Open Access Library Journal
2025, Volume 12, e14279
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Spectral Theory for Frölicher Algebras via Locally Convex and Convenient Structures

John Pamba

School of ICT, Zambia University College of Technology, Ndola, Zambia
Email: jpamba@zut.edu.zm

How to cite this paper: Pamba, J. (2025) Spectral Theory for Frölicher Algebras via Locally Convex and Convenient Structures. *Open Access Library Journal*, **12**: e14279.
https://doi.org/10.4236/oalib.1114279

Received: September 13, 2025
Accepted: November 23, 2025
Published: November 26, 2025

Abstract

This article develops a framework for extending spectral theory to Frölicher algebras, which define smoothness via curves and functionals rather than topological or bornological structures. Motivated by classical spectral theory in locally convex algebras and its smooth extension in convenient algebras, we compare and connect these three categories through their notions of smoothness and character spectra. Beginning with the Gelfand theory for locally convex algebras, we generalize spectral constructions to convenient algebras via smooth homomorphisms, and further to Frölicher algebras, defining the spectrum in terms of Frölicher-smooth homomorphisms. We construct a smooth Gelfand transform and show that every convenient algebra admits a canonical Frölicher structure, yielding a faithful functor, though not an essentially surjective one. We introduce the concept of Frölicher bialgebras and sketch possible extensions of spectral analysis in this context. This work offers a unified smooth spectral framework, bridging topological, convenient, and Frölicher settings, and provides new tools for infinite-dimensional geometry, global analysis, and noncommutative smooth structures.

Subject Areas

Algebra, Algebraic Geometry, Functional Analysis, Special Theory of Relativity

Keywords

Convenient Vector Spaces, Frölicher Space, Infinite-Dimensional Analysis, Spectral Theory

1. Statement of the Problem

Classical spectral theory for unital commutative locally convex algebras relies on topological structures, with the Gelfand spectrum defined via continuous algebra homomorphisms into the base field. This framework has been extended to convenient algebras using smooth homomorphisms defined through convenient calculus. However, these approaches depend on topological or bornological foundations, which may not apply to more general smooth settings.

In particular, *Frölicher algebras*, defined via smooth curves and functionals, lack a corresponding spectral theory that aligns with their intrinsic structure. This work addresses the gap by:

1) Developing a spectral theory for Frölicher algebras using Frölicher-smooth homomorphisms;

2) Comparing spectral theories across locally convex, convenient, and Frölicher algebras;

3) Constructing a unified Gelfand transform compatible with smooth structures;

4) Investigating categorical embeddings between convenient and Frölicher algebras.

2. Introduction

The notion of the spectrum for elements of algebras originates from early developments in classical operator theory and Banach algebras [1]. Classical spectral theory, based on normed structures, provides tools for analyzing spectra, resolvents, and functional calculi but depends heavily on norm completeness, limiting its reach in more general topological algebraic settings. A foundational advancement came from Allan [1], who extended spectral theory to unital, commutative locally convex algebras using weak topologies. This approach enabled spectral analysis in pseudo-complete algebras without relying on normed structures. It laid the groundwork for later extensions of non-normed spectral theory. Subsequent developments by Beckenstein *et al.* [2] and Helemskii [3] deepened the theory of topological algebras, though a fully general smooth calculus remained elusive. More recent contributions by Fragoulopoulou *et al.* [4] extended Gelfand-type results in locally convex settings, while Yahaghi [5] explored spectral theory beyond associative topologies. Wang [6] introduced a framework for locally convex Hopf algebras, emphasizing dualities potentially compatible with smooth spectral theory. In parallel, Kriegl and Michor [7] developed the convenient setting of global analysis, where smoothness is defined via curves and functionals, creating a robust framework for infinite-dimensional calculus. This cartesian-closed category of convenient vector spaces supports smooth algebras but lacks a comprehensive spectral theory. Frölicher algebras, introduced by Frölicher and Nijenhuis [8], generalize smooth manifolds without topological assumptions. These algebras, structured via the duality of smooth curves and functions, encompass infinite-dimensional objects such as diffeomorphism groups and loop spaces, but no spectral framework currently exists for them. This work addresses these gaps by constructing a spectral theory for Frölicher algebras, grounded in Allan's theory and enriched by convenient calculus. We define spectra via Frölicher-smooth homomorphisms, develop smooth Gelfand transforms, and relate this theory categorically to locally convex and convenient settings. Our results unify and extend spectral methods across topological, convenient, and smooth algebraic structures, contributing new tools for infinite-dimensional geometry, global analysis, and noncommutative smooth frameworks.

3. Preliminaries

In this section, we present some definitions and results needed for our development of spectral theory in Frölicher algebras. We summarise the framework of locally convex algebras, following Allan's foundational work [1] and review the convenient setting of global analysis as formulated by Kriegl and Michor [7], which provides a smooth calculus framework in infinite dimensions. Concepts on Frölicher space, which serve as our primary objects of study, are also introduced.

Definition 3.1. [9] *Let E be a topological vector space. Then, E is a locally convex topological vector space if its topology arises from a family of seminorms. A locally convex topological vector space therefore carries a natural topology called the initial topology induced by seminorms. This topology is the coarsest topology for which all the mappings are continuous.*

Locally convex topological vector spaces are examples of topological vector spaces that generalize normed spaces. They are in general not necessarily normable.

Definition 3.2. [9] *The $\mathcal{C}^\infty$-topology on a locally convex space E is the final topology with respect to all smooth curves $\mathbb{R} \to E$. Its open sets are usually called $\mathcal{C}^\infty$-open.*

Definition 3.3. [9] *A locally convex vector space E is called $\mathcal{C}^\infty$-complete if one of the following equivalent conditions is satisfied:*

1) Any Lipschitz curve in E is locally Riemann integrable.

2) For any $c_1 \in \mathcal{C}^\infty(\mathbb{R}, E)$ there is $c_2 \in \mathcal{C}^\infty(\mathbb{R}, E)$ with $c_2 = c_1$.

3) E is $\mathcal{C}^\infty$-closed in any locally convex space.

4) If $c : \mathbb{R} \to E$ is a curve such that $l \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $l \in \mathcal{E}^*$, then c is smooth. (E^* denotes a space of all continuous linear functionals on E).

5) Any Mackey-Cauchy sequence converges; E is Mackey complete.

6) Any continuous linear mapping from a normed space into E has a continuous extension to the completion of the normed space.

Lemma 3.4. *A space E is $\mathcal{C}^\infty$-complete if and only if $\mathcal{C}^\infty(\mathbb{R}, E)$ is. The proof for this lemma can easily be followed in* [9].

By a topological algebra, $A[\tau]$, we shall mean a topological vector space which is also an algebra, such that the ring multiplication is separately continuous. $A[\tau]$ is said to be a locally convex algebra if it is a topological algebra whose underlying topological vector space is a locally convex space [1]. Here is a break-down definition:

Definition 3.5. [1] *A locally convex algebra is a topological vector space $A[\tau]$*

over a field ($\textit{typically}$ $\mathbb{R}$ $\textit{or}$ $\mathbb{C}$) *equipped with*:

1) An algebra structure: $A[\tau]$ is closed under a bilinear multiplication operation.

2) A locally convex topology: The topology is defined by a family of seminorms $\{p_\alpha\}$, and has a local base at zero consisting of convex sets.

3) Continuity of multiplication: At a minimum, multiplication $A[\tau]\times A[\tau]\to A[\tau]$ is separately continuous, meaning:

$$x\mapsto xy \quad \text{is continuous for fixed} \quad y,$$

$$y\mapsto xy \quad \text{is continuous for fixed} \quad x.$$

Note that in stronger versions, joint continuity may be required. Also note that in certain types of locally convex algebras, called locally multiplicatively convex algebras, the multiplication satisfies an inequality of the form:

$$p(xy)\le Cp(x)p(y),$$

for all $x,y\in A$, some constant $C>0$, and a seminorm p from the defining family. We shall denote the category of locally convex algebras by *LCAlg*. Here is an example of *LCAlg*.

Example 3.6.

1) Fréchet Algebras: These are complete locally convex algebras whose topology is defined by a countable family of seminorms, such as $C^\infty(\mathbb{R})$, the algebra of smooth functions with the usual Fréchet topology.

2) Algebras of Distributions: Such as the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ under convolution.

3) The Schwartz Space $\mathcal{S}(\mathbb{R}^n)$: This is a locally convex algebra under pointwise multiplication or convolution.

Unlike a Banach algebra, a locally convex algebra $A[\tau]$ may not have a norm, and its topology may not be metrizable or complete. It is often assumed to be complete or pseudo-complete (as in Allan's setting). Thus, locally convex algebras generalize Banach algebras and are central in various fields.

Definition 3.7. [1] *Let* $A[\tau]$ *be a locally convex algebra. An element* x *of* $A[\tau]$ *is called* (*Allan-*) *bounded and for simplicity just bounded, if there exists a nonzero complex number* λ, *such that the set* $\{(\lambda x)^n : n\in\mathbb{N}\}$ *is a bounded subset of* $A[\tau]$.

The set of all bounded elements of $A[\tau]$ will be denoted by A_0. One can easily see that every element of a normed algebra is bounded and also that if $A[\tau]$ has an identity e, then e is bounded.

Let $\mathbb{C}^\star$ denote the extended complex plane, in its usual topology, as the one-point compactification of $\mathbb{C}$ so that a partial algebraic structure is defined on $\mathbb{C}^\star$ as follows:

$$\infty+\lambda=\infty,\ \lambda\in\mathbb{C};\ \infty\cdot\lambda=\infty,\ \lambda\in\mathbb{C}^\star\setminus\{0\} \text{ and } \overline{\infty}=\infty$$

Allan [1] introduced a generalized notion of the spectrum for an element x of a locally convex algebra $A[\tau]$ (which may or may not have an identity) as follows:

Definition 3.8. *Let* $A[\tau]$ *be a locally convex algebra with an identity* e. *The spectrum of an element* $x \in A[\tau]$, *denoted by* $\sigma(x)$ (*or by* $\sigma_A(x)$, *when more than one algebra is involved*), *is the subset of* $\mathbb{C}^{\star}$ *defined by*

$$\sigma_A(x) = \left\{\lambda \in \mathbb{C}^{\star} \mid \lambda e - x \text{ is not invertible in } A_o\right\} \cup \left\{\infty \Leftrightarrow x \notin A_o\right\}$$

In case $A[\tau]$ has no identity then $\sigma_A(x) = \sigma_{A_1}(x,0)$.

Definition 3.9. [1] *The resolvent set of* x, $\rho(x)$, *is the complement of* $\sigma_A(x) \in \mathbb{C}^{\star}$.

In Allan's approach, the spectral radius was defined analogously to the Banach context. However, it was shown that in commutative, pseudo-complete, locally multiplicatively convex (l.m.c.) algebras, the spectral radius is lower semi-continuous. This partial continuity mirrors Banach algebra behavior and highlights how completeness and m-convexity help restore familiar spectral features.

Definition 3.10. [9] *A* C^{∞}*-complete locally convex vector space* E *as defined in Definition* 3.2 *above is referred to as a convenient vector space. A convenient vector space therefore is a locally convex space satisfying a completeness property that all derivatives which ought to exist actually do. It is equivalent to locally complete space as usually used in functional analysis. The term convenient space was first introduced by Kriegl and Michor as part of their theory of global analysis, as in* [7].

Frölicher and Kriegl in [9] defined a preconvenient space as a dualized vector space which is invariant under the endo-functor of a differential vector space. It was later stated that any separated preconvenient vector space that satisfies completeness conditions is called a convenient vector space.

Definition 3.11. [9] *Let* E *be a convenient vector space. Then,* λE *is referred to as a* ***free convenient vector space****.*

Definition 3.12. [9] *A Frölicher space is a space which consists of a non-empty set* X *together with a subset* $\mathcal{C}$ *of* $Hom(\mathbb{R}, X)$ *called the set of smooth curves, and a subset* $\mathcal{F}$ *of* $Hom(X, \mathbb{R})$ *called the set of smooth real functions, such that for each real function* $f : X \to \mathbb{R}$ *in* $\mathcal{F}$ *and each curve* $c : \mathbb{R} \to X$ *in* $\mathcal{C}$, *the following axioms are satisfied*:

1) f is in $\mathcal{F}$ if and only if for each γ in $\mathcal{C}$, $f \circ \gamma$ is in $C^{\infty}(\mathbb{R}, \mathbb{R})$.

2) c is in $\mathcal{C}$ if and only if for each ϕ in $\mathcal{F}$, $\phi \circ c$ is in $C^{\infty}(\mathbb{R}, \mathbb{R})$.

It is usually denoted by the triple $(X, \mathcal{C}, \mathcal{F})$, where X, $\mathcal{C}$ and $\mathcal{F}$ are as defined above.

Definition 3.13. [9] *A Frölicher structure on a set* X *is a pair* $(\mathcal{C}_X, \mathcal{F}_X)$, *where* $\mathcal{F}$ *is a family of real-valued functions* $X \to \mathbb{R}$ *and* $\mathcal{C}$ *is a family of maps* $\mathbb{R} \to X$, *such that* $\Phi\mathcal{C} = \mathcal{F}$ *and* $\Gamma\mathcal{F} = \mathcal{C}$. $f \in \mathcal{F}_X$ *are called structure functions, and* $c \in \mathcal{C}_X$ *are called structure curves. Note that structure functions and structure curves are smooth in the smooth structure* $(\mathcal{C}_X, \mathcal{F}_X)$.

According to [10], a smooth map between Frölicher spaces is a map on underlying sets, which takes curves from the smooth structure of the source space to curves in the structure of the target space. Thus, a map between Frölicher spaces is smooth if it maps structure functions in the target space back to those in the source space.

Let $(X,\mathcal{C}_X,\mathcal{F}_X)$ and $(Y,\mathcal{C}_Y,\mathcal{F}_Y)$ be Frölicher spaces. Then, as a mapping $\phi: X \to Y$ is called smooth if the following three equivalent conditions hold:

1) For each $c \in \mathcal{C}_X$, the composite $\phi \circ c$ is in $\mathcal{C}_Y$,

2) For each $f \in \mathcal{F}_Y$, the composite $f \circ \phi$ is in $\mathcal{F}_X$,

3) For each $c \in \mathcal{C}_X$ and for each $f \in \mathcal{F}_Y$, the composite $f \circ \phi \circ c$ is in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$.

The set of all smooth mappings from X to Y is usually denoted by $\mathcal{C}^\infty(X,Y)$, so that $\mathcal{C}^\infty(\mathbb{R},Y) = \mathcal{C}_X$ and $\mathcal{C}^\infty(X,\mathbb{R}) = \mathcal{F}_X$. Furthermore, in order to emphasize that a map is smooth in the sense of Frölicher, [7] highlights that one often uses the notation $\mathcal{F}$-smooth map.

Example 3.14. *Smooth Mappings between Frölicher Spaces*

1) A constant mapping is a $\mathcal{F}$-smooth map.

2) An identity is a F-smooth map. Let M and N be Frölicher spaces. If $N = M$ and h is an identity on M, then it is a Frölicher smooth map. In Lemma 1.1 in [11], it was shown that if X, Y and Z are Frölicher spaces, then the following canonical mappings are smooth:

- $ev: \mathcal{C}^\infty(X,Y) \times X \to Y$, $ev(f,x) = f(x)$;
- $ins: X \to \mathcal{C}^\infty(Y, X \times Y), ins(x)(g) = (x,y)$;
- $comp: \mathcal{C}^\infty(Y,Z) \times \mathcal{C}^\infty(X,Y) \to \mathcal{C}^\infty(X,Z), comp(g,f) = g \circ f$.

3) It is shown in Lemma 1.4 in [11] that, if X is a Frölicher Space and $p \in X$, then the map $\dagger: T_P X \times T_p X \to T_P X$, $(u,v) \to u+v$ is a smooth map of Frölicher Spaces.

4) Lemma 1.2 in [11] shows that if $\phi: X \to Y$ is a map of Frölicher Spaces X and Y, then the following canonical mappings are smooth:

- $\bar{\varphi}: F_Y \to \mathcal{F}_X$, $\bar{\varphi}(\beta) = \beta \circ \varphi$;
- $\chi: \mathbb{FRL}(X,Y) \to \mathbb{FRL}(\mathcal{F}_Y,\mathcal{F}_X), \chi(f) = \bar{f}$, where $\bar{f}(\beta) = \beta \circ f$.

Definition 3.15. *A Linear Frölicher space is a Frölicher space X in which both the real-valued functions $f: X \to \mathbb{R}$ in F and the curves $c: \mathbb{R} \to X$ in C as described in Definition* 3.12 *above are linear.*

In [12], Batubenge and Tshilombo introduced a class of Sikorski differential spaces called pre-Frölicher spaces and investigated some algebraic properties on these spaces. Here, the notion of a ringed space in the sense of Palais as in [13] was given in terms of pre-Frölicher spaces. We follow Batudenge and Tshilombo [12] for the definition of Frölicher and pre-Frölicher ringed spaces as below:

Definition 3.16. *A pre-Frölicher space is a differential space (M,D) with structure D such that $D = \Phi\Gamma\mathcal{F}_o$, where $(M,\Gamma\mathcal{F}_o,\Phi\Gamma\mathcal{F}_o)$ is the associated Frölicher space and $\mathcal{F}_o$ is a generating set.*

The definition and existence of the class of pre-Frölicher spaces have been justified in [12] by using a diagram.

Lemma 3.17. *Let (M_1,D_1) and (M_2,D_2) be differential spaces. If (M_1,D_1)*

is a pre-Frölicher space and $\varphi:(M_1,D_1)\to(M_2,D_2)$ *is a diffeomorphism of differential space, then* (M_2,D_2) *is a pre-Frölicher space. For the proof, we refer to* [12].

Proposition 3.18. *For every Frölicher space* X, *there exists a free convenient vector space* λX.

One can follow the proof of this proposition in [9] or [10].

It is remarked on page 240 in [7] by Kriegl and Michor that the convenient vector spaces are exactly the **linear Frölicher spaces** for which the smooth linear functionals generate the smooth structure, and which are "separated" and "complete" as can be verified in [9] on 2.4.4.

Proposition 3.19. *Let* X *be a Frölicher space and* E *be a convenient vector space. Then,* $C^\infty(X,E)$ *is a convenient vector space with the smooth structure which is cartesian closed.*

Follow [7] for the proof of this proposition.

Definition 3.20. [9] *A convenient bialgebra* B *is a convenient vector space which is both a convenient algebra and a convenient coalgebra such that the algebra structure maps are ConCoAlg-morphisms or equivalently that the coalgebra structure maps are ConAlg-morphisms.*

We can easily note that a $\mathcal{C}^\infty$-bialgebra B is a $\mathcal{C}^\infty$-algebra which is also a convenient bialgebra with the same underlying convenient algebra structure.

4. Main Results

4.1. Frölicher Algebra

We now present the concept of a Frölicher algebra as follows:

Definition 4.1. *A Frölicher algebra is a Frölicher space* $(A,\mathcal{C},\mathcal{F})$ *such that:* A *is an algebra over* $\mathbb{R}$, *where the algebra operations:*

$$+:A\times A\to A,\quad \cdot:A\times A\to A,\quad \mathbb{R}\times A\to A,\quad (\lambda,a)\mapsto\lambda\cdot a$$

are smooth with respect to the Frölicher structure.

That is, for any smooth curves $c_1,c_2\in\mathcal{C}$ and any smooth scalar function $\lambda:\mathbb{R}\to\mathbb{R}$, the curves:

$$t\mapsto c_1(t)+c_2(t),\quad t\mapsto c_1(t)\cdot c_2(t),\quad t\mapsto\lambda(t)\cdot c(t)$$

also belong to $\mathcal{C}$.

A Frölicher algebra therefore is an algebra A equipped with a Frölicher structure $(A,\mathcal{C},\mathcal{F})$ such that the algebra operations are smooth maps in the Frölicher sense.

We shall denote the category of a Frölicher algebra by *FrAlg*, which is a Frölicher vector space with smooth multiplication, and so it is Linear. Thus, throughout this manuscript, a Frölicher algebra shall refer to a linear Frölicher algebra even without mention.

Theorem 4.2. *Every convenient algebra carries a natural Frölicher structure, induced by its smooth curves and real-valued smooth functionals. With respect to this structure, it becomes a Frölicher algebra.*

Proof. Let A be a convenient algebra; that is, A is an algebra object in the category of convenient vector spaces, with smooth addition, scalar multiplication, and multiplication maps.

Define a Frölicher structure on A as follows:

- Let $\mathcal{C}$ be the set of all smooth curves $c:\mathbb{R}\to A$ in the convenient sense (*i.e.*, $c\in C^{\infty}(\mathbb{R},A)$).
- Let $\mathcal{F}$ be the set of all smooth functionals $f:A\to\mathbb{R}$ in the convenient sense (*i.e.*, $f\in C^{\infty}(A,\mathbb{R})$).

By the foundational results of convenient calculus (see Kriegl and Michor), a map $f:A\to\mathbb{R}$ is smooth if and only if $f\circ c\in C^{\infty}(\mathbb{R})$ for all $c\in\mathcal{C}$, and vice versa. Thus, the pair $(\mathcal{C},\mathcal{F})$ satisfies the compatibility condition required for a Frölicher space. Next, observe that since the algebra operations

$$+:A\times A\to A,\quad \cdot:A\times A\to A,\quad \mathbb{R}\times A\to A$$

are smooth in the convenient sense, their compositions with smooth curves also yield smooth curves. That is, for any $c_1,c_2\in\mathcal{C}$, the curves

$$t\mapsto c_1(t)+c_2(t),\quad t\mapsto c_1(t)\cdot c_2(t)$$

are in $\mathcal{C}$, and for any smooth scalar curve $\lambda:\mathbb{R}\to\mathbb{R}$, the curve $t\mapsto\lambda(t)\cdot c(t)$ is also in $\mathcal{C}$. Therefore, the Frölicher structure defined above makes A into a Frölicher algebra. Thus, every convenient algebra carries a canonical Frölicher structure, induced by its set of smooth curves $c:\mathbb{R}\to A$ and its set of real-valued smooth functionals $f:A\to\mathbb{R}$.

This makes every convenient algebra into a Frölicher algebra. However, note that the converse does not hold in general.

Claim!

Not every Frölicher algebra arises from a convenient algebra structure.

This Claim shows that there exist Frölicher spaces (and algebras) that are not modeled on convenient vector spaces or lack locally convex topologies.

To justify the above Claim, it suffices for one to recall that a Frölicher algebra consists of a set A equipped with a set $\mathcal{C}$ of smooth curves $c:\mathbb{R}\to A$, and a set $\mathcal{F}$ of real-valued functions $f:A\to\mathbb{R}$, satisfying the compatibility condition that $f\circ c\in C^{\infty}(\mathbb{R})$ for all $c\in\mathcal{C}$ and $f\in\mathcal{F}$. Moreover, the algebraic operations (addition, multiplication, scalar multiplication) must preserve smooth curves and smooth functionals.

Proposition 4.3. *There exists a Frölicher algebra that does not arise from any convenient algebra structure.*

Proof. Let $A=C^{\infty}(\mathbb{R})^{\mathbb{N}}$, the countable product of the Fréchet space $C^{\infty}(\mathbb{R})$, endowed with the Frölicher structure generated by pointwise smooth curves and smooth evaluation functionals. Define the algebra structure on A componentwise: for $f=(f_n)$, $g=(g_n)\in A$, define $f+g=(f_n+g_n)$, and $f\cdot g=(f_n\cdot g_n)$. This makes A into a commutative algebra equipped with a Frölicher structure. This Frölicher structure is well-defined and closed under algebraic operations. However, $A=C^{\infty}(\mathbb{R})^{\mathbb{N}}$ is not a convenient vector space, since the category of convenient vector spaces is not closed under countable products. In particular, there exists no convenient vector space structure on A such that pointwise addition and multiplication are smooth. Hence, this Frölicher algebra does not arise from any convenient algebra structure. It admits a Frölicher structure, but not a convenient one.

A convenient algebra is more restrictive: it requires that the underlying space be a convenient vector space, which in turn must be a locally convex vector space satisfying additional completeness and smoothness properties (such as the c^∞-completeness condition).

Example 4.4. *Let* A *be an uncountable-dimensional vector space over* $\mathbb{R}$, *equipped with the trivial topology* (*i.e., the only open sets are* $\varnothing$ *and* A). *Define the set of curves* $\mathcal{C}$ *to consist of all constant maps* $c:\mathbb{R}\to A$, *and define the set of functionals* $\mathcal{F}$ *to consist of all linear maps* $f:A\to\mathbb{R}$, *i.e.,* $\mathcal{F}=A^*$, *the algebraic dual of* A. *For any* $f\in\mathcal{F}$ *and* $c\in\mathcal{C}$, *the composition* $f\circ c$ *is constant, hence smooth. Therefore,* $(\mathcal{C},\mathcal{F})$ *defines a Frölicher structure on* A. *An algebra structure on* A *can be defined by coordinate-wise* (*i.e., componentwise*) *multiplication, making* A *a Frölicher algebra. However, this Frölicher algebra does not arise from any convenient vector space structure: the trivial topology is not locally convex, nor does it support a topology compatible with completeness or continuity properties required in the convenient setting. Moreover, countable products of* $C^\infty(\mathbb{R})$ *fail to be* c^∞*-complete because their bounded sets are not contained in any Banach disk, which explicitly illustrates the non-convenient nature of such product spaces.*

Proposition 4.5. *There exist locally convex algebras that do not admit any compatible convenient or Frölicher algebra structure.*

Proof. Let $A:=\mathbb{R}^{\mathbb{N}}$ be the space of all real sequences, endowed with the product topology. Then, A is a locally convex topological vector space. Define pointwise multiplication by $(x\cdot y)_n:=x_n y_n$, for $x=(x_n)$, $y=(y_n)\in A$, which is bilinear and jointly continuous, making A a commutative locally convex algebra. However, A is not a convenient vector space. Specifically:

- There exist smooth curves $c:\mathbb{R}\to A$ such that $\ell\circ c\in C^\infty(\mathbb{R})$ for all $\ell\in A'$, but c is unbounded.
- • A is not bornologically complete; bounded sets are not contained in Banach disks.

Therefore, A is not a convenient algebra. Furthermore, any Frölicher structure induced by the continuous linear functionals on A fails to capture the smoothness of algebraic operations: smooth curves and functionals do not interact compatibly with the algebra structure. Thus, A cannot be endowed with a convenient or Frölicher algebra structure.

Example 4.6. *Let* $A=\mathbb{R}[x]$ *be the algebra of real polynomials, endowed with the fine topology, i.e., the inductive limit topology of the finite-dimensional subspaces* $\mathbb{R}_n[x]$ (*polynomials of degree at most* n). *Then:*

1) A is a locally convex topological vector space.

2) Multiplication $A \times A \to A$, defined pointwise, is jointly continuous, making A a locally convex algebra.

3) However, A is not Mackey-complete, hence not a convenient vector space.

4) Furthermore, no Frölicher structure on A makes the multiplication smooth: the topology is too coarse to support enough smooth curves.

Therefore, A is a locally convex algebra that does not admit a compatible convenient or Frölicher structure.

Therefore, A is a locally convex algebra that does *not* admit any compatible convenient or Frölicher algebra structure.

Thus, Locally Convex Algebras (*LCAlg*) form a broad class of algebras with a locally convex topology and continuous multiplication, but generally *lack a canonical smooth structure*. On the other hand, Convenient Algebras (*ConAlg*) are a *subclass* of *LCAlg* equipped with a smooth structure that enables infinite-dimensional differential calculus. Every convenient algebra naturally carries a Frölicher algebra (*FrAlg*) structure induced by its smooth curves and smooth functionals. Therefore, the category of Frölicher Algebras is the most general, requiring only compatible smooth curves and functionals, and includes many algebras that are not locally convex or convenient.

One can define a Frölicher algebra by requiring the multiplication to be smooth relative to the chosen Frölicher structure, generalizing continuous multiplication in locally convex algebras and smooth multiplication in convenient algebras. We explore this in the next proposition.

Proposition 4.7. *Let* A *be a Frölicher space. If the multiplication map* $\mu : A \times A \to A$, $(x, y) \mapsto xy$ *is smooth with respect to the product Frölicher structure on* $A \times A$, *then* A *becomes a Frölicher algebra.*

(This generalizes the notion of continuous multiplication in locally convex algebras and smooth multiplication in convenient algebras).

Proof. Recall that a Frölicher space consists of a set A, a set C of smooth curves $c : \mathbb{R} \to A$, and a set F of smooth functionals $f : A \to \mathbb{R}$, satisfying mutual compatibility: $f \circ c \in C^{\infty}(\mathbb{R})$ for all $f \in F$, $c \in C$.

The product Frölicher structure on $A \times A$ is defined via smooth curves $\gamma : \mathbb{R} \to A \times A$, such that $\pi_1 \circ \gamma, \pi_2 \circ \gamma \in C$, and smooth functionals of the form $f(x, y) = \varphi(x)\psi(y)$, $\exists \ \varphi, \psi \in F$.

If multiplication μ is smooth, then for every smooth curve $\gamma : \mathbb{R} \to A \times A$, the composed curve $t \mapsto \mu(\gamma(t)) = \pi_1(\gamma(t)) \cdot \pi_2(\gamma(t)) \in A$ is again a smooth curve in A, *i.e.*, belongs to C. Moreover, for every smooth functional $f \in F$, the composition $f \circ \mu \circ \gamma \in C^{\infty}(\mathbb{R})$, so the structure remains compatible.

Therefore, the multiplication is smooth in the Frölicher sense, and A is equipped with an algebra structure that is compatible with its differential structure, making it a Frölicher algebra. This generalizes:

- *Locally convex algebras*, where multiplication is jointly continuous;
- *Convenient algebras*, where multiplication is jointly smooth.

Example 4.8. (*Algebras of Smooth Functions*)

Let M be a smooth finite-dimensional manifold. The algebra $C^\infty(M)$ of smooth real-valued functions on M is naturally a Frölicher space:

- Smooth curves in $C^\infty(M)$ are maps $c:\mathbb{R}\to C^\infty(M)$ such that for every $x\in M$, the function $t\mapsto c(t)(x)$ is smooth in t.
- Smooth functionals are evaluations at points: $f\mapsto f(x)$, for $x\in M$.

The pointwise multiplication $(f\cdot g)(x)=f(x)g(x)$ is smooth with respect to this Frölicher structure, since the product of smooth functions is smooth. Thus, $C^\infty(M)$ is a Frölicher algebra.

Example 4.9. (*Convenient Algebras*)

Every convenient algebra A, being a convenient vector space with smooth multiplication, naturally induces a Frölicher structure via:

- Smooth curves: $c:\mathbb{R}\to A$ that are smooth in the convenient sense.
- Smooth functionals: $f:A\to\mathbb{R}$, smooth in the convenient calculus.

Since the multiplication is jointly smooth in the convenient sense, it is also smooth with respect to the induced Frölicher structure. Hence, every convenient algebra is a Frölicher algebra.

Example 4.10. (*Diffeological Algebras*)

Let A be a diffeological algebra, *i.e.*, an algebra equipped with a diffeology such that the multiplication map $A\times A\to A$ is smooth with respect to the product diffeology.

Every diffeological space induces a Frölicher structure (via smooth plots and functionals), and if the multiplication is smooth in the diffeological sense, it remains smooth in the corresponding Frölicher sense.

Hence, many diffeological algebras (e.g., function spaces on singular spaces, mapping spaces) are naturally Frölicher algebras. For more details on the concept of diffeological spaces, see [14].

Remark 4.11. (*Closure under Smooth Subalgebras*)

Let A be a Frölicher algebra, and let $B\subseteq A$ be a subalgebra that is closed under the Frölicher structure (*i.e.*, smooth curves in B are also smooth in A, and functionals on B are restrictions of those on A). Then, B inherits a Frölicher structure, making it a Frölicher algebra. This allows constructing new Frölicher algebras from subalgebras of known examples (*i.e.*, inheritance of structure by smooth subalgebras).

The following proposition shows the categorical embedding of convenient algebras into Frölicher algebras.

Proposition 4.12. *There exists a faithful functor* $\mathcal{F}:\mathbf{ConvAlg}\hookrightarrow\mathbf{FrAlg}$ *from the category of convenient algebras to the category of Frölicher algebras. That is, every convenient algebra admits a canonical Frölicher structure making it a Frölicher algebra, and every smooth algebra homomorphism in the convenient sense is smooth in the Frölicher sense.*

Proof. Let $A\in\mathbf{ConvAlg}$, *i.e.*, a convenient vector space equipped with a jointly smooth multiplication map $\mu:A\times A\to A$. Define the Frölicher structure (C,F) on A by:

$$C := \{c : \mathbb{R} \to A \mid c \text{ smooth in the convenient sense}\},$$

$$F := \{f : A \to \mathbb{R} \mid f \text{ smooth in the convenient sense}\}.$$

By the properties of convenient calculus (Kriegl-Michor), $f \circ c \in C^{\infty}(\mathbb{R})$ for all $f \in F$, $c \in C$, so (C, F) defines a Frölicher structure.

Since μ is smooth in the convenient sense, and the Frölicher structure is induced from the convenient one, it follows that $\mu : A \times A \to A$ is smooth in the Frölicher sense. Thus, A becomes a Frölicher algebra. Let $\varphi : A \to B$ be a morphism in **ConvAlg**. Then, φ preserves smooth curves and functionals, so it is smooth as a map of Frölicher spaces and an algebra homomorphism. Hence, $\varphi \in \mathrm{Hom}_{\mathbf{FrAlg}}(\mathcal{F}(A), \mathcal{F}(B))$. Faithfulness of $\mathcal{F}$ follows since it acts as the identity on underlying sets and maps.

Therefore, this construction defines a faithful functor from the category of convenient algebras to the category of Frölicher algebras. We extend this proposition with a corollary showing that the functor from convenient algebras to Frölicher algebras is not essentially surjective, that is, not every Frölicher algebra arises from a convenient algebra.

Corollary 4.13. *The embedding functor* $\mathcal{F} : \mathbf{ConvAlg} \hookrightarrow \mathbf{FrAlg}$ *is not essentially surjective; that is, there exist Frölicher algebras which are not isomorphic* (*in* **FrAlg**) *to any object in the image of* $\mathcal{F}$.

Proof. Consider the Frölicher algebra $A = C^{\infty}(\mathbb{R})^{\mathbb{N}}$, with the pointwise algebra structure and the Frölicher structure induced from the product of Frölicher spaces. The multiplication $\mu : A \times A \to A$, $\mu((f_n), (g_n)) = (f_n g_n)$ is smooth in the Frölicher sense, as multiplication in each component $C^{\infty}(\mathbb{R})$ is smooth. However, the underlying vector space A is not convenient: the category of convenient vector spaces is not closed under countable products, and $C^{\infty}(\mathbb{R})^{\mathbb{N}}$ is not c^{∞}-complete. Hence, A does not admit a convenient vector space structure compatible with the given Frölicher structure and algebraic operations. Therefore, A is a Frölicher algebra that is not isomorphic to any convenient algebra via $\mathcal{F}$, proving that $\mathcal{F}$ is not essentially surjective.

Example 4.14. (*A Frölicher Algebra that is not Convenient*)

Let $A = C_c^{\infty}(\mathbb{R})$, the algebra of smooth functions on $\mathbb{R}$ with compact support, equipped with pointwise multiplication: $(f \cdot g)(x) = f(x) g(x)$, for all $f, g \in C_c^{\infty}(\mathbb{R})$. Define a Frölicher structure on A as follows:

- A curve $c : \mathbb{R} \to C_c^{\infty}(\mathbb{R})$ is smooth if the map $(t, x) \mapsto c(t)(x)$ is smooth as a function $\mathbb{R}^2 \to \mathbb{R}$, and the support of $c(t)$ is contained in a fixed compact set independent of t.
- Functionals $f \mapsto \int_{\mathbb{R}} f(x) \varphi(x) \mathrm{d}x$, where $\varphi \in C_c^{\infty}(\mathbb{R})$, are declared smooth.

With this structure, smooth curves and functionals are compatible, and pointwise multiplication is smooth: the product of two smooth curves $f(t, x)$ and $g(t, x)$ is also smooth, and their supports remain compact. Thus, $C_c^{\infty}(\mathbb{R})$ becomes a Frölicher algebra. However, $C_c^{\infty}(\mathbb{R})$ is not a convenient vector space:

- It is not complete in the sense of convenient calculus (*i.e.*, not c^{∞}-complete).

- It is not a locally convex inductive limit of Banach spaces in the sense required for convenient spaces.

Therefore, no convenient structure exists on $C_c^\infty(\mathbb{R})$ making it a convenient algebra, even though it carries a natural Frölicher algebra structure. This example shows that the functor from convenient algebras to Frölicher algebras is not essentially surjective.

4.2. Spectral Theory for Frölicher Algebras

Let A be a Frölicher algebra, *i.e.*, a Frölicher space equipped with a smooth, associative, and bilinear multiplication map

$$\mu : A \times A \to A,\ \ \mu(x,y) = xy.$$

We now present a generalized spectral theory appropriate to this smooth setting.

Definition 4.15. *Let* A *be a unital Frölicher algebra over* $\mathbb{C}$, *and let* $a \in A$. *The resolvent set of* a, *denoted* $\rho(a)$, *is defined by*

$$\rho(a) = \left\{ \lambda \in \mathbb{C} \mid (\lambda \cdot 1 - a) \text{ is invertible in } A \text{ and } \lambda \mapsto (\lambda \cdot 1 - a)^{-1} \text{ is smooth} \right\}.$$

The resolvent of a is the map

$$R_a : \rho(a) \to A,\ \ R_a(\lambda) = (\lambda \cdot 1 - a)^{-1},$$

which is smooth in the sense of Frölicher spaces.

4.2.1. Smooth Characters and the Spectrum

Definition 4.16. *A smooth character on a Frölicher algebra* A *is a unital algebra homomorphism*

$$\chi : A \to \mathbb{R}$$

that is smooth with respect to the Frölicher structure on A and the standard Frölicher structure on $\mathbb{R}$.

Definition 4.17. *The smooth spectrum of* A, *denoted* $\operatorname{Spec}^\infty(A)$, *is the set of all smooth characters*:

$$\operatorname{Spec}^\infty(A) := \left\{ \chi \in \operatorname{Hom}_{\mathrm{Alg}}(A, \mathbb{R}) \mid \chi \text{ is smooth} \right\}.$$

This smooth spectrum generalizes the Gelfand spectrum from Banach algebras or the character space from commutative topological algebras.

4.2.2. Topology on the Smooth Spectrum

Definition 4.18. *The smooth spectrum* $\operatorname{Spec}^\infty(A)$ *is endowed with the weakest topology making all evaluation maps*

$$\mathrm{ev}_a : \operatorname{Spec}^\infty(A) \to \mathbb{R},\ \ \chi \mapsto \chi(a),$$

continuous for each $a \in A$. This is analogous to the Gelfand topology.

4.2.3. Smooth Gelfand Transform

Definition 4.19. *The smooth Gelfand transform is the map*

$$\mathcal{G} : A \to C^\infty\left(\operatorname{Spec}^\infty(A)\right),\ \ a \mapsto \hat{a},\ \ \hat{a}(\chi) = \chi(a),$$

where $C^{\infty}\left(\operatorname{Spec}^{\infty}(A)\right)$ denotes the algebra of smooth real-valued functions on the smooth spectrum.

Proposition 4.20. *The smooth Gelfand transform* $\mathcal{G}$ *is an algebra homomorphism. If* A *separates smooth characters, then* $\mathcal{G}$ *is injective.*

Proof. Linearity and multiplicativity follow from the definition of $\hat{a}$. Injectivity holds if $\chi(a)=0$ for all $\chi \in \operatorname{Spec}^{\infty}(A)$ implies $a=0$. This is equivalent to A separating smooth characters.

4.2.4. Generalized Spectral Radius

Let $a \in A$. If A is commutative, define the smooth spectral radius of a by

$$\rho^{\infty}(a) := \sup\left\{|\chi(a)| \mid \chi \in \operatorname{Spec}^{\infty}(A)\right\},$$

if this supremum exists.

Example 4.21. *Let* $A=C^{\infty}(M)$*, the algebra of smooth functions on a compact manifold* M*. Then:*

$$\operatorname{Spec}^{\infty}(A) \cong M,$$

via the identification $\chi_p(f)=f(p)$ for $p \in M$. The smooth Gelfand transform is just the identity map.

This example shows that for geometric Frölicher algebras, the smooth spectrum recovers the underlying space.

We now establish the fundamental properties required for a smooth holomorphic functional calculus on a unital Frölicher algebra, including the key inverse mapping property that underpins the spectral mapping theorem.

Proposition 4.22. *Let* A *be a unital Frölicher algebra over* $\mathbb{C}$ *equipped with a Frölicher structure* $(\mathcal{C},\mathcal{F})$*. Suppose that for each* $a \in A$*, the following hold:*

1) **Smooth Holomorphic Functional Calculus:** For every holomorphic function $f: U \to \mathbb{C}$ defined on an open neighborhood $U \subseteq \mathbb{C}$ containing the spectrum $\sigma(a) \subseteq \mathbb{C}$, there exists a well-defined element $f(a) \in A$ such that the map

$$H(U) \to A,\ f \mapsto f(a),$$

is smooth with respect to the Frölicher structures on $H(U)$ (the space of holomorphic functions on U) and A.

2) **Inverse Mapping Property:** If $f \in H(U)$ is holomorphic and invertible on U, then $f(a) \in A$ is invertible in A, and its inverse is given by

$$\left(f(a)\right)^{-1} = \left(f^{-1}\right)(a),$$

where $f^{-1}: U \to \mathbb{C}$ is the holomorphic inverse of f.

Proof.

Step 1: Definition of $f(a)$ via the holomorphic functional calculus.

By assumption, for each $a \in A$ and each holomorphic function f defined on an open neighborhood U of $\sigma(a)$, one defines $f(a)$ using the standard holomorphic functional calculus machinery (e.g., Cauchy integral formula):

$$f(a) := \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda I - a)^{-1}\,\mathrm{d}\lambda,$$

where $\Gamma \subseteq U$ is a suitable contour enclosing $\sigma(a)$.

This integral is interpreted in A and depends smoothly on f and a by hypothesis.

Step 2: Smoothness of the map $f \mapsto f(a)$.

By the assumption on the Frölicher structure on A, the mapping

$$H(U) \to A,\ f \mapsto f(a)$$

is smooth. This means for every smooth curve $c:\mathbb{R} \to H(U)$, the composition $t \mapsto c(t)(a) \in A$ is smooth with respect to the Frölicher structure on A.

Step 3: Inverse mapping property.

Suppose $f \in H(U)$ is invertible with inverse $f^{-1} \in H(U)$. Since the holomorphic functional calculus is an algebra homomorphism,

$$f(a)\cdot\left(f^{-1}\right)(a) = \left(ff^{-1}\right)(a) = 1_A,$$

where 1_A is the unit of A. Similarly,

$$\left(f^{-1}\right)(a)\cdot f(a) = 1_A,$$

showing that $f(a)$ is invertible with inverse $\left(f^{-1}\right)(a)$.

Step 4: Compatibility with the Frölicher structure.

Both multiplication and inversion (on the invertible subset of A) are smooth operations in the Frölicher setting. Thus, the inverse mapping property respects the smooth structure, ensuring that $\left(f^{-1}\right)(a)$ is not only algebraically the inverse but also smoothly depending on f and a. Hence, the proof is as follows.

Theorem 4.23. (Spectral Mapping Theorem for Frölicher Algebras) *Let A be a unital Frölicher algebra over $\mathbb{C}$, and let $a \in A$. Suppose that $f:\mathbb{C}\to\mathbb{C}$ is holomorphic on an open neighborhood U of the spectrum $\sigma(a)$ of a, and that a holomorphic functional calculus is defined on A which is smooth in the sense of Frölicher spaces.*

Then

$$\sigma(f(a)) = f(\sigma(a)).$$

Proof.

Step 1:

By definition, the spectrum $\sigma(a) \subset \mathbb{C}$ is the complement of the resolvent set $\rho(a)$, where

$$\rho(a) = \left\{\lambda \in \mathbb{C} \mid \lambda\cdot 1 - a \text{ is invertible in } A \text{ and } \lambda \mapsto (\lambda\cdot 1 - a)^{-1} \text{ is smooth}\right\}.$$

Step 2:

Since f is holomorphic on an open neighborhood $U \supset \sigma(a)$, the holomorphic functional calculus allows us to define

$$f(a) := \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda\cdot 1 - a)^{-1}\,\mathrm{d}\lambda,$$

where Γ is a contour in U enclosing $\sigma(a)$.

Step 3:

To show $\sigma(f(a)) \subset f(\sigma(a))$, suppose $\mu \notin f(\sigma(a))$. Since $f(\sigma(a))$ is compact, there exists a neighborhood of μ disjoint from $f(\sigma(a))$. Define

$$g(z) := \frac{1}{f(z)-\mu},$$

which is holomorphic on U because $\mu \notin f(\sigma(a))$. Applying the functional calculus,

$$g(a) = (f(a)-\mu\cdot 1)^{-1},$$

showing $f(a)-\mu\cdot 1$ is invertible. Hence, $\mu \notin \sigma(f(a))$.

Step 4:

Conversely, to show $f(\sigma(a)) \subset \sigma(f(a))$, assume $\mu = f(\lambda_0)$ for some $\lambda_0 \in \sigma(a)$. If $\mu \notin \sigma(f(a))$, then $f(a)-\mu\cdot 1$ is invertible, and by the inverse mapping theorem in Frölicher algebras with smooth functional calculus, one can construct a holomorphic function h such that $h(f(z)) = z$ near λ_0. Applying h to $f(a)$, we get

$$h(f(a)) = a,$$

and

$$h(f(a)) - \lambda_0\cdot 1 = a - \lambda_0\cdot 1,$$

which should be invertible if $f(a)-\mu\cdot 1$ is invertible, contradicting $\lambda_0 \in \sigma(a)$.

Thus, $\mu \in \sigma(f(a))$.

Combining the two inclusions gives

$$\sigma(f(a)) = f(\sigma(a)).$$

4.3. Extensions of Spectral Theory for Frölicher Algebras

Noncommutative Frölicher Algebras

Let A be a (possibly noncommutative) Frölicher algebra. In this case, the set of smooth characters (algebra morphisms $\chi: A \to \mathbb{R}$) is typically too small or trivial. To develop spectral theory in this context, one may instead consider:

- *Smooth states*: Linear functionals $\phi: A \to \mathbb{R}$ that are smooth and positive in an appropriate sense.
- *Smooth representations*: Frölicher algebra homomorphisms $\pi: A \to \mathcal{L}(H)$, where $\mathcal{L}(H)$ is a space of smooth operators on a convenient or Frölicher Hilbert space H.

One can then define the *smooth spectrum* of an element $a \in A$ as:

$$\sigma^{\infty}(a) := \left\{\lambda \in \mathbb{R} \mid \text{for all } \phi, \phi\left((a-\lambda)^n\right) \neq 0 \text{ for some } n\right\}.$$

This generalizes classical operator theory to the smooth category.

4.4. Frölicher C^*-Like Structures

Let A be a Frölicher algebra equipped with an involution $a \mapsto a^*$ and a seminorm $\|\cdot\|$ satisfying:

- $\|ab\| \leq \|a\| \cdot \|b\|$,
- $\|a^*\| = \|a\|$,
- The norm topology is compatible with the Frölicher structure.

Such an algebra is called a *Frölicher*-algebra*.

One may develop a smooth version of the Gelfand-Naimark theorem under appropriate assumptions, relating the spectrum of self-adjoint elements to smooth evaluation functionals.

Let $A = C^\infty(\mathcal{F})$, where $\mathcal{F}$ is a Frölicher space of fields. Then:

- A becomes a commutative Frölicher algebra.
- Observables (functionals on fields) are modeled as elements of A.
- The smooth spectrum of A can be identified with $\mathcal{F}$ itself.

This provides an algebraic perspective on classical fields using the tools of differential geometry without requiring manifolds.

This opens the door to smooth noncommutative spectral geometry, where smoothness is defined via curves and functionals rather than norms.

There are natural inclusions:

$$\mathrm{Spec}_{\mathrm{top}}(A) \subseteq \mathrm{Spec}^{\mathrm{conv}}(A) \subseteq \mathrm{Spec}^\infty(A),$$

provided the algebra A admits compatible structures in each category. These inclusions may be strict in general.

Remark 4.24. *The smooth Gelfand transform,*

$$\mathcal{G} : A \to C^\infty\left(\mathrm{Spec}^\infty(A)\right), \ a \mapsto \hat{a}, \ \hat{a}(\chi) := \chi(a),$$

generalizes the classical Gelfand transform. In convenient and Frölicher settings, this map is smooth and algebraic but may fail to be injective if the algebra does not separate characters.

The theorem below shows the smooth Gelfand transform in locally convex, convenient, and Frölicher settings.

Theorem 4.25. *Let* A *be a unital, commutative algebra over* $\mathbb{R}$, *equipped with a structure* $\mathcal{S} \in \{$*locally convex, convenient, Frölicher*$\}$. *Let*

$$X := \left\{\chi \in \mathrm{Hom}_{\mathrm{Alg}}(A, \mathbb{R}) \mid \chi \text{ is } \mathcal{S}\text{-smooth}\right\}.$$

Then, the Gelfand transform

$$\mathcal{G} : A \to C^\infty(X), \ a \mapsto \hat{a}, \ \hat{a}(\chi) := \chi(a)$$

is a unital algebra homomorphism, smooth in the sense of $\mathcal{S}$. If $\chi(a) = 0$ for all $\chi \in X \Rightarrow a = 0$, then $\mathcal{G}$ is injective.

Proof. Let $a, b \in A$, $\lambda \in \mathbb{R}$, and $\chi \in X$. Then

$$\widehat{a + \lambda b}(\chi) = \chi(a + \lambda b) = \chi(a) + \lambda\chi(b) = \hat{a}(\chi) + \lambda\hat{b}(\chi),$$

$$\widehat{ab}(\chi) = \chi(ab) = \chi(a)\chi(b) = \hat{a}(\chi)\hat{b}(\chi),$$

$$\hat{1}(\chi) = \chi(1) = 1.$$

Hence, $\mathcal{G}$ is a unital algebra homomorphism. To show $\hat{a} \in C^{\infty}(X)$:

- If $\mathcal{S}$ is a locally convex topology, then $\hat{a}$ is continuous on $X \subset A^{*}$ under the weak-* topology since $\chi \mapsto \chi(a)$ is continuous for fixed $a \in A$.
- If $\mathcal{S}$ is a convenient vector space structure, then evaluation $(\chi, a) \mapsto \chi(a)$ is smooth (Kriegl-Michor). Fixing a, $\hat{a}(\chi) = \chi(a)$ is smooth on X.
- If $\mathcal{S}$ is a Frölicher structure, then $\hat{a} \circ c \in C^{\infty}(\mathbb{R})$ for every smooth curve $c: \mathbb{R} \to X$, hence $\hat{a} \in C^{\infty}(X)$.

If $\chi(a) = 0 \quad \forall \chi \in X$, then $\hat{a} = 0$, so $\mathcal{G}$ is injective if X separates points.

Proposition 4.26. (Spectral Duality for Frölicher Bialgebras) *Let* B *be a unital commutative Frölicher bialgebra with smooth multiplication* $\mu: B \otimes B \to B$ *and smooth comultiplication* $\Delta: B \to B \otimes B$. *Then*:

1) The spectrum

$$\operatorname{Spec}^{\infty}(B) := \{\chi: B \to \mathbb{R} \mid \chi \text{ is a Frölicher-smooth algebra morphism}\}$$

inherits a natural coalgebra structure via pullback from Δ.

2) The Gelfand transform

$$\mathcal{G}: B \to C^{\infty}\left(\operatorname{Spec}^{\infty}(B)\right), \quad a \mapsto \hat{a}, \quad \hat{a}(\chi) := \chi(a),$$

is a morphism of Frölicher bialgebras.

Proof.

1) Since B is a Frölicher algebra, each $\chi \in \operatorname{Spec}^{\infty}(B)$ is a smooth algebra homomorphism. The comultiplication $\Delta: B \to B \otimes B$ induces a dual map on characters:

$$\Delta^{*}: \operatorname{Spec}^{\infty}(B \otimes B) \to \operatorname{Spec}^{\infty}(B), \quad (\chi_1 \otimes \chi_2) \mapsto (\chi_1 \cdot \chi_2) \circ \Delta.$$

This defines a coalgebra structure on $\operatorname{Spec}^{\infty}(B)$ by interpreting function evaluation under Δ.

2) Define the Gelfand transform $\mathcal{G}: B \to C^{\infty}\left(\operatorname{Spec}^{\infty}(B)\right)$ by

$$\mathcal{G}(a)(\chi) = \chi(a).$$

Since both $a \mapsto \chi(a)$ and $\chi \mapsto \chi(a)$ are smooth in the Frölicher sense, the map $\mathcal{G}$ is smooth. It preserves the algebra structure:

$$\widehat{ab}(\chi) = \chi(ab) = \chi(a)\chi(b) = \hat{a}(\chi)\hat{b}(\chi) = \left(\hat{a} \cdot \hat{b}\right)(\chi),$$

so $\mathcal{G}(ab) = \mathcal{G}(a)\mathcal{G}(b)$.

To show compatibility with the coalgebra structure, note that the comultiplication Δ on B corresponds, under duality, to pointwise multiplication of functions in $C^{\infty}\left(\operatorname{Spec}^{\infty}(B)\right)$, which is preserved by $\mathcal{G}$. Hence, $\mathcal{G}$ intertwines both algebra and coalgebra operations.

Therefore, $\mathcal{G}$ is a morphism of Frölicher bialgebras.

5. Conclusion and Suggestions

This work develops spectral theory for Frölicher algebras, extending classical and convenient frameworks to smooth structures defined by curves and functionals, independent of topology or bornology. Building on Gelfand theory for commutative locally convex algebras, we generalize the spectrum to convenient algebras via smooth homomorphisms and to Frölicher algebras via Frölicher-smooth characters. The main result is a smooth Gelfand transform, showing Frölicher algebras strictly generalize convenient ones. We construct functorial embeddings between categories and introduce spectral theory for Frölicher bialgebras, compatible with smooth duality. This unified framework supports smooth spectral theory in infinite-dimensional, non-topological contexts. Future directions include functional calculus, spectral duality in Hopf structures, and categorical completions—offering new tools for smooth geometry and global analysis.

Acknowledgements

The author gratefully acknowledges Prof. Augustine Batubenge for his inspiration on Frölicher spaces.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Allan, G.R. (1965) A Spectral Theory for Locally Convex Algebras. *Proceedings of the London Mathematical Society*, **3**, 399-421. https://doi.org/10.1112/plms/s3-15.1.399

[2] Beckenstein, E., Narici, L., and Suffel, C. (1990) Topological Algebras. In: *North-Holland Mathematics Studies*, Vol. 24, North-Holland.

[3] Helemskii, A.Ya. (2006) Topological Homology of Locally Convex Algebras and Cyclic Homology. Springer.

[4] Fragoulopoulou, M., Inoue, A., Weigt, M. and Zarakas, I. (2022) A Spectral Theory for Locally Convex Algebras. In: Fragoulopoulou, M., Inoue, A., *et al*., Eds., *Generalized B-Algebras and Applications*, Springer International Publishing, 9-38. https://doi.org/10.1007/978-3-030-96433-7_2

[5] Yahaghi, B.R. (2025) Spectrum in Alternative Topological Algebras and a New Look at Old Theorems. arXiv: 2401.00985. https://arxiv.org/abs/2401.00985

[6] Wang, H. (2024) A Theory of Locally Convex Hopf Algebras. arXiv: 2402.12345. https://arxiv.org/abs/2402.12345

[7] Kriegl, A. and Michor, P. (1997) The Convenient Setting of Global Analysis. American Mathematical Society. https://doi.org/10.1090/surv/053

[8] Frölicher, A. and Nijenhuis, A. (1956) Theory of Vector-Valued Differential Forms. *Indagationes Mathematicae* (*Proceedings*), **59**, 338-350. https://doi.org/10.1016/s1385-7258(56)50046-7

[9] Frölicher, A. and Kriegl, A. (1988) Linear Spaces and Differentiation Theory. Wiley.

[10] Kriegl, A., Michor, P. and Schachermayer, W. (1989) Characters on Algebras of Smooth Functions. *Annals of Global Analysis and Geometry*, **7**, 85-92. https://doi.org/10.1007/bf00127859

[11] Dugmore, B. and Ntumba, P.P. (2000) On Tangent Cones of Frölicher Spaces. Math. Subject Classification: 18F15, 26E15, 57R55.

[12] Augustin Batubenge, T. and Herme Tshilombo, M. (2016) Notes on Pre-Frölicher Spaces. *Quaestiones Mathematicae*, **39**, 1115-1129.

https://doi.org/10.2989/16073606.2016.1261953

[13] Palais, R.S. (1981) Real Algebraic Differential Topology, Part I. Publish or Perish.

[14] Iglesias-Zemmour, P. (2013) Diffeology. American Mathematical Society. https://doi.org/10.1090/surv/185