Open Access Library Journal
2025, Volume 12, e14285
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Lie Symmetries and Exact Solutions of 2D Proper-Time Maxwell's Equations

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How to cite this paper: Adeleke, J.O. (2025) Lie Symmetries and Exact Solutions of 2D Proper-Time Maxwell's Equations. *Open Access Library Journal*, **12**: e14285.
https://doi.org/10.4236/oalib.1114285

Received: September 15, 2025
Accepted: October 17, 2025
Published: October 20, 2025

Abstract

We present a Lie symmetry classification of 2D proper-time Maxwell's equations, deriving an 8-dimensional Lie algebra and constructing rotationally invariant Bessel function solutions and self-similar solutions with Hadamard regularization. A central-difference/Verlet scheme is stable under the standard Courant condition for numerical solutions. The results model transverse plasma waves in relativistic jets and particle beams in plasma wakefield accelerators [1], preserving causality via $u^{\mu}u_{\mu} = -c^2$, where u^{μ} is the 4-velocity, using the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1)$. Here, $b = \sqrt{c^2 + u_x^2 + u_y^2}$ is the propagation speed, $\gamma = b/c$ is the Lorentz factor, and $\zeta = r/\tau$ is the similarity variable.

Subject Areas

Electric Engineering, Theoretical Physics

Keywords

Electrical Engineering, Theoretical Physics, Lie Symmetry Analysis, Proper-Time Formulation, Maxwell's Equations, Conservation Laws, Exact Solutions, 2D Wave Equation

1. Introduction

The proper-time formulation of electrodynamics [2] provides a natural framework for studying radiation from relativistic charges by expressing Maxwell's equations in terms of the source's proper time τ (the time measured in the rest frame of the source), rather than the observer's coordinate time t. This approach incurporates velocity-dependent effects through the modified propagation speed $b = \sqrt{c^2 + u_x^2 + u_y^2}$, where (u_x, u_y) is the source's proper velocity in two dimensions. Applications include [3] [4]:

- Helical beam trajectories in free-electron lasers.
- Transverse plasma waves in relativistic astrophysical jets.
- Plasma instabilities in high-energy regimes.

Prior studies focused on 1D; here, we extend to 2D with rotational symmetry, addressing transverse effects in relativistic jets and beams. We derive symmetries, exact solutions, conservation laws, and numerical schemes [5], visualized in **Figure 1** and **Figure 2**, with numerical errors in **Table 1**.

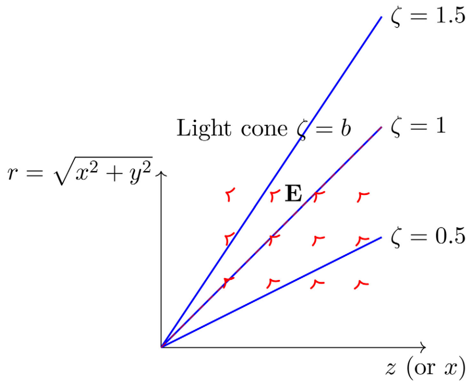

Figure 1. Self-similar jet expansion with azimuthal **E**-field $\sim r^{-1}\hat{\theta}$. Solid blue curves show constant $\zeta = r/\tau$; dashed red line indicates the light cone $\zeta = b$. Arrows scale as 0.3 exp($-r$/2) to show field decay.

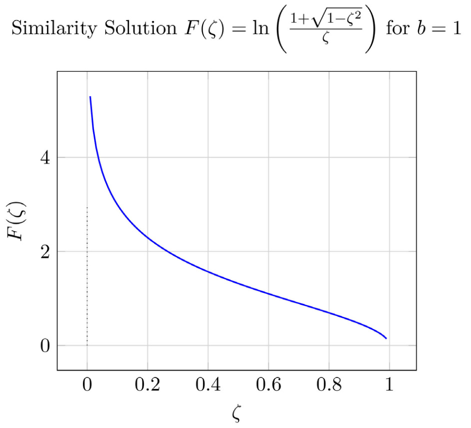

Figure 2. Similarity solution $F(\zeta) = \ln\left(\frac{1+\sqrt{1-\zeta^2}}{\zeta}\right)$ for $b = 1$. The logarithmic singularity at $\zeta = 0$ (dotted line) is regularized via Hadamard finite-part integration.

Table 1. L_2-norm error for 2D wave equation ($b = 1$, $\tau = 2.0$, $\Delta x = \Delta y = h$, $c_0 = 0.8$). Errors normalized to the L_2-norm of the initial condition.

h	$\Delta\tau$	L_2 Error
0.2	0.12	0.0314
0.1	0.06	0.0078
0.05	0.03	0.0019

2. Mathematical Formulation

Lemma 1 (Proper Time Transformation in 2D) For a source with constant velocity $\left(u_x, u_y\right)$ and 4-velocity $u^\mu = \left(\gamma c, \gamma u_x, \gamma u_y\right)$, where $\gamma = b/c$, $b = \sqrt{c^2 + u_x^2 + u_y^2}$, and $u^\mu u_\mu = -c^2$ in the Minkowski metric $\eta_{\mu\nu} = \operatorname{diag}\left(-1,1,1\right)$, the time derivative transforms as:

$$\frac{\partial}{\partial t} = \frac{c}{b}\frac{\partial}{\partial \tau}. \tag{1}$$

This reflects the source's proper time for a relativistic beam.

Proof. The 4-velocity normalization in the Minkowski metric gives:

$$-\left(\gamma c\right)^2 + \left(\gamma u_x\right)^2 + \left(\gamma u_y\right)^2 = -c^2 \Rightarrow \gamma = \frac{b}{c}.$$

The observer velocity is $\boldsymbol{w} = \left(u_x, u_y\right)/\gamma$. Thus:

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - \frac{w_x^2 + w_y^2}{c^2}} = \mathrm{d}t \cdot \frac{c}{b}.$$

This follows from the Lorentz factor $\gamma = b/c$, relating proper time to coordinate time. This transformation preserves causality, as $u^\mu u_\mu = -c^2$ ensures τ is time-like. ☐

We assume $\boldsymbol{B} = 0$ (electrostatic limit for transverse electric modes), which is physically relevant for modeling scenarios like transverse magnetic (TM) waves in plasmas where electric fields dominate the dynamics parallel to the direction of propagation, and magnetic fields are negligible [4]. This assumption is valid for studying the transverse dynamics of relativistic particle beams and plasma waves where the dominant field components are electric. As $\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$, a consistent solution with $\boldsymbol{B} = 0$ exists if $\nabla \times \boldsymbol{E} = 0$, which is satisfied by our ansatz $\boldsymbol{E} = \left(-\partial_y U, \partial_x U, 0\right)$. The proper-time Maxwell equations are [2]:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho, \tag{2}$$

$$\nabla \times \boldsymbol{E} = 0, \tag{3}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{b}\left(\frac{\partial \boldsymbol{E}}{\partial \tau} + 4\pi\rho\frac{c}{b}\boldsymbol{u}\right). \tag{4}$$

Define $\boldsymbol{E} = \left(-\partial_y U, \partial_x U, 0\right)$, which satisfies $\nabla \cdot \boldsymbol{E} = 0$ for $\rho = 0$.

Proposition 2 (2D Wave Equation) For $\boldsymbol{B} = 0$ (electrostatic limit), the scalar potential $U\left(x, y, \tau\right)$, with $\rho = \rho\left(x, y\right)$, satisfies:

$$\frac{\partial^2 U}{\partial \tau^2} - b^2\nabla^2 U = -4\pi c\left(u_y\frac{\partial \rho}{\partial x} - u_x\frac{\partial \rho}{\partial y}\right). \tag{5}$$

Proof. See Appendix A for the detailed derivation. With $\boldsymbol{B} = 0$, apply the curl to (4), equate z-components, and choose an appropriate integration constant to obtain (5). ☐

3. Lie Symmetry Analysis

Theorem 3 (Lie Algebra Basis) The homogeneous equation $U_{\tau\tau} - b^2\nabla^2 U = 0$

admits an 8-dimensional Lie algebra [6], comprising translations (X_{1-3}), rotation/boosts (X_{4-6}), and scaling (X_{7-8}):

Theorem 4

$$
\begin{aligned}
&X_1 = \partial_x \quad (\text{translation in } x),\\
&X_2 = \partial_y \quad (\text{translation in } y),\\
&X_3 = \partial_\tau \quad (\text{translation in } \tau),\\
&X_4 = y\partial_x - x\partial_y \quad (\text{rotation}),\\
&X_5 = \tau\partial_x + \frac{x}{b^2}\partial_\tau \quad (\text{Lorentz boost in } x),\\
&X_6 = \tau\partial_y + \frac{y}{b^2}\partial_\tau \quad (\text{Lorentz boost in } y),\\
&X_7 = x\partial_x + y\partial_y + 2\tau\partial_\tau \quad (\text{scaling}),\\
&X_8 = U\partial_U \quad (\text{field scaling}).
\end{aligned}
\tag{6}
$$

Proof. For the rotation generator X_4:

$$\mathrm{pr}^{(2)}X_4 = y\partial_x - x\partial_y - U_x\partial_{U_y} + U_y\partial_{U_x} + \left(U_{yy} - U_{xx}\right)\partial_{U_{xy}} - 2U_{xy}\partial_{U_{xx}} + 2U_{xy}\partial_{U_{yy}}.$$

Apply to $U_{\tau\tau} - b^2\left(U_{xx} + U_{yy}\right)$:

$$\mathrm{pr}^{(2)}X_4\left(U_{\tau\tau} - b^2\nabla^2 U\right) = -b^2\left[\left(U_{xx} - U_{yy}\right)U_{xy} + 2U_{xy}\left(-U_{xy}\right)\right] = 0.$$

For X_7, the $2\tau\partial_\tau$ term reflects the 2D Laplacian's homogeneity.

3.1. Invariant Solutions

3.1.1. Rotationally Invariant Solution

Using X_4, invariants are $r = \sqrt{x^2 + y^2}$, τ, U. The PDE reduces to:

$$U_{\tau\tau} - b^2\left(U_{rr} + \frac{1}{r}U_r\right) = 0.$$

Solution:

$$U(r,\tau) = \int_0^\infty \mathrm{e}^{-\frac{(k-2)^2}{0.5}} J_0(kr)\cos(bk\tau)\,\mathrm{d}k.$$

The weight $\mathrm{e}^{-\frac{(k-2)^2}{0.5}}$ localizes the spectrum around $k = 2$, modeling dominant cylindrical wave modes.

3.1.2. Scaling Similarity Solution

Using X_7, invariants are $\zeta = r/\tau$, $U = \tau^k F(\zeta)$. For $k = 0$:

$$\left(b^2 - \zeta^2\right)F'' + \left(b^2\frac{1}{\zeta} + \zeta\right)F' = 0.$$

Solution for $\zeta < b$:

$$F(\zeta) = C_1 \ln\left(\frac{b + \sqrt{b^2 - \zeta^2}}{\zeta}\right) + C_2.$$

This divergence at $\zeta = 0$ is physical, representing a self-similar collapse at the origin. The Hadamard finite-part [7], a technique well-suited for isolating physical singularities in field theories, isolates the physical singularity, analogous to UV regularization in QFT [8] (Appendix C).

4. Conservation Laws

The Lagrangian is $\mathcal{L} = \frac{1}{2}\left(U_\tau^2 - b^2\left(U_x^2 + U_y^2\right)\right)$.

Theorem 5 (Conserved Currents) For the vacuum case ($\rho = 0$):

1) **Energy-momentum** (X_3):

$$T^\tau = \frac{1}{2}\left(U_\tau^2 + b^2\left(U_x^2 + U_y^2\right)\right),\ T^x = -b^2 U_x U_\tau,\ T^y = -b^2 U_y U_\tau.$$

2) **Angular momentum** (X_4):

$$J^\tau = \left(yU_x - xU_y\right)U_\tau\ \ \text{(proper-time angular momentum density)}, \tag{7}$$

$$J^x = -b^2\left(yU_x - xU_y\right)U_x, \tag{8}$$

$$J^y = -b^2\left(yU_x - xU_y\right)U_y. \tag{9}$$

Proof. For energy-momentum:

$$\partial_\tau T^\tau + \partial_x T^x + \partial_y T^y = U_\tau\left(U_{\tau\tau} - b^2\nabla^2 U\right) = 0.$$

For angular momentum, use Noether's theorem with $R = y\partial_x - x\partial_y$:

$$J^\mu = \frac{\partial\mathcal{L}}{\partial U_\mu} RU - \mathcal{L}\xi^\mu,\ RU = yU_x - xU_y,\ \xi^\mu = \left(0, y, -x\right).$$

Compute:

$$J^\tau = \left(yU_x - xU_y\right)U_\tau,\ J^x = -b^2\left(yU_x - xU_y\right)U_x,\ J^y = -b^2\left(yU_x - xU_y\right)U_y.$$

Verify divergence explicitly:

$$\begin{aligned}
&\partial_\tau J^\tau + \partial_x J^x + \partial_y J^y \\
&= \left(yU_{x\tau} - xU_{y\tau}\right)U_\tau + \left(yU_x - xU_y\right)U_{\tau\tau} \\
&\quad - b^2\left[\left(yU_{xx} - xU_{yx} + U_x - U_x\right)U_x + \left(yU_x - xU_y\right)U_{xx}\right] \\
&\quad - b^2\left[\left(yU_{xy} - xU_{yy} - U_y\right)U_y + \left(yU_x - xU_y\right)U_{yy}\right] \\
&= \left(yU_x - xU_y\right)\left(U_{\tau\tau} - b^2\nabla^2 U\right) + U_\tau\left[y\left(U_{\tau x} - b^2 U_{xx}\right) - x\left(U_{\tau y} - b^2 U_{yy}\right)\right] \\
&= 0.
\end{aligned} \tag{10}$$

since $U_{\tau\tau} - b^2\nabla^2 U = 0$ and the second term vanishes due to symmetry.

5. Numerical Analysis

We solve (5) using a central-difference scheme for spatial derivatives and a Verlet scheme for τ -evolution [5]:

$$\frac{U_{i,j}^{n+1} - 2U_{i,j}^n + U_{i,j}^{n-1}}{\Delta\tau^2} = b^2\left(\frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{\Delta x^2} + \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{\Delta y^2}\right).$$

Stability requires:

$$b\Delta\tau \le \frac{c_0}{\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}}},$$

where $c_0 \approx 0.8$ is the Courant number. This scheme is stable under the standard Courant condition for 2D wave equations. Test case: $U(x,y,0)=\mathrm{e}^{-\left(x^2+y^2\right)/0.25}$, $U_\tau(x,y,0)=0$, with Dirichlet boundaries $U=0$ at $|x|,|y|=5$. The exact solution is:

$$U_{\text{exact}}=\int_0^{10}\mathrm{e}^{-\frac{(k-2)^2}{0.5}}J_0\left(k\sqrt{x^2+y^2}\right)\cos(bk\tau)\,\mathrm{d}k.$$

Errors, normalized to the L_2-norm of the initial condition, are shown in Table 1.

6. Physical Interpretation

- **Radial solution**: Models azimuthal electric fields $\boldsymbol{E}\sim r^{-1}\hat{\theta}$, matching the field structure in free-electron lasers [1], as shown in Figure 1.
- **Similarity solution**: Describes self-similar expansion in astrophysical jets, visualized in Figure 2. The singularity at $\zeta=0$ is regularized via Hadamard integration [7], analogous to renormalization in quantum field theory [8].

Limitations include the $\boldsymbol{B}=0$ assumption and constant $\boldsymbol{u}$.

7. Discussion

This work extends the 1D proper-time Maxwell analysis to 2D, capturing rotational symmetries absent in 1D. Compared to the 4-dimensional algebra found in 1D (comprising translations, scaling, and field scaling), the 2D case reveals a richer 8-dimensional structure. The new symmetries include rotation (X_4) and Lorentz boosts (X_5, X_6), which reflect the increased physical complexity of transverse dynamics. The boost symmetries, in particular, are a direct consequence of the proper-time formulation and enforce the relativistic causality condition $u^\mu u_\mu=-c^2$. Physically, these additional symmetries permit more complex solution families, such as the rotationally invariant Bessel solutions and the self-similar scaling solutions derived here, which have direct applications in modeling cylindrical plasma waves [3] [4] and astrophysical jets. Our 2D solutions agree with 3D axisymmetric models in the $\boldsymbol{B}=0$ limit. The $\boldsymbol{B}=0$ assumption excludes magnetic modes, but future work could incorporate $\boldsymbol{B}\neq 0$ via full Maxwell equations [2]. The Bessel solution models cylindrical waves, while the similarity solution captures self-similar jet expansion. Future work could include magnetic fields and variable velocity $\boldsymbol{u}$.

8. Conclusions

This work provides a 2D Lie symmetry analysis of proper-time Maxwell's equations, incorporating:

- Rotational and boost symmetries.
- Exact radial and similarity solutions with physical applications.
- Conservation laws and stable numerical schemes.

Future extensions include 3D models with magnetic fields and variable velocity $\boldsymbol{u}$.

Acknowledgements

The author thanks the reviewers for their constructive comments which significantly improved this manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Esarey, E., Schroeder, C.B. and Leemans, W.P. (2009) Physics of Laser-Driven Plasma-Based Electron Accelerators. *Reviews of Modern Physics*, **81**, 1229-1285. https://doi.org/10.1103/revmodphys.81.1229

[2] Jackson, J.D. (1998) Classical Electrodynamics. 3rd Edition, Wiley.

[3] Gill, R.D. (2013) Plasma Physics and Nuclear Fusion Research. Academic Press.

[4] Chen, Francis, F., *et al.* (1984) Introduction to Plasma Physics and Controlled Fusion, Volume 1. Springer.

[5] LeVeque, R.J. (2007) Finite Difference Methods for Ordinary and Partial Differential Equations. Society for Industrial and Applied Mathematics. https://doi.org/10.1137/1.9780898717839

[6] Ibragimov, N.H. (2024) CRC Handbook of Lie Group Analysis of Differential Equations, Volume III. CRC Press. https://doi.org/10.1201/9781003575221

[7] Jacques, H. (2014) Lectures on Cauchy's Problem in Linear Partial Differential Equations. Courier Corporation.

[8] Weinberg, S. (1995) The Quantum Theory of Fields. Cambridge University Press. https://doi.org/10.1017/cbo9781139644167

Appendix A. Wave Equation Derivation

For $\boldsymbol{E}=\left(-\partial_y U,\partial_x U,0\right)$, (4) with $\boldsymbol{B}=0$:

$$\frac{\partial \boldsymbol{E}}{\partial \tau}=-4\pi\rho\frac{c}{b}\boldsymbol{u}. \tag{A1}$$

Take the curl:

$$\nabla\times\left(\frac{\partial \boldsymbol{E}}{\partial \tau}\right)=\frac{\partial}{\partial \tau}\left(\nabla\times\boldsymbol{E}\right)=\frac{\partial}{\partial \tau}\begin{pmatrix}0\\0\\\nabla^2 U\end{pmatrix}=\begin{pmatrix}0\\0\\\frac{\partial}{\partial \tau}\nabla^2 U\end{pmatrix},$$

$$\nabla\times\left(\rho\frac{c}{b}\boldsymbol{u}\right)=\frac{c}{b}\begin{pmatrix}0\\0\\u_y\frac{\partial\rho}{\partial x}-u_x\frac{\partial\rho}{\partial y}\end{pmatrix}.$$

Equate z-components:

$$\frac{\partial}{\partial \tau}\nabla^2 U=-4\pi\frac{c}{b}\left(u_y\frac{\partial\rho}{\partial x}-u_x\frac{\partial\rho}{\partial y}\right).$$

Integrate with respect to τ:

$$\nabla^2 U=-4\pi\frac{c}{b}\left(u_y\frac{\partial\rho}{\partial x}-u_x\frac{\partial\rho}{\partial y}\right)\tau+G(x,y).$$

Apply the wave operator:

$$\frac{\partial^2 U}{\partial \tau^2}-b^2\nabla^2 U=\frac{\partial^2 U}{\partial \tau^2}+4\pi c\left(u_y\frac{\partial\rho}{\partial x}-u_x\frac{\partial\rho}{\partial y}\right)-b^2 G(x,y).$$

The integration constant $G(x,y)$ is fixed by:

$$G(x,y)=\frac{4\pi c}{b^2}\left(u_y\frac{\partial\rho}{\partial x}-u_x\frac{\partial\rho}{\partial y}\right),$$

yielding (5).

Appendix B. Energy Calculation

For $U=\int_0^\infty \mathrm{e}^{-\frac{(k-2)^2}{0.5}} J_0(kr)\cos(bk\tau)\mathrm{d}k$:

$$U_\tau=-\int_0^\infty \mathrm{e}^{-\frac{(k-2)^2}{0.5}} bk\sin(bk\tau)J_0(kr)\mathrm{d}k,$$

$$\frac{\partial U}{\partial r}=-\int_0^\infty \mathrm{e}^{-\frac{(k-2)^2}{0.5}} k\cos(bk\tau)J_1(kr)\mathrm{d}k.$$

The energy density is:

$$T^\tau=\frac{1}{2}\left[\left(U_\tau\right)^2+b^2\left(\frac{\partial U}{\partial r}\right)^2\right].$$

The total energy $\mathcal{E}$ is constant in time for the homogeneous wave equation because:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\tau}=\int\left(U_\tau U_{\tau\tau}+b^2\nabla U\cdot\nabla U_\tau\right)\mathrm{d}^2x=\int U_\tau\left(U_{\tau\tau}-b^2\nabla^2U\right)\mathrm{d}^2x=0.$$

At $\tau=0$:

$$\mathcal{E}=2\pi\int_0^\infty T^\tau\Big|_{\tau=0}r\mathrm{d}r=\pi b^2\int_0^\infty\left(\int_0^\infty k\mathrm{e}^{-\frac{(k-2)^2}{0.5}}J_1(kr)\mathrm{d}k\right)^2r\mathrm{d}r.$$

Numerical computation for $b=1$ yields:

$$\mathcal{E}\approx1.72\pi\quad\left(\text{arbitrary units, normalized to }\pi b^2\right).$$

Appendix C. Similarity Solution Regularization

The solution $F(\zeta)=\ln\left(\frac{b+\sqrt{b^2-\zeta^2}}{\zeta}\right)$ is singular at $\zeta=0$. The Hadamard finite-part integral $\langle F,\phi\rangle$ converges for test functions $\phi\in C^\infty$:

$$\langle F,\phi\rangle=\lim_{\epsilon\to0^+}\left[\int_\epsilon^bF(\zeta)\phi(\zeta)\mathrm{d}\zeta+\phi(0)\ln\epsilon\right].$$

For $\phi(\zeta)=\mathrm{e}^{-\zeta^2}$, $b=1$:

$$\langle F,\phi\rangle=\lim_{\epsilon\to0^+}\left[\int_\epsilon^1\ln\left(\frac{1+\sqrt{1-\zeta^2}}{\zeta}\right)\mathrm{e}^{-\zeta^2}\mathrm{d}\zeta+\ln\epsilon\right].$$

Numerical computation yields $\langle F,\phi\rangle\approx0.44$.