



Gauge Transformation Approach to a Dynamic Description in Dicke Model

Ahlem Abidi^{1,2}

¹Quantum and Nonlinear Optics Laboratory, Department of Physics, Nanostructured Materials, Faculty of Sciences of Tunis, Tunis El Manar University, El Manar, Tunisia

²Higher Institute of Technological Studies of Jendouba, Jendouba, Tunisia
Email: abidiahlem0@gmail.com

How to cite this paper: Abidi, A. (2025)
Gauge Transformation Approach to a
Dynamic Description in Dicke Model.
Open Access Library Journal, **12**: e14178.
<https://doi.org/10.4236/oalib.1114178>

Received: August 27, 2025

Accepted: November 3, 2025

Published: November 6, 2025

Copyright © 2025 by author(s) and Open
Access Library Inc.

This work is licensed under the Creative
Commons Attribution International
License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

Dynamic analysis can be an efficient tool to understand and characterize some physical systems at different stages of processing. This is the subject of this paper. We derive analytically the scale entropies, Fisher information and Wigner phase probability distribution of the Dicke model in a quantum phase transition. We focus the treatment on the Gauge transformation approach. The Dicke model is considered as two harmonic oscillators. For two different models, the first is two harmonic oscillators with angular frequencies varying for negative and positive time intervals, the second is two harmonic oscillators in a Paul Trap motion with a periodic quadrupole potential. The Gauge transformation approach is a path between classical and quantum such that the choice cosinusoidal form of the classical functions fully reflects the oscillatory behavior of entanglement, Fisher information and the Wigner phase probability distribution. Another interesting result, the time evolution of entanglement, Fisher information and the Wigner phase probability distribution are used to distinguish different quantum phase transitions. It is shown that the oscillations of the normal phase are always in phase advance compared to the critical phase, while the oscillations of the superradiant phase are in phase delay.

Subject Areas

Mathematical Physics, Quantum Information

Keywords

Dicke Model, Phase Transition, Entanglement, Fisher Information, Wigner Phase Probability Distribution

1. Introduction

For a long time, Lewis and Resenfield have developed a very powerful invariant method to find the exact quantum states, like the generalized invariant method [1]-[3], the path integral method [4] [5], the direct method to find quantum states in Gaussian form [6], Hermite polynomials [7] [8] etc, which is investigated well the one-dimensional harmonic oscillator. Also, quantum information is a further development of the Shannon's information theory. They are closely related to entropy mixing information and spirit. He ensures today the safety of Internet communications and opens a step to quantum computing, note for example: quantum cryptography, teleportation, coding [9]-[15]... The importance of the Lewis and Resenfield (LR) invariant method on quantum information processing has been considered just recently, specifying quantum entanglement. Entanglement is not only a qualitative phenomenon, it is then important to find quantitative observables for it. In fact, in recent years a reconsideration of quantum entanglement and some other concepts of quantum physics for dynamic systems has been made in terms of classical mathematical considerations. The trick is to introduce the invariant operator which is constructed on the basis of some solutions of differential equations. The most thorough analysis of this topic is due to refs [16]-[20], also this method is developed on the canonical non commutative phase space [21] [22]. In this regard, we have framed this work. We have introduced the gauge transformations approach to derive some properties of quantum entanglement, Fishier information and Wigner phase distribution probability on the Dicke model.

2. Gauge Transformation Approach to Entangled State Representation

The study of entangled dynamic systems is related to the time-dependent Hamiltonians by solving the usual Schrödinger equation. In this echo, we consider the single mode Dicke model with time-dependent angular frequencies $\omega(t)$ and atom-field coupling strength $\kappa(t)$ based on [23]. It is common to consider the situation where the resonance between the frequency of the field mode is equal to the transition frequency of the atoms. This simplifies the analysis and highlights the superradiance transition phenomenon, consequently we follow the Hamiltonian expression

$$H(t) = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2}\omega(t)(X_1^2 + X_2^2) + 2\kappa(t)\omega(t)X_1X_2 - \omega(t)(1+j). \quad (1)$$

The spatial coupling in (1) prevents us from studying directly the fundamental characteristics of the system. We will now consider the condition

$$Q_1 = \frac{1}{\sqrt{2}}(X_1 + X_2), \quad Q_2 = \frac{1}{\sqrt{2}}(X_1 - X_2), \quad (2)$$

and

$$p_1 = \frac{1}{\sqrt{2}}(P_1 + P_2), \quad p_2 = \frac{1}{\sqrt{2}}(P_1 - P_2), \quad (3)$$

to reach a satisfactory description. This therefore appears to be an entangled representation

$$H(t) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\Omega_1(t)Q_1^2 + \frac{1}{2}\Omega_2(t)Q_2^2 - \omega(t)(1+j), \tag{4}$$

where

$$\Omega_1(t) = 1 + \kappa(t)\omega(t), \quad \Omega_2(t) = 1 - \kappa(t)\omega(t). \tag{5}$$

Expression (4) is alike to the diagonal Hamiltonian of two entangled harmonic oscillators. Similarly, [24] find that the Dicke model in phase transition for a limited time can be described by an inverse harmonic oscillator model and that this time can be extended indefinitely in the thermodynamic limit. Substituting (4) in the Schrödinger equation $i\frac{\partial}{\partial t}\psi = H(t)\psi$, we obtain the solution as

$$\begin{aligned} \psi_{n,m}(Q_1, Q_2; t) = & \frac{1}{\sqrt{2^{n+m} n! m! \pi^2}} \left(\frac{\Omega_1(t)\Omega_2(t)}{\pi^2} \right)^{\frac{1}{4}} \times \exp \left[-\frac{\Omega_1(t)}{2} - \frac{\Omega_2(t)}{2} \right] \\ & \times H_n \left[\sqrt{\Omega_1(t)} Q_1 \right] H_m \left[\sqrt{\Omega_2(t)} Q_2 \right]. \end{aligned} \tag{6}$$

By referring to [25], we define a three consecutive transformations

$$U_{1,l}(t) = \exp \left[-\frac{i}{2} \int_0^t \Upsilon(t') (Q_l p_l + p_l Q_l) dt' \right], \quad U_{2,l}(t) = \exp \left[\frac{i}{4} \left(\frac{\dot{f}_j}{f_j} + \frac{\dot{f}_j^*}{f_j^*} \right) Q_l^2 \right]$$

and $U_{3,l}(t) = \exp \left[-\frac{1}{4} \ln |f(t)_j|^2 (Q_l p_l + p_l Q_l) \right]$ are applied to the form (4), where f_j are a complex functions, $l = 1, 2$ and $j = 1, 2$. Such that

$$\begin{aligned} H'(t) = & U_{1,1}(t)U_{2,1}(t)U_{3,1}(t) \frac{1}{2}(p_1^2 + \Omega_1(t)Q_1^2)U_{3,1}^{-1}(t)U_{2,1}^{-1}(t)U_{1,1}^{-1}(t) \\ & + U_{1,2}(t)U_{2,2}(t)U_{3,2}(t) \frac{1}{2}(p_2^2 + \Omega_2(t)Q_2^2)U_{3,2}^{-1}(t)U_{2,2}^{-1}(t)U_{1,2}^{-1}(t). \end{aligned} \tag{7}$$

This method shows an exact reformulation of a broad class of Dicke-type dynamics [26] [27], consequently expression (7) give as a solution the wave function $\Psi' = U_{1,1}(t)U_{2,1}(t)U_{3,1}(t)\psi$, and we obtain from expression (6)

$$\begin{aligned} \Psi'_{n,m}(Q_1, Q_2; t) = & \frac{(\beta_1(t)\beta_2(t))^{\frac{1}{4}}}{\sqrt{2^{n+m} n! m! \pi |f_1 f_2|}} \times \exp \left[-i \left(n + \frac{1}{2} \right) \int_0^t \frac{\beta_1(t')}{|f_1|^2} dt - i \left(m + \frac{1}{2} \right) \int_0^t \frac{\beta_2(t')}{|f_2|^2} dt \right] \\ & \times \exp \left[i \frac{Re(\dot{f}_1 f_1^*)}{2|f_1|^2} Q_1^2 + i \frac{Re(\dot{f}_2 f_2^*)}{2|f_2|^2} Q_2^2 - \frac{\beta_1(t)}{2|f_1|^2} Q_1^2 - \frac{\beta_2(t)}{2|f_2|^2} Q_2^2 \right] \\ & \times H_n \left[\sqrt{\frac{\beta_1(t)}{|f_1|}} Q_1 \right] H_m \left[\sqrt{\frac{\beta_2(t)}{|f_2|}} Q_2 \right]. \end{aligned} \tag{8}$$

Expression (8) is qualified by the phase factors $\left(n + \frac{1}{2} \right) \int_0^t \frac{\beta_1(t')}{|f_1|^2} dt$ and

$\left(m + \frac{1}{2}\right) \int_0^t \frac{\beta_2(t')}{|f_2|^2} dt$. The initial condition gives $\beta_1(t_0) = \Omega_1(t)$ and $\beta_2(t_0) = \Omega_2(t)$.

The quantum variables Q_1 and Q_2 can be checked using the homologous classical differential equations as

$$\ddot{Q}_j(t) + \Omega_j(t) Q_j(t) = 0. \quad (9)$$

The solution is given from the complex functions f_j . They are the basis of the dynamic description in the system and they are determined from expression

$$\frac{d}{dt}(f_j \dot{f}_j^* - \dot{f}_j f_j^*) = 0. \quad (10)$$

We have readily identified the entangled eigenstates going back to the variables (X_1, X_2) to achieve expression

$$\begin{aligned} \psi_{n,m}(X_1, X_2; t) &= \frac{(\beta_1(t)\beta_2(t))^{\frac{1}{4}}}{\sqrt{2^{n+m} n! m! \pi |f_1 f_2|}} \exp \left[-i \left(n + \frac{1}{2} \right) \int_0^t \frac{\beta_1(t')}{|f_1|^2} dt - i \left(m + \frac{1}{2} \right) \int_0^t \frac{\beta_2(t')}{|f_2|^2} dt \right] \\ &\times \exp \left[i (b_1 X_1^2 + b_2 X_2^2 + 2b_2 X_1 X_2) \right] \\ &\times \exp \left[-(c_1 + c_2) X_1^2 - (c_1 + c_2) X_2^2 - 2(c_1 - c_2) X_1 X_2 \right] \\ &\times H_n \left[\sqrt{\frac{\beta_1(t)}{2|f_1|}} (X_1 + X_2) \right] H_m \left[\sqrt{\frac{\beta_2(t)}{2|f_2|}} (X_1 - X_2) \right], \end{aligned} \quad (11)$$

where

$$\begin{aligned} b_1 &= \frac{\text{Re}(\dot{f}_1 f_1^*)^2}{4|f_1|^2} + \frac{\text{Re}(\dot{f}_2 f_2^*)^2}{4|f_2|^2}, \\ b_2 &= \frac{\text{Re}(\dot{f}_1 f_1^*)^2}{4|f_1|^2} - \frac{\text{Re}(\dot{f}_2 f_2^*)^2}{4|f_2|^2}, \\ c_1 &= \frac{\beta_1(t)}{4|f_1|^2} \quad \text{and} \quad c_2 = \frac{\beta_2(t)}{4|f_2|^2}. \end{aligned}$$

We have

$$\beta_j(t) = \text{Im} \left(f_j(t) (\dot{f}_j)^*(t) \right). \quad (12)$$

These parameters will be clarified later when we discuss their contribution to outcome evaluation. The present approach targets a time-dependent reformulation of the single-mode Dicke Hamiltonian via a structured LR-type gauge sequence, yielding a representation in terms of two time dependent Harmonic oscillators. This is distinct from many treatments that remain at the level of either the full Dicke Hamiltonian in a fixed basis or rely on perturbative/time-averaged approximations. The LR-guided transformation provides an analytic handle on the nonstationary dynamics and enables closed-form expressions for several information-theoretic quantities [28] [29].

3. Scale Entropies

The density matrix is the most immediate tool to describe an eigenstate. It is related to the entanglement entropy through the formula

$$S_j(t) = (1-j)^{-1} \ln \text{Tr}(\rho_{red}(x, x'; t))^j \quad \text{and it is given as}$$

$$\rho_{n,m}(X_1, X_2; X'_1, X'_2; t) = \psi_{n,m}(X_1, X_2; t) \psi_{n,m}^*(X'_1, X'_2; t). \quad (13)$$

$\psi_{n,m}(X_1, X_2; t)$ is the eigenstate defined in (11). Consequently

$$\begin{aligned} & \rho_{n,m}(X_1, X_2; X'_1, X'_2; t) \\ &= \frac{(\beta_1(t)\beta_2(t))^{\frac{1}{2}}}{2^{n+m} n! m! \pi |f_1 f_2|} \times H_n \left[\sqrt{\frac{\beta_1(t)}{2|f_1|}} (X_1 + X_2) \right] H_m \left[\sqrt{\frac{\beta_2(t)}{2|f_2|}} (X_1 - X_2) \right] \\ & \times H_n \left[\sqrt{\frac{\beta_1(t)}{2|f_1|}} (X'_1 + X'_2) \right] H_m \left[\sqrt{\frac{\beta_2(t)}{2|f_2|}} (X'_1 - X'_2) \right] \\ & \times \exp \left[i\epsilon(t)(X_1^2 - X_1'^2) - \frac{\mu(t)}{2}(X_1^2 + X_1'^2) + \nu(t) X_1 X_1' \right], \end{aligned} \quad (14)$$

To start, we specify the reduced density matrix of the vacuum states for one harmonic oscillator by the form

$$\begin{aligned} \rho_{(0,0)}(X_1, X'_1; t) &= \int dX_2 \rho_{0,0}(X_1, X_2; X'_1, X'_2; t) \\ &= \pi^{-1} (\mu(t) - \nu(t))^{\frac{1}{2}} \exp \left[i\epsilon(t)(X_1^2 - X_1'^2) - \frac{\mu(t)}{2}(X_1^2 + X_1'^2) + \nu(t) X_1 X_1' \right], \end{aligned} \quad (15)$$

where $\rho_{0,0}(X_1, X_2; X'_1, X'_2; t)$ is the density matrix of the overall system when ($n = m = 0$),

$$\mu(t) = 2(c_1 + c_2) - 16 \frac{(c_1 + c_2)^2 - b_2^2}{c_1 + c_2}, \quad (16)$$

$$\nu(t) = 16 \frac{(c_1 + c_2)^2 + b_2^2}{c_1 + c_2}, \quad (17)$$

and

$$\epsilon(t) = b_1 - 4 \frac{c_1 + c_2}{c_1 - c_2} b_2. \quad (18)$$

Then to find the entanglement entropies, we project expression (15) into the equation

$$\int_{-\infty}^{+\infty} \rho_{red}(X, X'; t) F_n(X'; t) = P_n(t) F_n(X; t), \quad (19)$$

to derive its eigenvalues. n is an integer number. The solution is provided with the following eigenvalues and eigenfunctions

$$P_n(t) = (1-j(t)) j(t)^n (t),$$

and

$$F_n(X; t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\sigma}{\pi} \right)^{\frac{1}{4}} H_n(\sqrt{\sigma} X) \exp \left[-\frac{\sigma}{2} X^2 + i\epsilon(t) X^2 \right]. \quad (20)$$

where

$$j(t) = \frac{\mu(t)}{\nu(t) + \frac{\sigma}{2}}, \quad \sigma = (\mu(t)^2 - \nu(t)^2)^{\frac{1}{2}}. \quad (21)$$

So we define the entanglement Rényi entropy in order j as

$$S_j(t) = \frac{1}{1-j} \left[\ln \left((1-j(t))^j \right) - \ln \left(1-j(t)^j \right) \right], \quad (22)$$

where

$$j(t) = \frac{\mu(t)}{\nu(t) + \frac{\sigma}{2}} \quad \text{and} \quad \sigma = (\mu(t)^2 - \nu(t)^2)^{\frac{1}{2}}. \quad (23)$$

The Rényi entropy is reduced to the von Neumann entropy in the boundary when $j \rightarrow 1$ and (22) becomes

$$S_1(t) = -\ln(1-j(t)) - j(t)(1-j(t))^{-1} \ln j(t). \quad (24)$$

We note that expressions (22) and (24) only depend on the solution of the classical Equation (9). Thus the choice of f_j give us information of the entanglement behavior. This expression is probably a good description of the entangled state. We expect that can be as a starting point between the classical and the quantum.

4. Fishier and Shannon Information

We can define the fishier information of an entangled Dicke model as [30]

$$F = \int f(X_1, X_2) \left[\frac{\partial}{\partial X_2} \left(\frac{\partial \ln(f(X_1, X_2))}{\partial X_1} \right) \right]^2 dX_1 dX_2. \quad (25)$$

$f(X_1, X_2)$ is the probability density, it is expressed from (11) as:

$$\begin{aligned} f(X_1, X_2) &= |\psi_{0,0}(X_1, X_2; t)|^2 \\ &= \frac{(\beta_1(t)\beta_2(t))^{\frac{1}{2}}}{\pi|f_1f_2|} \exp[-2(c_1+c_2)X_1^2 - 2(c_1+c_2)X_2^2 - 4(c_1-c_2)X_1X_2]. \end{aligned} \quad (26)$$

We have

$$\left[\frac{\partial}{\partial X_2} \left(\frac{\partial \ln(f(X_1, X_2))}{\partial X_1} \right) \right]^2 = 16((c_1+c_2)X_1 + (c_1-c_2))^2. \quad (27)$$

Consequently (25) become

$$\begin{aligned} F &= \frac{16(\beta_1(t)\beta_2(t))^{\frac{1}{2}}}{\pi|f_1f_2|} \left(\frac{\pi}{c_1+3c_2} \right)^{\frac{1}{2}} \left(\frac{1}{3c_1+c_2} \right)^{\frac{1}{2}} \\ &\times \left[\sqrt{\pi}(c_1-c_2)^2 + (c_1^2-c_2^2) \left(\frac{1}{c_1+3c_2} \right)^{\frac{1}{2}} + \frac{1}{4}(c_1+c_2)^2 \left(\frac{\pi}{(c_1+3c_2)^2} \right)^{\frac{1}{2}} \right]. \end{aligned} \quad (28)$$

The Shannon entropy is described as

$$S_s = -\int f(X_1, X_2) \ln f(X_1, X_2) dX_1 dX_2$$

$$= \frac{(\beta_1(t)\beta_2(t))^{\frac{1}{2}}}{|f_1 f_2|} \left(\frac{1}{3c_1 + c_2}\right)^{\frac{1}{2}} \left(\frac{1}{c_1 + 3c_2}\right)^{\frac{1}{2}} \left[\ln \left(\frac{(\beta_1(t)\beta_2(t))^{\frac{1}{2}}}{\pi |f_1 f_2|} \right) - \frac{1}{2} \right]. \quad (29)$$

5. Wigner Phase Probability Distribution of Entangled Dicke Model

We go further, we will study in this section the concept of phase state, its corresponding Wigner phase probability distribution, in order to clarify some properties of an entangled Dicke model. We start from expression (11), we can define the corresponding phase state of dimensions $(s_1 + 1)$ and $(s_2 + 1)$ as: [31] [32]

$$|\theta_k, \theta_l; t\rangle = \frac{1}{\sqrt{(s_1 + 1)(s_2 + 1)}} \sum_{n=0}^{s_1} \sum_{m=0}^{s_2} \exp[i(n\theta_k + m\theta_l)] |n, m; t\rangle, \quad (30)$$

and we can rewrite it as

$$\Psi_{\theta_k, \theta_l}(X_1 e^{i\theta_k}, X_2 e^{i\theta_l}; t)$$

$$= \frac{1}{\sqrt{(s_1 + 1)(s_2 + 1)}} \sum_{n=0}^{s_1} \sum_{m=0}^{s_2} \exp[i(n\theta_k + m\theta_l)] \psi_{n,m}(X_1 e^{i\theta_k}, X_2 e^{i\theta_l}; t). \quad (31)$$

Consequently the corresponding density matrix is defined as

$$\rho_{\theta_k, \theta_l}(X_1 e^{i\theta_k}, X_2 e^{i\theta_l}; X'_1 e^{-i\theta_k}, X'_2 e^{-i\theta_l}; t)$$

$$= \Psi_{\theta_k, \theta_l}(X_1 e^{i\theta_k}, X_2 e^{i\theta_l}; t) \Psi_{\theta_k, \theta_l}^*(X'_1 e^{-i\theta_k}, X'_2 e^{-i\theta_l}; t)$$

$$= \frac{(\beta_1(t)\beta_2(t))^{\frac{1}{2}}}{2^{n+m} n! m! \pi |f_1 f_2|} \sum_{n=0}^{s_1} \sum_{m=0}^{s_2} \frac{1}{(s_1 + 1)(s_2 + 1)}$$

$$\times H_n \left[\sqrt{\frac{\beta_1(t)}{2|f_1|}} (X_1 e^{i\theta_k} + X_2 e^{i\theta_l}) \right] \times H_n \left[\sqrt{\frac{\beta_1(t)}{2|f_1|}} (X'_1 e^{-i\theta_k} + X'_2 e^{-i\theta_l}) \right] \quad (32)$$

$$\times H_m \left[\sqrt{\frac{\beta_2(t)}{2|f_2|}} (X_1 e^{i\theta_k} - X_2 e^{i\theta_l}) \right] \times H_m \left[\sqrt{\frac{\beta_2(t)}{2|f_2|}} (X'_1 e^{-i\theta_k} - X'_2 e^{-i\theta_l}) \right]$$

$$\times \exp[-\sigma_1(t) e^{-2\theta_k} X_1^2 - \sigma_1(t) e^{-2\theta_l} X_2^2 - 2\sigma_2(t) e^{i\theta_k} e^{i\theta_l} X_1 X_2]$$

$$\times \exp[-\sigma_1^*(t) e^{-2\theta_k} X_1'^2 - \sigma_1^*(t) e^{-2\theta_l} X_2'^2 - 2\sigma_2^*(t) e^{-i\theta_k} e^{-i\theta_l} X'_1 X'_2].$$

We notice that

$$\sigma_1(t) = c_1 + c_2 - ib_1 \quad \text{and} \quad \sigma_2(t) = c_1 - c_2 - ib_2. \quad (33)$$

We can formally express the Wigner phase probability distribution as

$$P_{\theta_k, \theta_l} = \frac{1}{2\pi} \sum_{n,m=0}^{\infty} \sum_{n',m'=0}^{\infty} \exp[i(n-n')\theta_k] \exp[i(m-m')\theta_l]$$

$$\times \langle n, m | \rho_{\theta_k, \theta_l}(X_1 e^{i\theta_k}, X_2 e^{i\theta_l}; X'_1 e^{-i\theta_k}, X'_2 e^{-i\theta_l}; t) | n', m' \rangle. \quad (34)$$

Therefore, using (11) and (32); (34) can be computed go back to [33] as

$$\begin{aligned}
 & \langle n, m | \rho_{\theta_k, \theta_l} (X_1 e^{i\theta_k}, X_2 e^{i\theta_l}; X'_1 e^{-i\theta_k}, X'_2 e^{-i\theta_l}; t) | n', m' \rangle \\
 &= \frac{1}{2^{2\delta} \pi^2 |f_1 f_2|} \left(\frac{\beta_1(t) \beta_2(t)}{2 |f_1 f_2|} \right)^{2\delta+1} \sum_{n=0}^{s_1} \sum_{m=0}^{s_2} \sum_{k_1=0}^n \sum_{k_2=0}^m \sum_{k_3=0}^{n'} \sum_{k_4=0}^{m'} \frac{1}{(s_1+1)} \frac{1}{(s_2+1)} \binom{n}{k_1}^3 \binom{m}{k_2}^3 \binom{n'}{k_3} \binom{m'}{k_4} \\
 & \times \frac{(-1)^{\frac{3}{2}(n-k_1) + \frac{3}{2}(m-k_2) + \frac{1}{2}(n'-k_3) + \frac{1}{2}(m'-k_4)}}{\left(\left(\frac{n-k_1}{2} \right)! \right)^3 + \left(\left(\frac{m-k_2}{2} \right)! \right)^3 + \left(\left(\frac{n'-k_3}{2} \right)! \right) + \left(\left(\frac{m'-k_4}{2} \right)! \right)} \left(\sqrt{\frac{\beta_1}{2|f_1|}} \right)^{3k_1+k_3} \left(-\sqrt{\frac{\beta_2}{2|f_2|}} \right)^{3k_2+k_4} \\
 & \times \Gamma\left(\frac{2k_1+1}{2} + \delta\right) \Gamma\left(\frac{2k_2+1}{2} + \delta\right) \Gamma\left(\frac{k_1+k_3+1}{2} + \delta\right) \Gamma\left(\frac{k_2+k_4+1}{2} + \delta\right) \\
 & \times \frac{\left(((n-k_1) + \delta)! \right)^3 \left(((m-k_2) + \delta)! \right)^3 \left(((n'-k_3) + \delta)! \right) \left(((m'-k_4) + \delta)! \right)}{2 \left[\mathcal{G}_1^{\frac{2k_1+1}{2} + \delta} \mathcal{G}_2^{\frac{2k_2+1}{2} + \delta} \mathcal{G}_3^{\frac{k_1+k_3+1}{2} + \delta} \mathcal{G}_4^{\frac{k_2+k_4+1}{2} + \delta} \right]} \tag{35} \\
 & \times F_2 \left(\frac{-n+k_1}{2}, \frac{-n+k_1}{2}, \frac{2k_2+1}{2} + \delta, \frac{1}{2} + \delta, \frac{1}{2} + \delta; \frac{\beta_1(t)}{2|f_1|v_2}, \frac{\beta_1(t)e^{i\theta_l}}{2|f_1|v_2} \right) \\
 & \times F_2 \left(\frac{-m+k_2}{2}, \frac{-m+k_2}{2}, \frac{2k_1+1}{2} + \delta, \frac{1}{2} + \delta, \frac{1}{2} + \delta; \frac{\beta_2(t)}{2|f_2|v_1}, \frac{\beta_2(t)e^{i\theta_k}}{2|f_2|v_1} \right) \\
 & \times F_2 \left(\frac{-n+k_1}{2}, \frac{-n'+k_1}{2}, \frac{k_2+k_4+1}{2} + \delta, \frac{1}{2} + \delta, \frac{1}{2} + \delta; \frac{\beta_1(t)}{2|f_1|v_4}, \frac{\beta_1(t)e^{-i\theta_l}}{2|f_1|v_4} \right) \\
 & \times F_2 \left(\frac{-m+k_2}{2}, \frac{-m'+k_2}{2}, \frac{k_1+k_3+1}{2} + \delta, \frac{1}{2} + \delta, \frac{1}{2} + \delta; \frac{\beta_1(t)}{2|f_1|v_3}, \frac{\beta_2(t)e^{-i\theta_k}}{2|f_2|v_3} \right),
 \end{aligned}$$

where

$$\begin{aligned}
 v_1 &= \frac{\sigma_1(t)}{2} (2 + e^{-2\theta_k} + e^{-2\theta_l}) - \frac{1}{2} \sqrt{\sigma_1^2(t) (e^{-2\theta_k} - e^{-2\theta_l})^2 + 4\sigma_2^2(t) (1 + e^{i\theta_k} e^{i\theta_l})^2}, \\
 v_2 &= \frac{\sigma_1(t)}{2} (2 + e^{-2\theta_k} + e^{-2\theta_l}) + \frac{1}{2} \sqrt{\sigma_1^2(t) (e^{-2\theta_k} - e^{-2\theta_l})^2 + 4\sigma_2^2(t) (1 + e^{i\theta_k} e^{i\theta_l})^2}, \\
 v_3 &= \frac{\sigma_1^*(t)}{2} (2 + e^{-2\theta_k} + e^{-2\theta_l}) - \frac{1}{2} \sqrt{\sigma_1^{*2}(t) (e^{-2\theta_k} - e^{-2\theta_l})^2 + 4\sigma_2^{*2}(t) (1 + e^{-i\theta_k} e^{-i\theta_l})^2}, \\
 v_4 &= \frac{\sigma_1^*(t)}{2} (2 + e^{-2\theta_k} + e^{-2\theta_l}) + \frac{1}{2} \sqrt{\sigma_1^{*2}(t) (e^{-2\theta_k} - e^{-2\theta_l})^2 + 4\sigma_2^{*2}(t) (1 + e^{-i\theta_k} e^{-i\theta_l})^2}.
 \end{aligned}$$

F_2 is the Appell hypergeometric function and $\delta = \begin{cases} 1 \\ 0 \end{cases}$ is the Kronecker delta symbol.

Expression (35) one depend of f_j and the phase factors $e^{i\theta_k}$, $e^{i\theta_l}$; so it is easy to control it by setting those two parameters. This will be studied below.

6. Numerical Results and Discussion

Two Different Models

It is appropriate for further considerations that we limit ourselves to the discussion of the equilibrium properties of the system. We define the critical point

$\kappa_c = \frac{1}{2}\omega$. We note that when the coupling crosses the critical point $\kappa > \kappa_c$, the system undergoes a phase transition. It is in the superradiant phase transition. In the opposite $\kappa < \kappa_c$, the system is in the normal phase. We will present herein two applications to implement the analytical results of (22), (24) and (35) to a problem of two entangled harmonic oscillators. Two entangled harmonic oscillators with angular frequencies varying between negative and positive time intervals has been the subject of the first example. We address the particular case of sudden jump [34], expression (5) becomes:

$\Omega_{ji} = \Omega_j$, and $\Omega_{jf} = \Omega_j$ when we replace ω with $\omega_f = (1 + \beta_0)^{\frac{1}{2}} \omega$. (i, f) means respectively initial and final time.

We choose f_j in (10) as

$$f_{1k}(t) = \begin{cases} f_{11i}(t) = e^{i\Omega_{1i}t} \cos \Omega_{1i}t & t < 0 \\ f_{12f}(t) = e^{i\Omega_{1f}t} \cos \Omega_{1f}t & t > 0, \end{cases} \quad (36)$$

$$f_{2k}(t) = \begin{cases} f_{21i}(t) = e^{i\Omega_{2i}t} \sin \Omega_{2i}t & t < 0 \\ f_{22f}(t) = \frac{\Omega_{2i}}{\Omega_{2f}} e^{i\Omega_{2f}t} \sin \Omega_{2f}t & t > 0. \end{cases} \quad (37)$$

Consequently $\beta_j(t)$ in (12) become

$$\beta_{1k}(t) = \begin{cases} \beta_{11i}(t) = -\Omega_{1i} \cos \Omega_{1i}^2 t & t < 0 \\ \beta_{12f}(t) = -\Omega_{1f} \cos \Omega_{1f}^2 t & t > 0, \end{cases} \quad (38)$$

$$\beta_{2k}(t) = \begin{cases} \beta_{21i}(t) = -\Omega_{2i} \sin \Omega_{2i}^2 t & t < 0 \\ \beta_{22f}(t) = -\frac{\Omega_{2i}^2}{\Omega_{2f}} \sin \Omega_{2f}^2 t & t > 0. \end{cases} \quad (39)$$

where $(k = 1, 2)$.

The phases θ_k, θ_l reads:

$$\theta_k = \theta_0 + \frac{2\pi n}{s_1 + 1} \quad \text{and} \quad \theta_l = \theta_0 + \frac{2\pi m}{s_2 + 1}. \quad (40)$$

We set $\beta_0 = 0.3, \omega = 0.6$ gives reason to $\kappa_c = 0.3$ and $\theta_0 = \frac{\pi}{12}$.

To ease the numerical calculation, we assume that $(n', m') = (0, 1)$.

We will treat for the second case, two harmonic oscillators in a Paul trap, thus providing a quadrupole potential. Typically for this purpose, we used a periodic potential of the form [35]. In fact, the angular frequencies in (5) are:

$$\Omega_j = a + b \cos(\omega_j t), \quad (41)$$

and we have

$$f_j(t) = e^{i\Omega_j t} \left[1 + \frac{b}{\Omega_{0j}(t)} \cos(\Omega_{0j} t) \right]. \quad (42)$$

Ω_{0_j} is given by replacing ω_j in Ω_j by $\omega_0 = \frac{2\pi}{T_0}$. Consequently $\beta_j(t)$ in (12) become

$$\beta_j(t) = -\Omega_j \left(1 + \frac{b}{\Omega_{0_j}(t)} \cos(\Omega_{0_j} t) \right)^2. \tag{43}$$

We set $a = 0.2$ and $b = 0.45$.

The cosine-sine structure in expressions (36), (37) and (42) give reason to the oscillatory behavior of entanglement, Fisher information and Wigner phase probability distribution of **Figures 1-6**. We notice that even its expressions differ

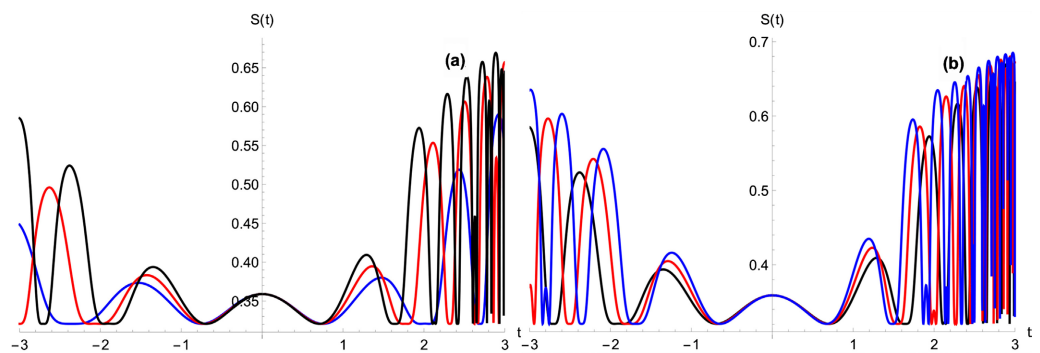


Figure 1. Plot of expression (24). (a): normal phase for $\kappa = 0.1$ (blue solid line), $\kappa = 0.2$ (red solid line), $\kappa = 0.3$ (black solid line). (b): superradiant phase for $\kappa = 0.3$ (black solid line), $\kappa = 0.4$ (red solid line), $\kappa = 0.5$ (blue solid line).

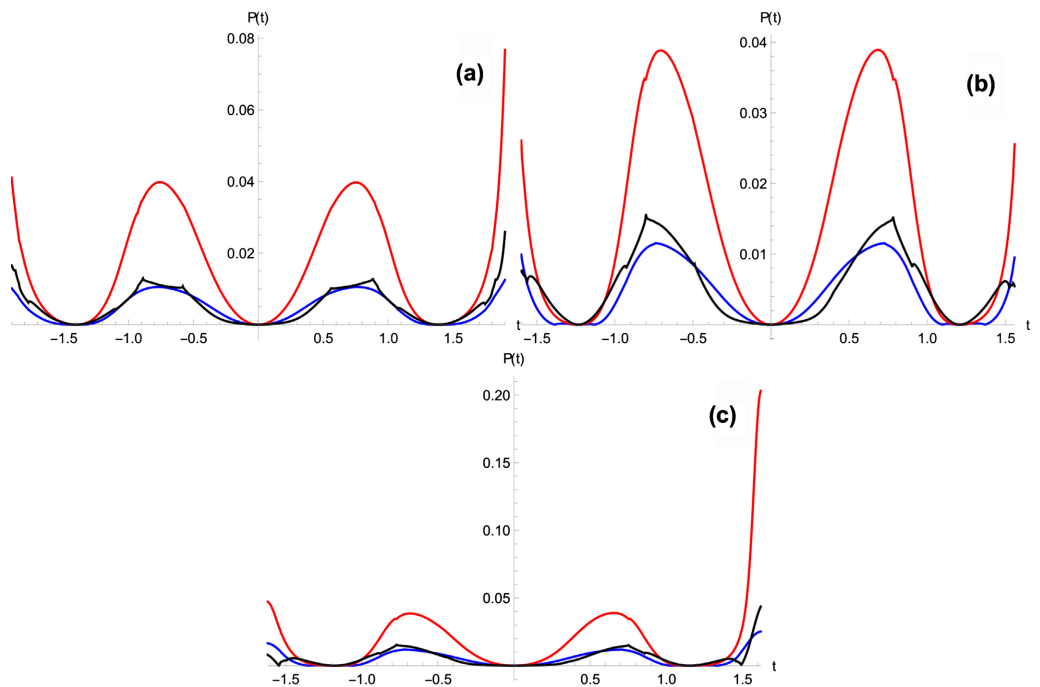


Figure 2. Plot of expression (35). (a): normal phase $\kappa = 0.1$, (b): critical phase $\kappa = 0.3$, (c): super-radiant phase $\kappa = 0.4$ and different values of the couple $(s_1, s_2) : \{(0,1)$ (blue solid line), $(0,2)$ (red solid line), $(0,3)$ (black solid line) $\}$.

formally, they can behave very similar, this means that f_j clearly reflects the common features between these concepts. In general, by increasing κ , the oscillations become more quickly, consequently the oscillations of the critical phase act as a good description of transition between phases. The oscillations of the normal phase oscillate in phase advance compared to the normal phase and in phase delay compared to the super radiant phase. With respect **Figure 2**, going beyond this interval, P_{θ_k, θ_l} becomes very large, the quick increase eliminates the oscillatory behavior and an increase of the dimension $(s_2 + 1)$ of the phase state gives reason to an interference between the different parts of the Wigner phase probability distribution. $s \rightarrow \infty$ does not mean that P_{θ_k, θ_l} can have a value towards infinity, consequently the limit $s \rightarrow \infty$ is also available. These non-classical properties can also be spanned by the arbitrary choice of the phases θ_k, θ_l . By increasing the coupling atom-field κ , entanglement and Fisher information increase **Figure 1** and **Figure 5**, or it presents entanglement interference dominated by large values when κ is large **Figure 3** and **Figure 6**.

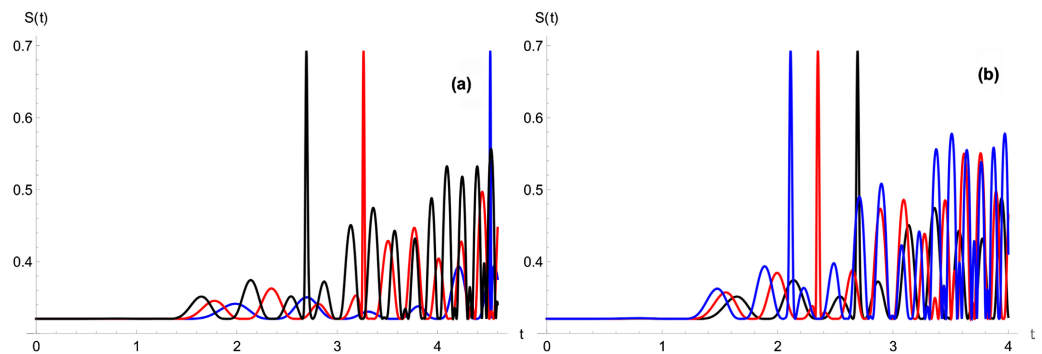


Figure 3. Plot of expression (24). (a): normal phase for $\kappa = 0.1$ (blue solid line), $\kappa = 0.2$ (red solid line), $\kappa = 0.3$ (black solid line). (b): superradiant phase for $\kappa = 0.3$ (black solid line), $\kappa = 0.4$ (red solid line), $\kappa = 0.5$ (blue solid line).

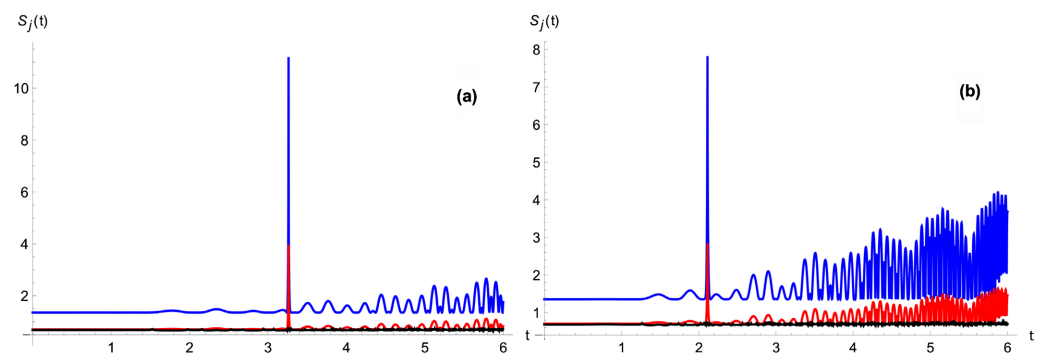


Figure 4. Plot of expression (22). (a): normal phase for $\kappa = 0.1$ and different values of $\beta : \{ \beta = 2$ (blue solid line), $\beta = 4$ (red solid line), $\beta = 100$ (black solid line)}. (b): superradiant phase same as Fig 4(a) but for $\kappa = 0.5$.

The combination of Rényi and von Neumann entropies, Fisher information, Shannon information, and Wigner-phase distribution in a single analytic framework is, to our knowledge, not simultaneously available in prior literature for this specific

mapping. This unifies entropic diagnostics with phase-space diagnostics under a common nonstationary, gauge-driven transformation. Unified, analytically tractable treatment of entropies, information, and phase-space under nonadiabatic driving.

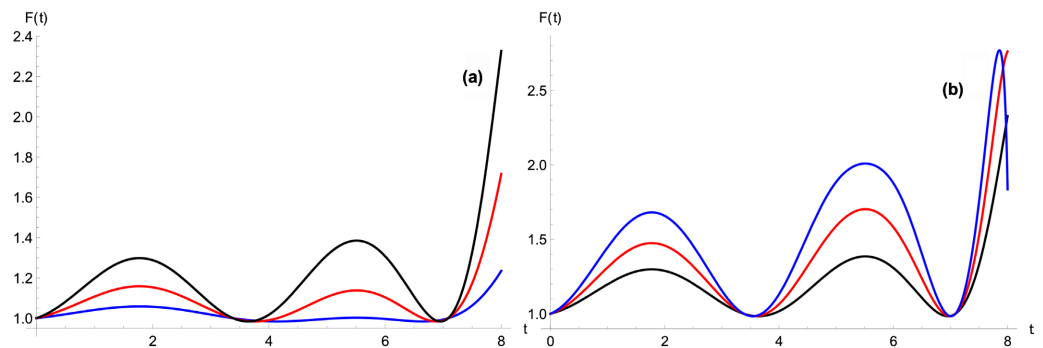


Figure 5. Plot of expression (28). (a): normal phase for $\kappa = 0.1$ (blue solid line), $\kappa = 0.2$ (red solid line) and $\kappa = 0.3$ (black solid line). (b): superradiant phase $\kappa = 0.3$ (black solid line), $\kappa = 0.4$ (red solid line) and $\kappa = 0.5$ (blue solid line).

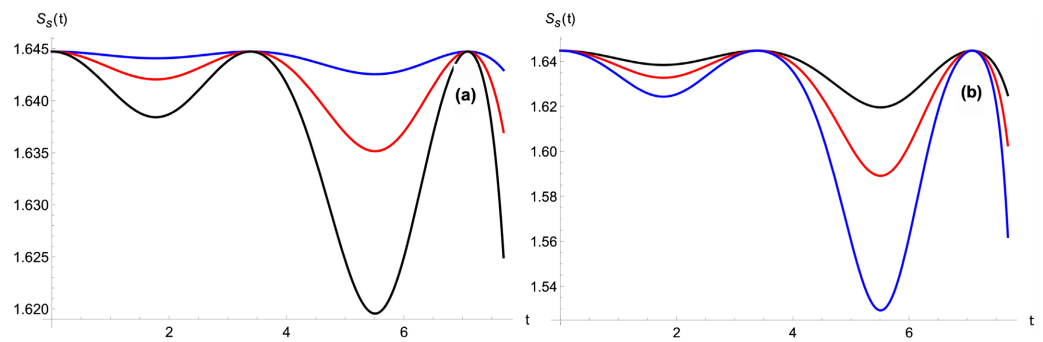


Figure 6. Plot of expression (29). (a): normal phase for $\kappa = 0.1$ (blue solid line), $\kappa = 0.2$ (red solid line) and $\kappa = 0.3$ (black solid line). (b): superradiant phase $\kappa = 0.3$ (black solid line), $\kappa = 0.4$ (red solid line) and $\kappa = 0.5$ (blue solid line).

7. Conclusion

This work has discussed entanglement, Fisher, Shannon information and Wigner phase probability distribution in a dynamic process of the Dicke model based on the Gauge transformation approach. The first is two harmonic oscillators with angular frequencies defined between negative and positive time intervals. The gauge transformation approach is a melting point between classical and quantum. The choice of the classical functions interprets the oscillatory behavior of entanglement, Fisher information and Wigner phase probability distribution. Following the evolution of the phase transition between two different models, the phase shift between oscillations is an attribute to identify the different phases of the model. The oscillations of the normal phase are in phase advance with respect to the critical phase while the superradiant phase is in phase delay.

Conflicts of Interest

The author declares no conflicts of interest.

References

- [1] Pedrosa, I.A., Serra, G.P. and Guedes, I. (1997) Wave Functions of a Time-Dependent Harmonic Oscillator with and without a Singular Perturbation. *Physical Review A*, **56**, 4300-4303. <https://doi.org/10.1103/physreva.56.4300>
- [2] Dantas, C.M.A., Pedrosa, I.A. and Baseia, B. (1992) Harmonic Oscillator with Time-Dependent Mass and Frequency and a Perturbative Potential. *Physical Review A*, **45**, 1320-1324. <https://doi.org/10.1103/physreva.45.1320>
- [3] Lewis, H.R. and Riesenfeld, W.B. (1969) An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field. *Journal of Mathematical Physics*, **10**, 1458-1473. <https://doi.org/10.1063/1.1664991>
- [4] Ling, R. and Feng, J. (1998) An Exact Wavefunction of Damped Harmonic Oscillator. *Acta Physica Sinica*, **47**, 1952-1956. <https://wulixb.iphy.ac.cn/en/article/doi/10.7498/aps.47.1952>
- [5] Yeon, K.H., Lee, K.K., Um, C.I., George, T.F. and Pandey, L.N. (1993) Exact Quantum Theory of a Time-Dependent Bound Quadratic Hamiltonian System. *Physical Review A*, **48**, 2716-2720. <https://doi.org/10.1103/physreva.48.2716>
- [6] Oh, H.G., Lee, H.R., George, T.F. and Um, C.I. (1989) Exact Wave Functions and Coherent States of a Damped Driven Harmonic Oscillator. *Physical Review A*, **39**, 5515-5522. <https://doi.org/10.1103/physreva.39.5515>
- [7] Pedrosa, I.A. (1997) Exact Wave Functions of a Harmonic Oscillator with Time-Dependent Mass and Frequency. *Physical Review A*, **55**, 3219-3221. <https://doi.org/10.1103/physreva.55.3219>
- [8] Lanfear, N., López, R.M. and Suslov, S.K. (2011) Exact Wave Functions for Generalized Harmonic Oscillators. *Journal of Russian Laser Research*, **32**, 352-361. <https://doi.org/10.1007/s10946-011-9223-1>
- [9] Ekert, A.K. (1991) Quantum Cryptography Based on Bell's Theorem. *Physical Review Letters*, **67**, 661-663. <https://doi.org/10.1103/physrevlett.67.661>
- [10] Kollmitzer, C. and Pivk, M. (2010) Applied Quantum Cryptography. [https://books.google.tn/books?hl=ar&lr=&id=hluC-QAAQBAJ&oi=fnd&pg=PR1&dq=Kollmitzer,+C.+and+Pivk,+M.+\(2010\)+Applied+Quantum+Cryptography.&ots=97MH1ctUzp&sig=fl4ua_F-gX3ojAGKz7Xir669KcE&redir_esc=y#v=onepage&q=Kollmitzer%2C%20C.%20and%20Pivk%2C%20M.%20\(2010\)%20Applied%20Quantum%20Cryptography.&f=false](https://books.google.tn/books?hl=ar&lr=&id=hluC-QAAQBAJ&oi=fnd&pg=PR1&dq=Kollmitzer,+C.+and+Pivk,+M.+(2010)+Applied+Quantum+Cryptography.&ots=97MH1ctUzp&sig=fl4ua_F-gX3ojAGKz7Xir669KcE&redir_esc=y#v=onepage&q=Kollmitzer%2C%20C.%20and%20Pivk%2C%20M.%20(2010)%20Applied%20Quantum%20Cryptography.&f=false)
- [11] Bennett, C.H., Brassard, G., Crépeau, C., Jozsa, R., Peres, A. and Wootters, W.K. (1993) Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels. *Physical Review Letters*, **70**, 1895-1899. <https://doi.org/10.1103/physrevlett.70.1895>
- [12] Ikram, M., Zhu, S. and Zubairy, M.S. (2000) Quantum Teleportation of an Entangled State. *Physical Review A*, **62**, Article ID: 022307. <https://doi.org/10.1103/physreva.62.022307>
- [13] Furusawa, A., Sørensen, J.L., Braunstein, S.L., Fuchs, C.A., Kimble, H.J. and Polzik, E.S. (1998) Unconditional Quantum Teleportation. *Science*, **282**, 706-709. <https://doi.org/10.1126/science.282.5389.706>
- [14] van Loock, P. and Braunstein, S.L. (2000) Multipartite Entanglement for Continuous Variables: A Quantum Teleportation Network. *Physical Review Letters*, **84**, 3482-3485. <https://doi.org/10.1103/physrevlett.84.3482>

- [15] Bennett, C.H. and Wiesner, S.J. (1992) Communication via One- and Two-Particle Operators on Einstein-Podolsky-Rosen States. *Physical Review Letters*, **69**, 2881-2884. <https://doi.org/10.1103/physrevlett.69.2881>
- [16] Ghosh, S., Gupta, K.S. and Srivastava, S.C.L. (2017) Entanglement Dynamics Following a Sudden Quench: An Exact Solution. *EPL (Europhysics Letters)*, **120**, Article No. 50005. <https://doi.org/10.1209/0295-5075/120/50005>
- [17] Park, D. (2018) Dynamics of Entanglement and Uncertainty Relation in Coupled Harmonic Oscillator System: Exact Results. *Quantum Information Processing*, **17**, Article No. 147. <https://doi.org/10.1007/s11128-018-1914-x>
- [18] Abidi, A., Trabelsi, A. and Krichene, S. (2021) Coupled Harmonic Oscillators and Their Application in the Dynamics of Entanglement and the Nonadiabatic Berry Phases. *Canadian Journal of Physics*, **99**, 898-906. <https://doi.org/10.1139/cjp-2020-0410>
- [19] Abidi, A. and Trabelsi, A. (2022) Dynamics of Entanglement in Coherent States, Entangled Schrödinger Cat State and Distribution Function. *Reports on Mathematical Physics*, **90**, 123-140. [https://doi.org/10.1016/s0034-4877\(22\)00054-4](https://doi.org/10.1016/s0034-4877(22)00054-4)
- [20] Abidi, A. and Trabelsi, A. (2024) The Schmidt Decomposition for Entangled System and Nonadiabatic Berry Phases. *American Journal of Physics and Applications*, **12**, 27-39. <https://doi.org/10.11648/j.ajpa.20241202.12>
- [21] Fiore, G. and Gouba, L. (2011) Class of Invariants for the Two-Dimensional Time-Dependent Landau Problem and Harmonic Oscillator in a Magnetic Field. *Journal of Mathematical Physics*, **52**, Article ID: 103509. <https://doi.org/10.1063/1.3653486>
- [22] Dey, S. and Fring, A. (2014) Noncommutative Quantum Mechanics in a Time-Dependent Background. *Physical Review D*, **90**, Article ID: 084005. <https://doi.org/10.1103/physrevd.90.084005>
- [23] Romera, E. and Nagy, Á. (2011) Rényi Entropy and Quantum Phase Transition in the Dicke Model. *Physics Letters A*, **375**, 3066-3069. <https://doi.org/10.1016/j.physleta.2011.06.046>
- [24] Gietka, K. and Busch, T. (2021) Inverted Harmonic Oscillator Dynamics of the Nonequilibrium Phase Transition in the Dicke Model. *Physical Review E*, **104**, Article ID: 034132. <https://doi.org/10.1103/physreve.104.034132>
- [25] Lambert, N., Emary, C. and Brandes, T. (2005) Entanglement and Entropy in a Spin-Boson Quantum Phase Transition. *Physical Review A*, **71**, Article ID: 053804. <https://doi.org/10.1103/physreva.71.053804>
- [26] Liu, W. and Duan, L. (2023) Quantum Phase Transitions in a Generalized Dicke Model. *Entropy*, **25**, Article No. 1492. <https://doi.org/10.3390/e25111492>
- [27] Li, F., Wang, S.J., Weiguny, A. and Lin, D.L. (1994) Gauge Transformation Approach to the Exact Solution of a Generalized Harmonic Oscillator. *Journal of Physics A: Mathematical and General*, **27**, 985-992. <https://doi.org/10.1088/0305-4470/27/3/036>
- [28] Hall, M.J.W. (2000) Quantum Properties of Classical Fisher Information. *Physical Review A*, **62**, Article ID: 012107. <https://doi.org/10.1103/physreva.62.012107>
- [29] Nagy, Á. and Romera, E. (2012) Fisher Information, Rényi Entropy Power and Quantum Phase Transition in the Dicke Model. *Physica A: Statistical Mechanics and its Applications*, **391**, 3650-3655. <https://doi.org/10.1016/j.physa.2012.02.024>
- [30] Herrera Romero, R. and Bastarrachea-Magnani, M.A. (2024) Phase and Amplitude Modes in the Anisotropic Dicke Model with Matter Interactions. *Entropy*, **26**, Article No. 574. <https://doi.org/10.3390/e26070574>
- [31] Buzek, V. and Knight, P.L. (1995) I: Quantum Interference, Superposition States of

- Light, and Nonclassical Effects. *Progress in Optics*, **34**, 1-158.
- [32] Pegg, D.T. (1998) Complement of the Hamiltonian. *Physical Review A*, **58**, 4307-4313. <https://doi.org/10.1103/physreva.58.4307>
- [33] Prudnikov, A.P., Brychkov, Yu.A. and Marichev, O.I. (1986) Integrals and Series, Vol. 1: Elementary Functions. OPA.
<https://www.scribd.com/document/831302248/Integrals-and-Series-Vol-1-Elementary-Functions-Compress>
- [34] Ji, J., Kim, J.K. and Kim, S.P. (1995) Heisenberg-Picture Approach to the Exact Quantum Motion of a Time-Dependent Harmonic Oscillator. *Physical Review A*, **51**, 4268-4271. <https://doi.org/10.1103/physreva.51.4268>
- [35] Brown, L.S. (1991) Quantum Motion in a Paul Trap. *Physical Review Letters*, **66**, 527-529. <https://doi.org/10.1103/physrevlett.66.527>