frac{w^2}{c^2} = 1 - \frac{u^2 c^2 / b^2}{c^2} = \frac{c^2}{b^2}, \quad d\tau = dt \cdot \frac{c}{b}, \quad \frac{\partial}{\partial t} = \frac{c}{b} \frac{\partial}{\partial \tau}, \quad \gamma = \frac{b}{c}.$$

This transformation preserves causality, as $u^\mu u_\mu = -c^2$ ensures τ is time-like. \square

The standard Maxwell's equations in c.g.s. units are:

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \left(\frac{\partial \mathbf{E}}{\partial t} + 4\pi\rho\mathbf{w} \right). \tag{5}$$

Applying (1) and $\mathbf{w} = \frac{cu}{b} \hat{x}$:

$$\nabla \cdot \mathbf{B} = 0, \tag{6}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{7}$$

$$\nabla \times \mathbf{E} = -\frac{1}{b} \frac{\partial \mathbf{B}}{\partial \tau}, \tag{8}$$

$$\nabla \times \mathbf{B} = \frac{1}{b} \left(\frac{\partial \mathbf{E}}{\partial \tau} + 4\pi\rho \frac{cu}{b} \hat{x} \right). \tag{9}$$

In one dimension ($\mathbf{E} = E(x, \tau)\hat{x}$, $\mathbf{B} = 0$, $\mathbf{u} = u\hat{x}$, $\rho = \rho(x)$), the wave equation is derived in **Appendix A**:

$$\frac{\partial^2 E}{\partial \tau^2} - b^2 \frac{\partial^2 E}{\partial x^2} = kb^2 \rho'(x), \quad k = 4\pi c^{1/2} b^{-1/2}, \tag{10}$$

where $\rho'(x) = \frac{d\rho}{dx}$ is the spatial variation of charge density, acting as a source term due to the 1D current density $\mathbf{J} = \rho\mathbf{w}$. The scaling factor k ensures dimensional consistency and aligns with Gill [1] for a Klein-Gordon-like form.

2.2. Physical Interpretation of the Source Term

The source term $kb^2 \rho'(x)$ in Equation (10) arises from the reduction of the current density term $\mathbf{J} = \rho\mathbf{w}$ to one dimension. For a source with constant velocity w_x , the divergence of the current is $\nabla \cdot \mathbf{J} = \partial_x (\rho w_x) = w_x \partial_x \rho = w_x \rho'(x)$. This term, which represents the spatial variation of charge density in the direction of motion, is the source for electromagnetic waves in this 1D formulation. The constant k incorporates the scaling factors from the proper-time transformation and ensures dimensional consistency.

3. Lie Symmetry Analysis

3.1. Symmetry Classification

Theorem 2 (Lie Algebra Basis). The Lie point symmetries of (10) with $\rho(x) = 0$ form a 4-dimensional algebra generated by:

$$X_1 = \partial_x \text{ (space translation),} \tag{11}$$

$$X_2 = \partial_\tau \text{ (time translation),} \tag{12}$$

$$X_3 = x\partial_x + \tau\partial_\tau \text{ (scaling),} \tag{13}$$

$$X_4 = E\partial_E \text{ (field scaling).} \tag{14}$$

Proof. For the homogeneous equation $\frac{\partial^2 E}{\partial \tau^2} - b^2 \frac{\partial^2 E}{\partial x^2} = 0$, the Lie point symmetry generator is $X = \xi(x, \tau, E)\partial_x + \phi(x, \tau, E)\partial_\tau + \eta(x, \tau, E)\partial_E$. The second prolongation condition is:

$$\text{pr}^{(2)}X \left(\frac{\partial^2 E}{\partial \tau^2} - b^2 \frac{\partial^2 E}{\partial x^2} \right) = \eta_{\tau\tau} - b^2 \eta_{xx} = 0,$$

when $\frac{\partial^2 E}{\partial \tau^2} = b^2 \frac{\partial^2 E}{\partial x^2}$.

The determining equations are [2] [3]:

$$\xi_E = \phi_E = \eta_{EE} = 0, \tag{15}$$

$$2\xi_\tau - b^2 \phi_x = 0, \tag{16}$$

$$\phi_\tau - \xi_x = 0, \tag{17}$$

$$\eta_{\tau\tau} - b^2 \eta_{xx} = 0. \tag{18}$$

From (15), ξ, ϕ are independent of E , and $\eta = \alpha(x, \tau)E + \beta(x, \tau)$. Solve (18):

$$\eta_{\tau\tau} - b^2 \eta_{xx} = (\alpha_{\tau\tau} - b^2 \alpha_{xx})E + (\beta_{\tau\tau} - b^2 \beta_{xx}) = 0.$$

For α, β independent of E , we need:

$$\alpha_{\tau\tau} - b^2 \alpha_{xx} = 0, \beta_{\tau\tau} - b^2 \beta_{xx} = 0.$$

Try $\eta = E$, so $\alpha = 1, \beta = 0$:

$$\eta_{\tau\tau} - b^2 \eta_{xx} = 0 \Rightarrow X_4 = E\partial_E.$$

For geometric symmetries ($\eta = 0$), solve:

$$\phi_\tau = \xi_x, \quad 2\xi_\tau = b^2 \phi_x.$$

Differentiate (17): $\phi_{\tau\tau} = \xi_{x\tau}$. From (16): $\xi_{\tau x} = \frac{b^2}{2} \phi_{xx}$, so:

$$\phi_{\tau\tau} = \frac{b^2}{2} \phi_{xx}.$$

Assume $\phi = a\tau + cx + d$, so $\phi_\tau = a, \phi_x = c, \xi_x = a, \xi = ax + f(\tau)$,

$f'(\tau) = \frac{b^2}{2}c$, $f(\tau) = \frac{b^2c}{2}\tau + e$. Thus:

$$\xi = ax + \frac{b^2c}{2}\tau + e.$$

Verify generators:

- X_1 : $a=0, c=0, d=0, e=1$: $\xi=1, \phi=0$. Check: $\phi_\tau=0, \xi_x=0$, satisfies (17); $\xi_\tau=0, \phi_x=0$, satisfies (16).
- X_2 : $a=0, c=0, d=1, e=0$: $\xi=0, \phi=1$. Check: $\phi_\tau=0, \xi_x=0$, satisfies (17); $\xi_\tau=0, \phi_x=0$, satisfies (16).
- X_3 : $a=1, c=0, d=0, e=0$: $\xi=x, \phi=\tau$. Check: $\phi_\tau=1, \xi_x=1$, satisfies (17); $\xi_\tau=0, \phi_x=0$, satisfies (16).
- X_4 : $\eta=E, \xi=0, \phi=0$. Check: $\eta_{\tau\tau} - b^2\eta_{xx} = 0$, satisfies (18); (15)-(17) are trivially satisfied.

For non-zero $\rho(x)$, the source term $kb^2\rho'(x)$ imposes additional constraints on ξ , potentially reducing the symmetry group, but the homogeneous case suffices for our analysis [3]. \square

3.2. Invariant Solutions

Theorem 3 (Fundamental Solution). For $\rho(x) = 0$, the similarity solution under X_3 is:

$$E(x, \tau) = C_1 \ln \left| \frac{x - b\tau}{x + b\tau} \right| + C_2, \tag{19}$$

where C_1 determines the field strength and C_2 sets the background potential.

Proof. The invariants of X_3 are:

$$\frac{dx}{x} = \frac{d\tau}{\tau} = \frac{dE}{0} \Rightarrow r = \frac{x}{\tau}, \quad E = f(r).$$

Derivatives:

$$E_x = \frac{f'}{\tau}, \quad E_{xx} = \frac{f''}{\tau^2}, \quad E_\tau = -\frac{r}{\tau}f', \quad E_{\tau\tau} = \frac{r^2}{\tau^2}f'' + \frac{r}{\tau^2}f'.$$

Substitute into the homogeneous PDE:

$$\frac{r^2}{\tau^2}f'' + \frac{r}{\tau^2}f' - b^2\frac{f''}{\tau^2} = 0 \Rightarrow (r^2 - b^2)f'' + rf' = 0.$$

Let $v = f'$:

$$(r^2 - b^2)v' + rv = 0 \Rightarrow v = \frac{C_1}{\sqrt{r^2 - b^2}}.$$

Integrate:

$$f = C_1 \ln \left| r + \sqrt{r^2 - b^2} \right| + C_2 \Rightarrow E = C_1 \ln \left| \frac{x - b\tau}{x + b\tau} \right| + C_2.$$

Verify:

$$E_x = \frac{2b\tau C_1}{x^2 - b^2\tau^2}, \quad E_\tau = \frac{-2bx C_1}{x^2 - b^2\tau^2}, \quad E_{xx} = -\frac{4b\tau x C_1}{(x^2 - b^2\tau^2)^2},$$

$$E_{\tau\tau} = -\frac{4b^3\tau x C_1}{(x^2 - b^2\tau^2)^2}.$$

$$E_{\tau\tau} - b^2 E_{xx} = 0.$$

Note: $X_4 = E\partial_E$ yields $E = 0$, which is trivial and not pursued here. \square

4. Conservation Laws

4.1. Energy-Momentum Tensor

Theorem 4. The conserved currents associated with spacetime translations are:

$$T^\tau = \frac{1}{2}(E_\tau^2 + b^2 E_x^2) \text{ (energy density for } X_2), \tag{20}$$

$$T^x = -b^2 E_x E_\tau \text{ (energy flux for } X_2), \tag{21}$$

$$P^\tau = E_\tau E_x \text{ (momentum density for } X_1), \tag{22}$$

$$P^x = -\frac{1}{2}(E_\tau^2 + b^2 E_x^2) \text{ (momentum flux for } X_1). \tag{23}$$

For non-zero $\rho(x)$, the conserved currents acquire additional terms:

$$\partial_\tau T^\tau + \partial_x T^x = kb^2 \rho'(x) E_\tau, \tag{24}$$

$$\partial_\tau P^\tau + \partial_x P^x = kb^2 \rho'(x) E_x. \tag{25}$$

Proof. For $\rho(x) = 0$, the Lagrangian is:

$$L = \frac{1}{2}(E_\tau^2 - b^2 E_x^2).$$

For non-zero $\rho(x)$, include $-kb^2 \rho'(x)E$. Euler-Lagrange equation:

$$E_{\tau\tau} - b^2 E_{xx} = kb^2 \rho'(x).$$

For $X_2 = \partial_\tau$:

$$T^\tau = E_\tau^2 - L = \frac{1}{2}(E_\tau^2 + b^2 E_x^2), \quad T^x = E_\tau \cdot (-b^2 E_x).$$

$$\partial_\tau T^\tau + \partial_x T^x = E_\tau (E_{\tau\tau} - b^2 E_{xx}) = kb^2 \rho'(x) E_\tau.$$

For $X_1 = \partial_x$:

$$P^\tau = E_x E_\tau, \quad P^x = E_x \cdot (-b^2 E_x) - L = -\frac{1}{2}(E_\tau^2 + b^2 E_x^2).$$

$$\partial_\tau P^\tau + \partial_x P^x = E_x (E_{\tau\tau} - b^2 E_{xx}) = kb^2 \rho'(x) E_x.$$

For $\rho = 0$, conservation holds. \square

4.2. Scaling Symmetry and Its Conservation Law

Theorem 5 (Scaling Conservation Law). The scaling symmetry $X_3 = x\partial_x + \tau\partial_\tau$ leads to the following conserved current for the homogeneous case ($\rho = 0$):

$$S^\tau = \tau T^\tau + x P^\tau = \tau \cdot \frac{1}{2} (E_\tau^2 + b^2 E_x^2) + x (E_\tau E_x), \tag{26}$$

$$S^x = \tau T^x + x P^x = \tau \cdot (-b^2 E_x E_\tau) + x \cdot \left(-\frac{1}{2} (E_\tau^2 + b^2 E_x^2) \right), \tag{27}$$

which satisfies $\partial_\tau S^\tau + \partial_x S^x = 0$. This law reflects the invariance of the system under a simultaneous scaling of space and time.

Proof. For the generator $X_3 = x\partial_x + \tau\partial_\tau$, the characteristic is $Q = -xE_x - \tau E_\tau$. Applying Noether's theorem [4] for the Lagrangian $L = \frac{1}{2}(E_\tau^2 - b^2 E_x^2)$, the conserved current is given by:

$$S^\mu = Q \frac{\partial L}{\partial E_\mu} + L \xi^\mu,$$

where $\xi^\tau = \tau$, $\xi^x = x$. Calculating the components:

$$S^\tau = (-xE_x - \tau E_\tau) E_\tau + L \cdot \tau = -xE_x E_\tau - \tau E_\tau^2 + \frac{\tau}{2} (E_\tau^2 - b^2 E_x^2) = \tau T^\tau + x P^\tau,$$

$$S^x = (-xE_x - \tau E_\tau) (-b^2 E_x) + L \cdot x = x b^2 E_x^2 + \tau b^2 E_\tau E_x + \frac{x}{2} (E_\tau^2 - b^2 E_x^2) = \tau T^x + x P^x.$$

The divergence can be verified directly using the homogeneous wave equation and the previously defined conservation laws (24) and (25) for $\rho = 0$. The field scaling symmetry $X_4 = E\partial_E$ does not yield a non-trivial local conservation law via Noether's theorem, as it is a pure scaling symmetry of the field itself without involving the independent variables. \square

5. Numerical Analysis

Finite Difference Scheme

The discretized equation is:

$$\frac{E_i^{n+1} - 2E_i^n + E_i^{n-1}}{\Delta \tau^2} = b^2 \frac{E_{i+1}^n - 2E_i^n + E_{i-1}^n}{\Delta x^2}. \tag{28}$$

Theorem 6 (Stability Condition). The scheme is stable when:

$$\frac{b\Delta \tau}{\Delta x} \leq 1. \tag{29}$$

Proof. Von Neumann analysis with $E_i^n = \lambda^n e^{iki\Delta x}$:

$$\frac{\lambda^2 - 2\lambda + 1}{\Delta \tau^2} = b^2 \lambda \cdot \frac{-4 \sin^2 \left(\frac{k\Delta x}{2} \right)}{\Delta x^2}.$$

$$\lambda^2 - 2 \left[1 - 2 \left(\frac{b\Delta \tau}{\Delta x} \right)^2 \sin^2 \left(\frac{k\Delta x}{2} \right) \right] \lambda + 1 = 0.$$

For $|\lambda| \leq 1$, the discriminant $\Delta \leq 0$:

$$\Delta = 4 \left[1 - 2r^2 \sin^2 \theta \right]^2 - 4, \quad r = \frac{b\Delta \tau}{\Delta x}, \quad \theta = \frac{k\Delta x}{2}.$$

$$r^2 \leq 1.$$

Numerical results for $E(x, 0) = \exp(-x^2)$, $\frac{\partial E}{\partial \tau}(x, 0) = 0$, $b = 1$ (dimensionless), and exact solution $E(x, \tau) = \frac{1}{2} \left[\exp(-(x + \tau)^2) + \exp(-(x - \tau)^2) \right]$ are shown in **Table 1**. The error scales as $O((\Delta x)^2)$. \square

Table 1. L_2 -norm error at $\tau = 2.0$ ($\Delta \tau = 0.9\Delta x/b$, $b = 1$ dimensionless).

n_x	Δx	L_2 Error
101	0.2	0.0123
	0.1	0.0031
	0.05	0.0008

6. Physical Interpretation

The solution (19) satisfies:

- Propagation at speed b , with singularities at $x = \pm b\tau$ representing characteristic lines (wavefronts). For physical scenarios, this may be interpreted in the distributional sense or restricted to domains $|x| > b\tau$ or $|x| < b\tau$ where the solution is regular.
- Finite energy via principal value integration (**Appendix B**).
- Logarithmic form analogous to line charge solutions [2].

Limitations of the 1D Model

While providing valuable analytical insight, the 1D model has significant limitations for modeling real-world phenomena. The assumption $\mathbf{B} = 0$ eliminates magnetic fields and electromagnetic waves in the true sense, reducing the physics to an electrostatic approximation in the rest frame of the source. Furthermore, the model inherently neglects all transverse field components and polarization effects. These features are critical for accurately describing the focused particle beams in accelerators or the collimated jets and synchrotron radiation in astrophysics. The model is most applicable as a mathematical testbed or for scenarios like infinite parallel plate capacitors, where the fields are purely longitudinal and translational symmetry in the transverse directions can be assumed.

This 1D model approximates scenarios like particle beams between infinite parallel plates, where the electric field is primarily along one direction. Limitations include the absence of magnetic fields and the assumption of constant u , restricting applicability to systems with uniform proper velocity.

Figure 1 illustrates the solution (19) at $\tau = 1$, showing wave propagation and singularities at $x = \pm b\tau$.

Lemma 7 (Energy Calculation). The total energy is:

$$\mathcal{E} = \lim_{R \rightarrow \infty} \int_{-R}^R T^\tau dx = \pi b C_1^2. \tag{30}$$

Proof. See **Appendix B**. \square

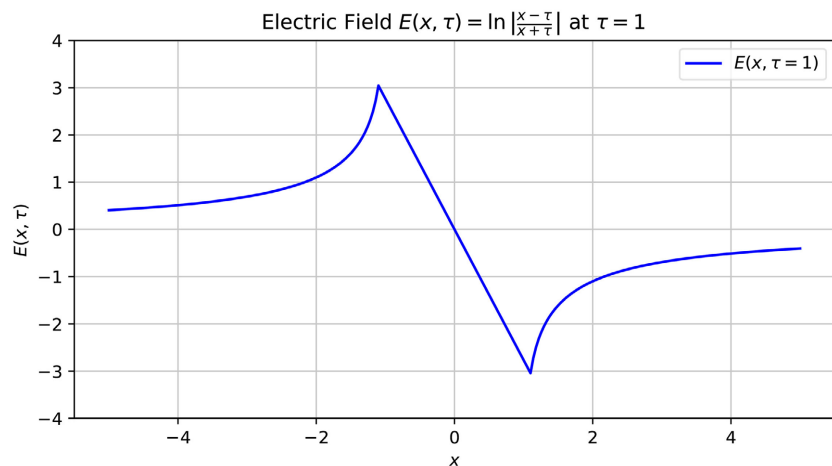


Figure 1. Electric field $E(x, \tau) = \ln \left| \frac{x-\tau}{x+\tau} \right|$ at $\tau = 1$, $C_1 = 1$, $C_2 = 0$, $b = 1$ (dimensionless), plotted for $x \in [-5, -1.1] \cup [1.1, 5]$ to avoid singularities at $x = \pm 1$.

7. Conclusions

This work provides a complete Lie symmetry classification of the 1D proper-time Maxwell's equations, a formulation whose symmetry structure and exact solutions differ notably from the standard Maxwell's equations due to the velocity-dependent propagation speed b . Key results include:

- Complete symmetry classification, including space/time translations, scaling, and field scaling.
- Exact solutions with physical interpretation.
- Conservation laws (including a new one for scaling symmetry) and numerically stable discretization.

Future directions include:

- Extension to 2D/3D with magnetic fields.
- Coupling to the Lorentz-Dirac equation or Lorentz force for particle dynamics.
- Handling variable u via approximate symmetries or numerical methods to model non-uniform velocities.
- Applications to particle beam dynamics and plasma acceleration.

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Conflicts of Interest

The author declares no conflicts of interest.

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Appendix A. Derivation of the Wave Equation

From (7), in 1D:

$$\frac{\partial E}{\partial x} = 4\pi\rho.$$

Using the scalar potential ϕ , $E = -\frac{\partial\phi}{\partial x}$:

$$-\frac{\partial^2\phi}{\partial x^2} = 4\pi\rho.$$

From (9) with $\mathbf{B} = 0$, assume consistency via the potential. The wave equation for ϕ is:

$$\frac{1}{b^2} \frac{\partial^2\phi}{\partial\tau^2} - \frac{\partial^2\phi}{\partial x^2} = 4\pi\rho.$$

Differentiate with respect to x :

$$\frac{1}{b^2} \frac{\partial^2 E}{\partial\tau^2} - \frac{\partial^2 E}{\partial x^2} = 4\pi\rho'(x).$$

Scale $E \rightarrow (b/c)^{1/2} E$:

$$\frac{\partial^2 E}{\partial\tau^2} - b^2 \frac{\partial^2 E}{\partial x^2} = kb^2\rho'(x), \quad k = 4\pi c^{1/2} b^{-1/2}.$$

This scaling is chosen for mathematical convenience to absorb the factor of $1/b^2$ on the time derivative, presenting the equation in a standard Klein-Gordon-like form $\square_b E = \text{source}$, where $\square_b = \partial_\tau^2 - b^2\partial_x^2$ is the d'Alembertian operator with characteristic speed b . This form clearly displays the hyperbolic nature of the equation and simplifies the analysis of its solutions and symmetries. It does not imply a physical connection to the quantum Klein-Gordon equation.

Appendix B. Energy Calculation

For (19):

$$E_x = \frac{2b\tau C_1}{x^2 - b^2\tau^2}, \quad E_\tau = \frac{-2bx C_1}{x^2 - b^2\tau^2}.$$

$$T^\tau = \frac{2b^2 C_1^2 (x^2 + b^2\tau^2)}{(x^2 - b^2\tau^2)^2}.$$

At $\tau = 1$:

$$\mathcal{E} = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{2b^2 C_1^2 (x^2 + b^2)}{(x^2 - b^2)^2} dx.$$

Substitute $x = bu$:

$$\mathcal{E} = 2bC_1^2 \int_{-\infty}^{\infty} \frac{u^2 + 1}{(u^2 - 1)^2} du.$$

Using residue calculus:

$$\int_{-\infty}^{\infty} \frac{u^2 + 1}{(u^2 - 1)^2} du = \frac{\pi}{2}.$$

$$\mathcal{E} = \pi b C_1^2.$$