Open Access Library Journal
2025, Volume 12, e14046
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Normalized Solutions to Fractional Kirchhoff-Choquard Type Equations with the Lower Critical Exponent

Haojia Zhang

School of Mathematics, Liaoning Normal University, Dalian, China
Email: zhanghaojia0603@163.com

How to cite this paper: Zhang, H.J. (2025) Normalized Solutions to Fractional Kirchhoff-Choquard Type Equations with the Lower Critical Exponent. *Open Access Library Journal*, **12**: e14046.
https://doi.org/10.4236/oalib.1114046

Received: July 30, 2025
Accepted: September 22, 2025
Published: September 25, 2025

Abstract

This paper is concerned with the normalized ground states to the following lower critical fractional Kirchhoff-Choquard type equations under the constraint $\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=c^2$,

$$\left(a+b\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)(-\Delta)^s u-\lambda u =\left(I_\theta*|u|^{\frac{N+\theta}{N}}\right)|u|^{\frac{N+\theta}{N}-2}u+\mu\left(I_\theta*|u|^q\right)|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

where $s\in(0,1)$, $N\in(2,4]$, $\theta\in(0,N)$, $a,b,c,\mu>0$, $q\in\left(\frac{N+\theta}{N},\frac{N+\theta+2s}{N}\right)$, $\lambda\in\mathbb{R}$ appears as a Lagrange multiplier and I_θ is the Riesz potential. Using the constraint variational method, we establish the existence of normalized ground states and analyze their asymptotic properties as $\mu\to0^+$ or $c\to0^+$.

Subject Areas

Partial Differential Equation

Keywords

Fractional Kirchhoff-Choquard Equation, Normalized Solution, Lower Critical Exponent

1. Introduction

In this paper, we study the following lower critical fractional Kirchhoff-Choquard type equation

$$\begin{aligned}&\left(a+b\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)(-\Delta)^s u-\lambda u\\&=\left(I_\theta*|u|^{\frac{N+\theta}{N}}\right)|u|^{\frac{N+\theta}{N}-2}u+\mu\left(I_\theta*|u|^q\right)|u|^{q-2}u\quad\text{in }\mathbb{R}^N\end{aligned}\tag{1.1}$$

with prescribed L^2-norm constraint

$$\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=c^2,\tag{1.2}$$

where $s\in(0,1)$, $N\in(2,4]$, $\theta\in(0,N)$, $a,b,c,\mu>0$, $q\in\left(\frac{N+\theta}{N},\frac{N+\theta+2s}{N}\right)$ and $\lambda\in\mathbb{R}$ appears as an unknown Lagrange multiplier. In particular, $\frac{N+\theta}{N}$ is the lower critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality and $\frac{N+\theta+2s}{N}$ is the L^2-critical exponent. The function $I_\theta:\mathbb{R}^N\to\mathbb{R}$ is called the Riesz potential and is defined as follows,

$$I_\theta(x):=\frac{A_\theta}{|x|^{N-\theta}},\quad\text{where }A_\theta:=\frac{\Gamma\left(\frac{N-\theta}{2}\right)}{\Gamma\left(\frac{\theta}{2}\right)\pi^{\frac{N}{2}}2^\theta}.$$

For convenience, we drop A_θ in what follows. The symbol $(-\Delta)^s$ is the fractional Laplace operator defined as

$$(-\Delta)^s u(x)=C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y,\ u\in H^s\left(\mathbb{R}^N\right),$$

where $C(N,s)$ is a dimensional constant and P.V. means the Cauchy principal value of the singular integral. As usual, the fractional Sobolev space $H^s\left(\mathbb{R}^N\right)$ is defined for any $s\in(0,1)$ as

$$H^s\left(\mathbb{R}^N\right)=\left\{u\in L^2\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}}\mathrm{d}y\in L^2\left(\mathbb{R}^N\right)\right\}.$$

Note from [1] that

$$2C^{-1}(N,s)\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

Hence, we denote the scalar product by

$$\langle u,v\rangle=\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v\,\mathrm{d}x+\int_{\mathbb{R}^N}uv\,\mathrm{d}x,\ \forall u,v\in H^s\left(\mathbb{R}^N\right)$$

and the norm by

$$\|u\|=\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x\right)^{\frac{1}{2}},\ \forall u\in H^s\left(\mathbb{R}^N\right).$$

The Kirchhoff problem arises in multiple areas of mathematical physics. The Kirchhoff problem was proposed in [2] as a generalization of the classical D'Alembert wave equations when researching the changes in the length of the string during vibrations. Additionally, the Kirchhoff problem also appears in biological systems (for example, population density). The Kirchhoff problem also appears in other fields like biological systems, such as population density. In [3], Lions proposed an abstract functional analysis framework to deal with the stationary analogue of the equation. After the work of Lions, the Kirchhoff problem began to receive more attention, and many physicists are more interested in normalized solutions. For instance, Ye [4] proved the existence of solutions with the constraint (1.2) for the following Kirchhoff equation:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u-\lambda u=|u|^{p-2}u,\ \ x\in\mathbb{R}^N, \tag{1.3}$$

where $N\le 3$, $p\in\left(2,2^*\right)$, $2^*=6$ if $N=3$ and $2^*=+\infty$ if $N=1,2$. In [4], the primary approach is the analysis of excluding the dichotomy of the minimizing sequences for the related constraint minimization problem. In [5], Ye continued to study problem (1.3) for the critical case, *i.e.* $p=2+\frac{8}{N}$. Ye proved that the functional has a critical point with a mountain pass geometry with the constraint (1.2) if $c>c^*:=\left(2^{-1}b\|Q\|_2^{\frac{8}{N}}\right)^{\frac{N}{8-2N}}$, where Q is the unique positive radial solution of $-2\Delta Q+\left(\frac{4}{N}-1\right)Q=|Q|^{\frac{8}{N}}Q$ in $\mathbb{R}^N$. If $c\in\left(0,c^*\right)$, the functional has no critical point on the constraint (1.2). Li and Ye [6] studied the existence and concentration phenomenon of normalized ground states to the following Kirchhoff equations with an external potential $V(x)$:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+V(x)u-\lambda u=|u|^{p-2}u,\ \ x\in\mathbb{R}^N,$$

where $N\le 3$, $p\in\left(2,2^*\right)$ and the potential $V:\mathbb{R}^N\to\mathbb{R}$. Li *et al.* [7] used a perturbed Pohozaev constraint approach to consider the existence and asymptotic behaviors of solutions to the following Kirchhoff equation:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)\Delta u-\lambda u=|u|^{p-2}u+\mu|u|^{r-2}u,\ \ x\in\mathbb{R}^3,$$

where $2<p<\frac{14}{3}<r\le 6$ or $\frac{14}{3}<r<p\le 6$ and $\mu>0$.

The Choquard equation arose from the works of Fröhlich [8] and Pekar [9]. It is used to reveal the quantum theory of a polaron, which states that free electrons in an ionic lattice interact with phonons. In 1976, the Choquard equation was also introduced by Ph. Choquard in the modelling of a one-component plasma [10]. In 1996, the Choquard equation also appeared in quantum gravity within the framework of Schrödinger-Newton systems, describing a self-gravitating quantum particle interacting with its own gravitational field [11]. In the past years, Choquard equation has attracted more and more attention from researchers. For example, Moroz and Schaftingen [12] studied the following semilinear elliptic problem:

$$-\Delta u+u=\left(I_\theta*|u|^p\right)|u|^{p-2}u,\ \ x\in\mathbb{R}^N, \tag{1.4}$$

where $p>1$. Moroz and Schaftingen studied the existence of ground states for problem (1.4) and established the regularity, positivity, symmetry, monotonicity and decay asymptotics of ground states. Li and Ma [13] studied the following Brezis-Nirenberg type problem for Choquard equations:

$$-\Delta u+u=\left(I_\theta*|u|^p\right)|u|^{p-2}u+\mu|u|^{r-2}u,\ \ x\in\mathbb{R}^N, \tag{1.5}$$

where $N\geq 3$, $p=\frac{N+\theta}{N}$ or $\frac{N+\theta}{N-2}$ and $r\in\left(2,2^*\right)$, and proved that problem (1.5) has positive and radially nonincreasing ground states by using the subcritical approximation and the Pohozaev constraint method. Yao *et al.* [14] considered the existence and nonexistence of normalized ground states for the following lower critical Choquard equations with the local perturbation:

$$-\Delta u+\lambda u=\gamma\left(I_\theta*|u|^{\frac{N+\theta}{N}}\right)|u|^{\frac{N+\theta}{N}-2}u+\mu|u|^{r-2}u,\ \ x\in\mathbb{R}^N,$$

where $r\in\left(2,2^*\right)$. In particular, Yao considered the limiting case $r=\frac{2N}{N-2}$ corresponding to the double critical exponent. Meng and He [15] studied the existence and the qualitative behavior of normalized ground states for the nonlinear fractional Choquard equations with Hardy-Littlewood-Sobolev upper critical exponent. Zhang *et al.* [16] proved the existence and asymptotic behaviors of the normalized ground states for a lower critical Choquard equation with a nonlocal perturbation.

Recently, the normalized ground states for Kirchhoff-Choquard type equations have been extensively concerned. For example, Liu [17] studied the following Kirchhoff-Choquard type equation:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u-\lambda u=\left(I_\theta*|u|^p\right)|u|^{p-2}u,\ \ x\in\mathbb{R}^N,$$

where $N\geq 3$ and $p\in\left(\frac{N}{N+\theta},\frac{N+\theta}{N-2}\right)$, and proved the threshold values separating the existence and nonexistence of the critical points of functional on constraint (1.2) and also proved behaviors of the Lagrange multipliers and the energies in relation to the constrained critical points of functional as $c\to 0^+$ or $c\to\infty$. Zhu *et al.* [18] considered the following Kirchhoff-Choquard type equation:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u-\lambda u=\left(I_\theta*|u|^p\right)|u|^{p-2}u+\left(I_\theta*|u|^r\right)|u|^{r-2}u,\ \ x\in\mathbb{R}^N,$$

where $N\geq 3$ and $\frac{N+\theta}{N}<r<\frac{N+\theta+4}{N}<p<\frac{N+\theta}{N-2}$ or $\frac{N+\theta+4}{N}<r<p<\frac{N+\theta}{N-2}$. Zhu *et al.* established the existence of two normalized ground states in the mixed critical case where $\frac{N+\theta}{N}<r<\frac{N+\theta+4}{N}<p<\frac{N+\theta}{N-2}$ and the existence of ground states in the L^2 supercritical case where

$\frac{N+\theta+4}{N}<r<p<\frac{N+\theta}{N-2}$. Liang *et al.* [19] studied the following lower critical Kirchhoff-Choquard type equations:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u=\alpha k(x)|u|^{r-2}u+\beta\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{2_\mu^*-2}u,\ x\in\mathbb{R}^N,$$

where $N\geq 3$, α and β are positive real parameters, $2_\mu^*=\frac{2N-\mu}{N-2}$ and $k\in L^t(\mathbb{R}^N)$ with $t=\frac{2^*}{2^*-r}$ if $1<r<2^*$ and $t=\infty$ if $r\geq 2^*$, and discussed the multiplicity of solutions by using concentration compactness principle and variational methods.

Finally, this paper is compared with several recent publications. In [20], Liu *et al.* studied the fractional Kirchhoff problem with combined nonlinearities, and we changed the general nonlinearities to the convolution nonlinearities. Although the nonlinear term of the problem in [15] is a convolution term, it employs the upper critical exponent, and its operator also differs from that in this paper. In contrast to [18], we studied the Kirchhoff-Choquard equation with fractional fields. Thus, our problem presents considerable research worth.

Motivated by the above works, in this paper, we intend to study the existence and asymptotic properties of the normalized ground states of problem (1.1). The energy functional associated with problem (1.1) is given by

$$\begin{aligned}E(u)=&\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)^2\\&-\frac{N}{2(N+\theta)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{\frac{N+\theta}{N}}|u(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\&-\frac{\mu}{2q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^q|u(y)|^q}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y.\end{aligned}\tag{1.6}$$

We define the following constraint set

$$\mathcal{S}_c=\left\{u\in H^s(\mathbb{R}^N):\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=c^2\right\}$$

and focus on studying the following constraint minimization problem:

$$e(c):=\inf_{u\in\mathcal{S}_c}E(u).\tag{1.7}$$

Now, we state our main results.

Theorem 1.1. *The problem* (1.7) *possesses a ground state* $u_{c,\mu}\in\mathcal{S}_c$ *and satisfies that*

$$E(u_{c,\mu})=e(c)<-\frac{N}{2(N+\theta)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}},$$

where S_θ is given in (2.1). Moreover, we have the following asymptotic behaviors:

$$\lim_{\mu\to 0^+} E\left(u_{c,\mu}\right) = \lim_{\mu\to 0^+} e(c) = -\frac{N}{2(N+\theta)} S_\theta^{-\frac{N+\theta}{N}} c^{\frac{2(N+\theta)}{N}},$$

$$\lim_{c\to 0^+} E\left(u_{c,\mu}\right) = \lim_{c\to 0^+} e(c) = 0.$$

Corollary 1.2. *The problem* (1.1) *possesses a ground state* $u_{c,\mu} \in \mathcal{S}_c$ *and the corresponding Lagrange multiplier* $\lambda_{c,\mu} < 0$.

In this paper, we don't prove that the ground states $u_{c,\mu}$ are positive, radially symmetric. We treat this as an open problem for readers. If the readers are interested in this problem, we refer them to [13] [18] [20] and the references therein for similar proofs.

The rest of this paper is organized as follows. In Section 2, we recall some preliminary facts that will be used in proving main results. In Section 3, we complete the proof of Theorem 1.1 and Corollary 1.2.

Throughout this paper, we adopt the following notations.

- $B_r(y) := \left\{x \in \mathbb{R}^N : |x-y| < r\right\}$.
- C denotes any positive constants that may be different in different places.
- $\|\cdot\|_p$ is the standard norm in Lebesgue space $L^p\left(\mathbb{R}^N\right)$ for $p \in [1,\infty)$.
- The symbol $\rightharpoonup$ denotes weak convergence and the symbol $\to$ denotes strong convergence.
- $o_n(1)$ denotes a real sequence tending to 0 as $n \to \infty$.
- $H^{-s}\left(\mathbb{R}^N\right)$ is the dual space of $H^s\left(\mathbb{R}^N\right)$.
- $2_s^* = \dfrac{2N}{N-2s}$ is the fractional critical Sobolev exponent.

2. Preliminaries

In this section, we summarize several preliminary results. First, let us recall the well-known Hardy-Littlewood-Sobolev inequality.

Lemma 2.1. [21] *Let* $r,t>1, \theta \in (0,N)$ *satisfy* $\frac{1}{r}+\frac{1}{t}=\frac{N+\theta}{N}$, $f \in L^r\left(\mathbb{R}^N\right)$ *and* $g \in L^t\left(\mathbb{R}^N\right)$. *Then, there exists a sharp constant* $C(N,\theta,r,t)>0$, *independent of* f *and* g, *such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y\right| \le C(N,\theta,r,t)\|f\|_r\|g\|_t.$$

If $r = t = \dfrac{2N}{N+\theta}$, then

$$C(N,\theta,r,t) := C(N,\theta) = \pi^{\frac{N-\theta}{2}} \frac{\Gamma\left(\frac{\theta}{2}\right)}{\Gamma\left(\frac{N+\theta}{2}\right)} \left[\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right]^{-\frac{\theta}{N}}.$$

Owing to Lemma 2.1, we derive that

$$S_\theta := \inf\left\{\|u\|_2^2 : u \in L^2\left(\mathbb{R}^N\right), \int_{\mathbb{R}^N}\left(I_\theta * |u|^{\frac{N+\theta}{N}}\right)|u|^{\frac{N+\theta}{N}} \mathrm{d}x = 1\right\} > 0. \tag{2.1}$$

Equivalently, for any $u \in H^s\left(\mathbb{R}^N\right)$, there holds

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{\frac{N+\theta}{N}}|u(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y \le S_\theta^{-\frac{N+\theta}{N}}\left(\int_{\mathbb{R}^N}|u|^2\mathrm{d}x\right)^{\frac{N+\theta}{N}}. \tag{2.2}$$

Moreover, from [22], we find that, for some fixed $C > 0$, $\varepsilon > 0$ and $z \in \mathbb{R}^N$, S_θ is achieved if and only if

$$u(x) = U_{\varepsilon,z}(x) := C\left(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2}\right)^{\frac{N}{2}},$$

which means that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|U_{\varepsilon,z}(x)|^{\frac{N+\theta}{N}}|U_{\varepsilon,z}(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y = S_\theta^{-\frac{N+\theta}{N}}\left(\int_{\mathbb{R}^N}|U_{\varepsilon,z}|^2\mathrm{d}x\right)^{\frac{N+\theta}{N}}. \tag{2.3}$$

If $f = g = |u|^p \in L^{\frac{2N}{N+\theta}}\left(\mathbb{R}^N\right)$ in Lemma 2.1, the following result is true by the Hölder inequality.

Lemma 2.2. [23] *Let* $\theta \in (0, N)$. *Then, for any* $u \in L^{\frac{2Np}{N+\theta}}\left(\mathbb{R}^N\right)$, *there exists a constant* $C(N,\theta) > 0$ *such that*

$$\int_{\mathbb{R}^N}\left(I_\theta * |u|^p\right)|u|^p\mathrm{d}x \le C(N,\theta)\left(\int_{\mathbb{R}^N}|u|^{\frac{2Np}{N+\theta}}\mathrm{d}x\right)^{\frac{N+\theta}{N}}.$$

Then, we introduce the fractional Gagliardo-Nirenberg inequality.

Lemma 2.3. [24] *Let* $N \ge 2$ *and* $p \in (2, 2_s^*)$. *Then, there exists a constant* $C(N,\theta,s) > 0$ *such that*

$$\int_{\mathbb{R}^N}|u(x)|^p\mathrm{d}x \le C(N,\theta,s)\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^{\frac{N(p-2)}{2s}}\|u\|_2^{p-\frac{N(p-2)}{2s}}, \quad \forall u \in H^s\left(\mathbb{R}^N\right).$$

Lemma 2.4. [15] *Let* $N > 2$ *and* $\frac{N+\theta}{N} < p < 2_{\theta,s}^* := \frac{N+\theta}{N-2s}$. *Then, there exists a constant* $C(N,\theta,s,p) > 0$ *such that*

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(I_\theta * |u|^p\right)|u|^p\mathrm{d}x \\ &\le C(N,\theta,s,p)\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^{\frac{Np-N-\theta}{s}}\|u\|_2^{2p\left(1-\frac{Np-N-\theta}{2ps}\right)}, \quad \forall u \in H^s\left(\mathbb{R}^N\right).\end{aligned}$$

Next, we give Brezis-Lieb lemma for the convolution term of the functional.

Lemma 2.5. [12] *Let* $N > 2$, $p \in [1, \infty)$ *and* $\{u_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^{\frac{2Np}{N+\theta}}\left(\mathbb{R}^N\right)$. *If* $u_n \to u$ *a.e. on* $\mathbb{R}^N$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\left(I_\theta*|u_n|^p\right)|u_n|^p\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(I_\theta*|u_n-u|^p\right)|u_n-u|^p\,\mathrm{d}x\right)=\int_{\mathbb{R}^N}\left(I_\theta*|u|^p\right)|u|^p\,\mathrm{d}x.$$

We also have classical Brezis-Lieb lemma for the nonlinear local term.

Lemma 2.6. *Let* $\Omega\subseteq\mathbb{R}^N$ *be a domain,* $p\in[1,\infty)$ *and* $\{u_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^r(\Omega)$. *If* $u_n\to u$ *a.e. on* Ω *as* $n\to\infty$, *then for every* $p\in[1,r]$,

$$\lim_{n\to\infty}\int_\Omega\left||u_n|^p-|u_n-u|^p-|u|^p\right|^{\frac{r}{p}}\mathrm{d}x=0.$$

Lemma 2.7. *If* $\{u_n\}\subset H^s(\mathbb{R}^N)$ *satisfies* $u_n\rightharpoonup u$ *in* $H^s(\mathbb{R}^N)$ *and* $u_n(x)\to u(x)$ *a.e. in* $\mathbb{R}^N$ *for some* $u\in H^s(\mathbb{R}^N)$ *as* $n\to\infty$, *there holds that*

$$E(u_n)=E(u)+E(u_n-u)+\frac{b}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2\left\|(-\Delta)^{\frac{s}{2}}(u_n-u)\right\|_2^2+o_n(1).$$

Proof. Since $u_n\rightharpoonup u$ as $n\to\infty$ in $H^s(\mathbb{R}^N)$, in view of Lemma 2.6, it implies that

$$\|u_n\|^2=\|u_n-u\|^2+\|u\|^2+o_n(1) \tag{2.4}$$

and

$$\int_{\mathbb{R}^N}|u_n|^2\,\mathrm{d}x=\int_{\mathbb{R}^N}|u_n-u|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x+o_n(1). \tag{2.5}$$

Combining (2.4) with (2.5), we have

$$\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2=\left\|(-\Delta)^{\frac{s}{2}}(u_n-u)\right\|_2^2+\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+o_n(1).$$

Hence, we can get that

$$\begin{aligned}\frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2&=\frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}}(u_n-u)\right\|_2^2+\frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2+o_n(1)\\ \frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^4&=\frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}}(u_n-u)\right\|_2^4+\frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^4\\ &\quad+\frac{b}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2\left\|(-\Delta)^{\frac{s}{2}}(u_n-u)\right\|_2^2+o_n(1).\end{aligned} \tag{2.6}$$

It follows Lemma 2.5 and (2.6) that

$$E(u_n)=E(u)+E(u_n-u)+\frac{b}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2\left\|(-\Delta)^{\frac{s}{2}}(u_n-u)\right\|_2^2+o_n(1).$$

Hence, the proof is complete. □

Lemma 2.8. [25] *Let* $R>0$ *and* $p\in[2,2_s^*)$. *If* $\{u_n\}$ *is bounded in* $H^s(\mathbb{R}^N)$ *and*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|u_n(x)|^p\,\mathrm{d}x=0,$$

then $u_n\to0$ in $L^t(\mathbb{R}^N)$ as $n\to\infty$ for any $t\in(2,2_s^*)$.

Note that for $q\in\left(\frac{N+\theta}{N},\frac{N+\theta+2s}{N}\right)$, we have $Nq-N-\theta\in(0,2s)$.

Lemma 2.9. *Let* $N\in(2,4]$. *Then,* $E(u)$ *is bounded from below and coercive*

$$\textit{on}\ \ \mathcal{S}_c\ \ \textit{and}\ \ e(c) < -\frac{N}{2(N+\theta)} S_\theta^{-\frac{N+\theta}{N}} c^{\frac{2(N+\theta)}{N}}.$$

Proof. Note from (2.2), Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} E(u) \geq{}& \frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^2 + \frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^4 - \frac{N}{2(N+\theta)} S_\theta^{-\frac{N+\theta}{N}} c^{\frac{2(N+\theta)}{N}} \\ &- \frac{\mu}{2q} C(N,\theta,s) \|u\|_{\frac{2Nq}{N+\theta}}^{2q} \\ \geq{}& \frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^4 - \frac{N}{2(N+\theta)} S_\theta^{-\frac{N+\theta}{N}} c^{\frac{2(N+\theta)}{N}} \\ &- \frac{\mu}{2q} C(N,\theta,s) \left\|(-\Delta)^{\frac{s}{2}} u\right\|_2^{\frac{Nq-N-\theta}{s}} c^{2q-\frac{Nq-N-\theta}{s}}. \end{aligned} \tag{2.7}$$

Since $Nq-N-\theta \in (0,2s)$, we obtain that $\frac{Nq-N-\theta}{s} \in (0,2)$. Then, (2.7) is obvious to obtain that $E(u) > -\infty$. Thus, we prove that $E(u)$ is bounded from below and coercive on $\mathcal{S}_c$.

For $u \in \mathcal{S}_c$ and $t \in \mathbb{R}$, we define

$$\varphi := \frac{c}{\left\|U_{\varepsilon,z}\right\|_2} U_{\varepsilon,z} \ \text{ and } \ (t \star \varphi)(x) := e^{\frac{N}{2}t} \varphi\left(e^t x\right) \ \text{ for } x \in \mathbb{R}^N.$$

Obviously, we observe that $\varphi \in \mathcal{S}_c$ and $t \star \varphi \in \mathcal{S}_c$.

In view of (2.3), we derive that

$$\begin{aligned} E(t\star\varphi) ={}& \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}(t\star\varphi)\right|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}(t\star\varphi)\right|^2 \mathrm{d}x\right)^2 \\ &- \frac{N}{2(N+\theta)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|(t\star\varphi)(x)\right|^{\frac{N+\theta}{N}}\left|(t\star\varphi)(y)\right|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y \\ &- \frac{\mu}{2q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|(t\star\varphi)(x)\right|^{q}\left|(t\star\varphi)(y)\right|^{q}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y \\ ={}& \frac{a}{2}e^{2st}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}\varphi\right|^2 \mathrm{d}x + \frac{b}{4}e^{4st}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}\varphi\right|^2 \mathrm{d}x\right)^2 \\ &- \frac{N}{2(N+\theta)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi(x)|^{\frac{N+\theta}{N}}|\varphi(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y \\ &- \frac{\mu}{2q}e^{(Nq-N-\theta)t}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi(x)|^{q}|\varphi(y)|^{q}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y \\ ={}& \frac{a}{2}e^{2st}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}\varphi\right|^2 \mathrm{d}x + \frac{b}{4}e^{4st}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}\varphi\right|^2 \mathrm{d}x\right)^2 \\ &- \frac{N}{2(N+\theta)} S_\theta^{-\frac{N+\theta}{N}} c^{\frac{2(N+\theta)}{N}} \\ &- \frac{\mu}{2q}e^{(Nq-N-\theta)t}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi(x)|^{q}|\varphi(y)|^{q}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y \ \text{ for } t \ll -1. \end{aligned} \tag{2.8}$$

Since $Nq-N-\theta \in (0,2s)$, we have $(Nq-N-\theta)t \in (0,2st)$. Due to the condition of $t \ll -1$, the dominant term of (2.8) is $-\frac{\mu}{2q} e^{(Nq-N-\theta)t} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi(x)|^q |\varphi(y)|^q}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y$, which means that

$$\begin{aligned} &\frac{a}{2} e^{2st} \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} \varphi\right|^2 \mathrm{d}x + \frac{b}{4} e^{4st} \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} \varphi\right|^2 \mathrm{d}x \right)^2 \\ &-\frac{\mu}{2q} e^{(Nq-N-\theta)t} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi(x)|^q |\varphi(y)|^q}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y < 0. \end{aligned}$$

Therefore, we can easily get that $e(c) < -\frac{N}{2(N+\theta)} S_\theta^{-\frac{N+\theta}{N}} c^{\frac{2(N+\theta)}{N}}$. □

Lemma 2.10. *For* $0<c_1<c_2$, *there holds* $c_1^2 e(c_2) < c_2^2 e(c_1)$.

Proof. Choose $\{u_n\} \subset \mathcal{S}_{c_1}$ as a minimizing sequence of $e(c_1)$. Now, we prove there are $C>0$ and $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\frac{\mu}{2q} \int_{\mathbb{R}^N} \left(I_\theta * |u_n|^q\right) |u_n|^q \mathrm{d}x \geq C \ \text{ or} \\ &\frac{N}{2(N+\theta)} \int_{\mathbb{R}^N} \left(I_\theta * |u_n|^{\frac{N}{N+\theta}} \right) |u_n|^{\frac{N}{N+\theta}} \mathrm{d}x \geq C, \ \forall n \geq n_0. \end{aligned} \tag{2.9}$$

Assume by contradiction that $\frac{\mu}{2q} \int_{\mathbb{R}^N} \left(I_\theta * |u_n|^q\right) |u_n|^q \mathrm{d}x \to 0$ and $\frac{N}{2(N+\theta)} \int_{\mathbb{R}^N} \left(I_\theta * |u_n|^{\frac{N}{N+\theta}} \right) |u_n|^{\frac{N}{N+\theta}} \mathrm{d}x \to 0$ as $n \to \infty$, up to a subsequence if necessary. Since

$$\begin{aligned} e(c_1) + o_n(1) &= E(u_n) \\ &\geq -\frac{\mu}{2q} \int_{\mathbb{R}^N} \left(I_\theta * |u_n|^q\right) |u_n|^q \mathrm{d}x - \frac{N}{2(N+\theta)} \int_{\mathbb{R}^N} \left(I_\theta * |u_n|^{\frac{N}{N+\theta}} \right) |u_n|^{\frac{N}{N+\theta}} \mathrm{d}x, \end{aligned}$$

we have $e(c_1) \geq 0$ by letting $n \to \infty$, which contradicts with Lemma 2.9. Set $v_n(x) := u_n\left(t^{-\frac{2}{N}} x \right)$, where $t = \frac{c_2}{c_1}$. Then, $v_n \in \mathcal{S}_{c_2}$. So, in view of (2.9), we obtain that

$$\begin{aligned} e(c_2) \leq E(v_n) &\\ = \frac{a}{2} &\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} v_n\right|^2 \mathrm{d}x + \frac{b}{4} \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} v_n\right|^2 \mathrm{d}x \right)^2 \\ &- \frac{N}{2(N+\theta)} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)|^{\frac{N+\theta}{N}} |v_n(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y \\ &- \frac{\mu}{2q} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)|^q |v_n(y)|^q}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= t^{2-\frac{4s}{N}} \frac{a}{2} \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 \mathrm{d}x + t^{4-\frac{8s}{N}} \frac{b}{4} \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 \mathrm{d}x \right)^2 \\
&\quad - t^{2+\frac{2\theta}{N}} \frac{N}{2(N+\theta)} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{\frac{N+\theta}{N}} |u_n(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y \\
&\quad - t^{2+\frac{2\theta}{N}} \frac{\mu}{2q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{q} |u_n(y)|^{q}}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y \\
&= t^2 E(u_n) + t^2 \left[\left(t^{-\frac{4s}{N}} - 1 \right) \frac{a}{2} \left\| (-\Delta)^{\frac{s}{2}} u_n \right\|_2^2 + \left(t^{2-\frac{8s}{N}} - 1 \right) \frac{b}{4} \left\| (-\Delta)^{\frac{s}{2}} u_n \right\|_2^4 \right] \\
&\quad + \frac{N}{2(N+\theta)} t^2 \left(1 - t^{\frac{2\theta}{N}} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{\frac{N+\theta}{N}} |u_n(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y \\
&\quad + \frac{\mu}{2q} t^2 \left(1 - t^{\frac{2\theta}{N}} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{q} |u_n(y)|^{q}}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y \\
&\le t^2 E(u_n) + t^2 \left(1 - t^{\frac{2\theta}{N}} \right) C, \quad \text{for } t > 1.
\end{aligned}
$$

Let $n \to \infty$, we obtain that $e(c_2) \le t^2 e(c_1) + t^2 \left(1 - t^{\frac{2\theta}{N}} \right) C$. As a consequence, we can easily get $c_1^2 e(c_2) < c_2^2 e(c_1)$. □

According to [12] and [20], we have a proof similar to the fractional Pohozaev identity.

Lemma 2.11. *Let* $u \in H^s(\mathbb{R}^N)$ *be a weak solution of problem* (1.1), *then* u *satisfies the following Pohozaev identity*

$$
\begin{aligned}
&\frac{N-2s}{2} \left(a + b \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x \right) \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x - \frac{N\lambda}{2} \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \\
&= \frac{N}{2} \int_{\mathbb{R}^N} \left(I_\theta * |u|^{\frac{N+\theta}{N}} \right) |u|^{\frac{N+\theta}{N}} \mathrm{d}x + \frac{\mu(N+\theta)}{2q} \int_{\mathbb{R}^N} \left(I_\theta * |u|^q \right) |u|^q \mathrm{d}x.
\end{aligned}
$$

In the end, we consider the Pohozaev manifold

$$
u \in \mathcal{P}_c := \left\{ u \in \mathcal{S}_c : P_\mu(u) = 0 \right\}
$$

with

$$
P_\mu(u) := a \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 + b \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^4 - \frac{\mu(Nq - N - \theta)}{2qs} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^{N-\theta}} \mathrm{d}x\mathrm{d}y.
$$

Lemma 2.12. *Let* $u \in H^s(\mathbb{R}^N)$ *be a weak solution of problem* (1.1), *then* $u \in \mathcal{P}_c$.

Proof. Note that all nontrivial critical points belong to the corresponding Nehari manifold, *i.e.*

$$
\begin{aligned}
&a \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 + b \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^4 - \lambda \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left(I_\theta * |u|^{\frac{N+\theta}{N}} \right) |u|^{\frac{N+\theta}{N}} \mathrm{d}x + \mu \int_{\mathbb{R}^N} \left(I_\theta * |u|^q \right) |u|^q \mathrm{d}x.
\end{aligned}
$$

Together Lemma 2.11 with above identity, we can easily compute that any nontrivial solution satisfies $P_\mu(u)=0$. ☐

Therefore, we have completed the important proofs in the preliminary part.

3. Proof of Theorem 1.1

In this section, we prove the existence of minimizers of problem (1.1) and analyze its asymptotic behaviors as $\mu\to 0^+$ or $c\to 0^+$ through the following three lemmas.

Lemma 3.1. *For any* $\mu,c>0$, $e(c)$ *has at least one minimizer.*

Proof. According to Lemma 2.9, we have $E(u)$ is bounded from below. Thus, we can choose a minimizer sequence that $\{u_n\}\subset\mathcal{S}_c$ such that

$$E(u_n)\to e(c) \text{ as } n\to\infty. \tag{3.1}$$

It follows from $Nq-N-\theta\in(0,2s)$ and (2.7) that $\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2$ is bounded. Hence, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Thus, we can assume that for some $u\in H^s(\mathbb{R}^N)$ and up to a subsequence as $n\to\infty$,

$$\begin{aligned}&u_n\rightharpoonup u \text{ in } H^s(\mathbb{R}^N),\\ &u_n\to u \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N)\ \forall p\in[2,2_s^*),\\ &u_n\to u \text{ a.e. in } \mathbb{R}^N.\end{aligned}$$

In what follows, we distinguish the proof into two cases.

Case (1): $u=0$. Then, $u_n\rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ as $n\to\infty$. We show that there exist $R,\delta>0$ and $y_n\in\mathbb{R}^N$ such that

$$\int_{B_R(y_n)}|u_n|^2\,\mathrm{d}x\geq\delta,\quad\forall\, n\in\mathbb{N}. \tag{3.2}$$

Otherwise, in view of Lemma 2.8, we have $u_n\to 0$ in $L^p(\mathbb{R}^N)$ for any $p\in(2,2_s^*)$ as $n\to\infty$, which together with Lemma 2.2, implies that

$$-\frac{\mu}{2q}\int_{\mathbb{R}^N}\left(I_\theta*|u|^q\right)|u|^q\,\mathrm{d}x\to 0 \text{ as } n\to\infty. \tag{3.3}$$

Then, combining with (2.2), (3.1) and (3.3), we observe that

$$\begin{aligned}&e(c)+o_n(1)=E(u_n)\\ &=\frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2+\frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^4\\ &\quad-\frac{N}{2(N+\theta)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{\frac{N+\theta}{N}}|u_n(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y+o_n(1)\\ &\geq\frac{a}{2}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^2+\frac{b}{4}\left\|(-\Delta)^{\frac{s}{2}}u_n\right\|_2^4-\frac{N}{2(N+\theta)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}}+o_n(1)\\ &\geq-\frac{N}{2(N+\theta)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}}+o_n(1).\end{aligned} \tag{3.4}$$

Let $n\to\infty$, we obtain that $e(c)\geq -\frac{N}{2(N+\theta)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}}$, which contradicts to Lemma 2.9. Therefore, from (3.2), we define a sequence $\hat{u}_n(x):=u_n(x+y_n)$. Obviously, $\{\hat{u}_n\}\subset\mathcal{S}_c$ and

$$E(\hat{u}_n)\to e(c)\ \text{as}\ n\to\infty. \tag{3.5}$$

Thus, up to a subsequence, if necessary, we obtain that as $n\to\infty$

$$\begin{aligned}&\hat{u}_n\rightharpoonup\hat{u}\ \text{ in } H^s(\mathbb{R}^N),\\&\hat{u}_n\to\hat{u}\ \text{ in } L^p_{\text{loc}}(\mathbb{R}^N),\ \forall p\in[2,2_s^*),\\&\hat{u}_n\to\hat{u}\ \text{ a.e. in } \mathbb{R}^N.\end{aligned}$$

1) $\|\hat{u}\|_2^2=l^2<c^2$. We can get $0<l<c$. Set $\hat{v}_n:=\hat{u}_n-\hat{u}$ and $d_n:=\|\hat{v}_n\|_2^2$. Obviously, by Lemma 2.6, we have

$$\|\hat{u}_n\|_2^2=\|\hat{u}\|_2^2+\|\hat{v}_n\|_2^2+o_n(1).$$

Equivalently,

$$c^2=l^2+d_n^2+o_n(1),$$

which states that $0<d_n<c$ for n large enough and $\|\hat{v}_n\|_2\to d$ with $c^2=l^2+d^2$. Therefore, in view of (3.5), Lemma 2.7 and Lemma 2.10, we observe that

$$\begin{aligned}e(c)+o_n(1)&=E(\hat{u}_n)\\&=E(\hat{u})+E(\hat{v}_n)+\frac{b}{2}\left\|(-\Delta)^{\frac{s}{2}}\hat{u}\right\|_2^2\left\|(-\Delta)^{\frac{s}{2}}\hat{v}_n\right\|_2^2+o_n(1)\\&\geq e(l)+e(d_n)+\frac{b}{2}\left\|(-\Delta)^{\frac{s}{2}}\hat{u}\right\|_2^2\left\|(-\Delta)^{\frac{s}{2}}\hat{v}_n\right\|_2^2+o_n(1)\\&\geq e(l)+\frac{d_n^2}{c^2}e(c)+o_n(1).\end{aligned}$$

Letting $n\to\infty$ and using again Lemma 2.10, we obtain that

$$e(c)\geq e(l)+\frac{d^2}{c^2}e(c)>\frac{l^2}{c^2}e(c)+\frac{d^2}{c^2}e(c)=e(c),$$

which is impossible.

2) $\|\hat{u}\|_2^2=c^2$. Then, $\hat{u}_n\to\hat{u}$ in $L^2(\mathbb{R}^N)$ as $n\to\infty$. Therefore, in view of Gagliardo-Nirenberg inequality, we know that $\hat{u}_n\to\hat{u}$ in $L^p(\mathbb{R}^N)$ for any $p\in[2,2_s^*)$ as $n\to\infty$. In view of Lemma 2.4, we can see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(I_\theta*|\hat{u}_n-\hat{u}|^p\right)|\hat{u}_n-\hat{u}|^p\,\mathrm{d}x=0. \tag{3.6}$$

Then, by (3.6) and Lemma 2.5, we have as $n\to\infty$

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(I_\theta*|\hat{u}_n|^{\frac{N+\theta}{N}}\right)|\hat{u}_n|^{\frac{N+\theta}{N}}\,\mathrm{d}x=\int_{\mathbb{R}^N}\left(I_\theta*|\hat{u}|^{\frac{N+\theta}{N}}\right)|\hat{u}|^{\frac{N+\theta}{N}}\,\mathrm{d}x\\&\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(I_\theta*|\hat{u}_n|^q\right)|\hat{u}_n|^q\,\mathrm{d}x=\int_{\mathbb{R}^N}\left(I_\theta*|\hat{u}|^q\right)|\hat{u}|^q\,\mathrm{d}x.\end{aligned} \tag{3.7}$$

Thus, by using (3.5), (3.7) and the weak semicontinuity of norm, we derive that

$$e(c)=\lim_{n\to\infty}E(\hat{u}_n)\geq E(\hat{u})\geq e(c),$$

which leads to $e(c)=E(\hat{u})$. Hence, $\hat{u}$ is a minimizer of $e(c)$ for any $c>0$.

Case (2): $u\neq 0$. Thus, up to a subsequence, if necessary, we obtain that as $n\to\infty$

$$\begin{aligned}&u_n\rightharpoonup u \text{ in } H^s(\mathbb{R}^N),\\ &u_n\to u \text{ in } L^p_{\text{loc}}(\mathbb{R}^N)\ \forall p\in[2,2_s^*),\\ &u_n\to u \text{ a.e. in } \mathbb{R}^N.\end{aligned}$$

1) $\|u\|_2^2=l^2<c^2$. We can get $0<l<c$. Setting $v_n:=u_n-u$, and $d_n:=\|v_n\|_2^2$. Obviously, by Lemma 2.6, we have

$$\|u_n\|_2^2=\|u\|_2^2+\|v_n\|_2^2+o_n(1).$$

Equivalently,

$$c^2=l^2+d_n^2+o_n(1),$$

which states that $0<d_n<c$ for n large enough and $\|v_n\|_2\to d$ with $c^2=l^2+d^2$. Therefore, by using Lemma 2.7, Lemma 2.10 and (3.1), we observe that

$$\begin{aligned}e(c)+o_n(1)&=E(u_n)\\ &=E(u)+E(v_n)+\frac{b}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2\left\|(-\Delta)^{\frac{s}{2}}v_n\right\|_2^2+o_n(1)\\ &\geq e(l)+e(d_n)+\frac{b}{2}\left\|(-\Delta)^{\frac{s}{2}}u\right\|_2^2\left\|(-\Delta)^{\frac{s}{2}}v_n\right\|_2^2+o_n(1)\\ &\geq e(l)+\frac{d_n^2}{c^2}e(c)+o_n(1).\end{aligned}$$

Letting $n\to\infty$ and using again Lemma 2.10, we observe that

$$e(c)\geq e(l)+\frac{d^2}{c^2}e(c)>\frac{l^2}{c^2}e(c)+\frac{d^2}{c^2}e(c)=e(c),$$

which is impossible.

2) $\|u\|_2^2=c^2$. We have $u_n\to u$ in $L^2(\mathbb{R}^N)$ as $n\to\infty$. Therefore, by Gagliardo-Nirenberg inequality, we know that $u_n\to u$ in $L^p(\mathbb{R}^N)$ for any $p\in[2,2_s^*)$ as $n\to\infty$. In view of Lemma 2.4, we can see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(I_\theta*|u_n-u|^p\right)|u_n-u|^p\,\mathrm{d}x=0. \tag{3.8}$$

Then, by (3.8) and Lemma 2.5, we have as $n\to\infty$

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(I_\theta*|u_n|^{\frac{N+\theta}{N}}\right)|u_n|^{\frac{N+\theta}{N}}\,\mathrm{d}x=\int_{\mathbb{R}^N}\left(I_\theta*|u|^{\frac{N+\theta}{N}}\right)|u|^{\frac{N+\theta}{N}}\,\mathrm{d}x\\ &\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(I_\theta*|u_n|^q\right)|u_n|^q\,\mathrm{d}x=\int_{\mathbb{R}^N}\left(I_\theta*|u|^q\right)|u|^q\,\mathrm{d}x.\end{aligned} \tag{3.9}$$

Thus, in light of (3.1), (3.9) and the weak semicontinuity of norm, we derive that

$$e(c)=\lim_{n\to\infty}E(u_n)\geq E(u)\geq e(c),$$

which leads to $e(c)=E(u)$. Hence, u is a minimizer of $e(c)$ for any $c>0$.

□

Lemma 3.2. *For any* $c>0$, *the corresponding Lagrange multiplier* $\lambda_{c,\mu}<0$.

Proof. In view of Lemma 3.1, there exists $u_{c,\mu}\in\mathcal{S}_c$ such that $E(u_{c,\mu})=e(c)$. According to the Lagrange multiplier theorem, there exists $\lambda_{c,\mu}\in\mathbb{R}$ to satisfy

$$E'_{\gamma}(u_{c,\mu})=\lambda_{c,\mu}\Psi'(u_{c,\mu})\ \text{ in } H^{-s}(\mathbb{R}^N),\tag{3.10}$$

where $\Psi:H^s(\mathbb{R}^N)\to\mathbb{R}$ is given by

$$\Psi(u)=\frac{1}{2}\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x,\ \ u\in H^s(\mathbb{R}^N).$$

It follows from (3.10), we obtain that $u_{c,\mu}$ satisfies problem (1.1) for $\lambda=\lambda_{c,\mu}$. Then, due to definition of $P_\mu(u)$ and $Nq-N-\theta\in(0,2s)$, we deduce that

$$\begin{aligned}
\lambda_{c,\mu}\|u_{c,\mu}\|_2^2 &= a\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2+b\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^4\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{c,\mu}(x)|^{\frac{N+\theta}{N}}|u_{c,\mu}(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\
&\quad-\mu\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{c,\mu}(x)|^{q}|u_{c,\mu}(y)|^{q}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\
&=\frac{\mu(Nq-N-\theta)}{2qs}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{c,\mu}(x)|^{q}|u_{c,\mu}(y)|^{q}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\
&\quad-\mu\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{c,\mu}(x)|^{q}|u_{c,\mu}(y)|^{q}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{c,\mu}(x)|^{\frac{N+\theta}{N}}|u_{c,\mu}(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\
&<\mu\left(\frac{1}{q}-1\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{c,\mu}(x)|^{q}|u_{c,\mu}(y)|^{q}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{c,\mu}(x)|^{\frac{N+\theta}{N}}|u_{c,\mu}(y)|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Since $q>\frac{N+\theta}{N}$, we have $\frac{1}{q}<1$. Hence, it is obvious to obtain that $\lambda_{c,\mu}<0$.

□

Lemma 3.3. *Let* $u_{c,\mu}\in\mathcal{S}_c$ *is the minimizer of* $e(c)$, *then we have the following asymptotic behaviors which*

$$\lim_{\mu\to0^+}E(u_{c,\mu})=\lim_{\mu\to0^+}e(c)=-\frac{N}{2(N+\theta)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}}$$

and

$$\lim_{c\to 0^+} E\left(u_{c,\mu}\right)=\lim_{c\to 0^+} e(c)=0.$$

Proof. According to the Lemma 2.12, we have $u_{c,\mu}\in\mathcal{P}_c$. Thus, by (2.2), Lemma 2.2 and $Nq-N-\theta\in(0,2s)$, we obtain that

$$\begin{aligned}
0&\le a\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2+b\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^4\\
&=\frac{\mu(Nq-N-\theta)}{2qs}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_{c,\mu}(x)\right|^q\left|u_{c,\mu}(y)\right|^q}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\
&\le\frac{\mu}{qs}C(N,\theta)\left\|u_{c,\mu}\right\|_{\frac{2Nq}{N+\theta}}^{2q}\\
&\le\frac{\mu}{qs}C(N,\theta,s)c^{2q-\frac{Nq-N-\theta}{s}}\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^{\frac{Nq-N-\theta}{s}}.
\end{aligned}$$

Hence, we have

$$\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2\le\frac{\mu}{aqs}C(N,\theta,s)c^{2q-\frac{Nq-N-\theta}{s}}\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^{\frac{Nq-N-\theta}{s}}.$$

Letting use again $Nq-N-\theta\in(0,2s)$, the above inequality brings that

$$0\le\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2\le\left(\frac{C(N,\theta,s)}{aqs}\right)^{\frac{2s}{2s-Nq+N+\theta}}\mu^{\frac{2s}{2s-Nq+N+\theta}}c^{\frac{2(2qs-Nq+N+\theta)}{2s-Nq+N+\theta}}. \tag{3.11}$$

Case (1): $\mu\to 0^+$. In this case, for any given $c>0$, in light of (3.11), letting $\mu\to 0^+$, we have $\lim_{\mu\to 0^+}\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2=0$. Hence, it is obvious to obtain that

$$\lim_{\mu\to 0^+} a\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2+b\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^4=0.$$

Finally, in view of $P_\mu\left(u_{c,\mu}\right)=0$, we infer that

$$\begin{aligned}
&\lim_{\mu\to 0^+}\frac{\mu(Nq-N-\theta)}{2qs}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_{c,\mu}(x)\right|^q\left|u_{c,\mu}(y)\right|^q}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\
&=\lim_{\mu\to 0^+} a\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2+b\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^4=0.
\end{aligned} \tag{3.12}$$

Therefore, according to (2.2), Lemma 2.9 and (3.12), we infer that

$$\begin{aligned}
&-\frac{N}{2(N+\theta)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}}>e(c)=E\left(u_{c,\mu}\right)\\
&=-\frac{N}{2(N+\theta)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_{c,\mu}(x)\right|^{\frac{N+\theta}{N}}\left|u_{c,\mu}(y)\right|^{\frac{N+\theta}{N}}}{|x-y|^{N-\theta}}\mathrm{d}x\mathrm{d}y+o_\mu(1)\\
&\ge-\frac{N}{2(N+\theta)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}}+o_\mu(1),
\end{aligned}$$

which gives that

$$\lim_{\mu\to0^+} E\left(u_{c,\mu}\right)=\lim_{\mu\to0^+} e\left(c\right)=-\frac{N}{2\left(N+\theta\right)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}}.$$

Case (2): $c\to0^+$. Since μ is fixed, in light of (3.11), letting $c\to0^+$, we have $\lim_{c\to0^+}\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2=0$. Hence, it is obvious to obtain that

$$\lim_{c\to0^+} a\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2+b\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^4=0.$$

Finally, in view of $P_\mu\left(u_{c,\mu}\right)=0$, we deduce that

$$\begin{aligned}&\lim_{c\to0^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_{c,\mu}\left(x\right)\right|^q\left|u_{c,\mu}\left(y\right)\right|^q}{\left|x-y\right|^{N-\theta}}\mathrm{d}x\mathrm{d}y\\&=\lim_{c\to0^+} a\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^2+b\left\|(-\Delta)^{\frac{s}{2}}u_{c,\mu}\right\|_2^4=0.\end{aligned}\tag{3.13}$$

Consequently, in view of (2.2), Lemma 2.9 and (3.13), we have

$$\begin{aligned}-\frac{N}{2\left(N+\theta\right)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}}&>e\left(c\right)=E\left(u_{c,\mu}\right)\\&=-\frac{N}{2\left(N+\theta\right)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_{c,\mu}\left(x\right)\right|^{\frac{N+\theta}{N}}\left|u_{c,\mu}\left(y\right)\right|^{\frac{N+\theta}{N}}}{\left|x-y\right|^{N-\theta}}\mathrm{d}x\mathrm{d}y+o_c\left(1\right)\\&\geq-\frac{N}{2\left(N+\theta\right)}S_\theta^{-\frac{N+\theta}{N}}c^{\frac{2(N+\theta)}{N}}+o_c\left(1\right),\end{aligned}$$

which signifies that

$$\lim_{c\to0^+} E\left(u_{c,\mu}\right)=\lim_{c\to0^+} e\left(c\right)=0.$$

□

Finally, we conduct the principal contributions of this paper as follows: By establishing minimizing sequence to $e\left(c\right)$ and fractional Pohozaev identity, we prove the existence of minimizers of problem (1.1) and analyze its asymptotic behaviors as $\mu\to0^+$ or $c\to0^+$.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Di Nezza, E., Palatucci, G. and Valdinoci, E. (2012) Hitchhiker's Guide to the Fractional Sobolev Spaces. *Bulletin des Sciences Mathématiques*, **136**, 521-573. https://doi.org/10.1016/j.bulsci.2011.12.004

[2] Kirchhoff, G. (1877) Mechanik (Vol. 1). Рипол Классик. https://sc.panda985.com/extdomains/books.google.com/books?hl=zh-CN&lr=&id=BJ4MAwAAQBAJ&oi=fnd&pg=PA1&dq=Mechanik&ots=9ME-

2wjVPg&sig=5BAI6vV0KHcPT4Z86I-E8Wm9Ras

[3] Lions, J.L. (1978) On Some Questions in Boundary Value Problems of Mathematical Physics. *North-Holland Mathematics Studies*, **30**, 284-346. https://doi.org/10.1016/s0304-0208(08)70870-3

[4] Ye, H. (2014) The Sharp Existence of Constrained Minimizers for a Class of Nonlinear Kirchhoff Equations. *Mathematical Methods in the Applied Sciences*, **38**, 2663-2679. https://doi.org/10.1002/mma.3247

[5] Ye, H. (2014) The Existence of Normalized Solutions for L^2-Critical Constrained Problems Related to Kirchhoff Equations. *Zeitschrift für angewandte Mathematik und Physik*, **66**, 1483-1497. https://doi.org/10.1007/s00033-014-0474-x

[6] Li, G. and Ye, H. (2019) On the Concentration Phenomenon of L^2-Subcritical Constrained Minimizers for a Class of Kirchhoff Equations with Potentials. *Journal of Differential Equations*, **266**, 7101-7123. https://doi.org/10.1016/j.jde.2018.11.024

[7] Li, G., Luo, X. and Yang, T. (2022) Normalized Solutions to a Class of Kirchhoff Equations with Sobolev Critical Exponent. *Annales Fennici Mathematici*, **47**, 895-925. https://doi.org/10.54330/afm.120247

[8] Fröhlich, H. (1937) Theory of Electrical Breakdown in Ionic Crystal. *Proceedings of the Royal Society of London. Series A—Mathematical and Physical Sciences*, **160**, 230-241.

[9] Pekar, S. (1954) Untersuchung über die Elektronentherorie der Kristalle. Akademie Verlag.

[10] Lieb, E.H. (1977) Existence and Uniqueness of the Minimizing Solution of Choquard's Nonlinear Equation. *Studies in Applied Mathematics*, **57**, 93-105. https://doi.org/10.1002/sapm197757293

[11] Moroz, I.M., Penrose, R. and Tod, P. (1998) Spherically-Symmetric Solutions of the Schrödinger-Newton Equations. *Classical and Quantum Gravity*, **15**, 2733-2742. https://doi.org/10.1088/0264-9381/15/9/019

[12] Moroz, V. and Van Schaftingen, J. (2013) Groundstates of Nonlinear Choquard Equations: Existence, Qualitative Properties and Decay Asymptotics. *Journal of Functional Analysis*, **265**, 153-184. https://doi.org/10.1016/j.jfa.2013.04.007

[13] Li, X. and Ma, S. (2019) Choquard Equations with Critical Nonlinearities. *Communications in Contemporary Mathematics*, **22**, Article ID: 1950023. https://doi.org/10.1142/s0219199719500238

[14] Yao, S., Chen, H., Rădulescu, V.D. and Sun, J. (2022) Normalized Solutions for Lower Critical Choquard Equations with Critical Sobolev Perturbation. *SIAM Journal on Mathematical Analysis*, **54**, 3696-3723. https://doi.org/10.1137/21m1463136

[15] Meng, Y. and He, X. (2023) Normalized Solutions for the Fractional Choquard Equations with Hardy-Littlewood-Sobolev Upper Critical Exponent. *Qualitative Theory of Dynamical Systems*, **23**, Article No. 19. https://doi.org/10.1007/s12346-023-00875-z

[16] Zhang, Z., Liu, J. and Sun, H. (2024) Existence and Asymptotical Behavior of L^2-Normalized Standing Wave Solutions to HLS Lower Critical Choquard Equation with a Nonlocal Perturbation. *Qualitative Theory of Dynamical Systems*, **23**, Article No. 206. https://doi.org/10.1007/s12346-024-01060-6

[17] Liu, Z. (2019) Multiple Normalized Solutions for Choquard Equations Involving Kirchhoff Type Perturbation. *Topological Methods in Nonlinear Analysis*, **54**, 297-319. https://doi.org/10.12775/tmna.2019.046

[18] Zhu, S., Che, G. and Chen, H. (2024) Existence and Asymptotic Behavior of Normalized

Solutions for Choquard Equations with Kirchhoff Perturbation. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, **119**, Article No. 2. https://doi.org/10.1007/s13398-024-01668-y

[19] Liang, S., Pucci, P. and Zhang, B. (2020) Multiple Solutions for Critical Choquard-Kirchhoff Type Equations. *Advances in Nonlinear Analysis*, **10**, 400-419. https://doi.org/10.1515/anona-2020-0119

[20] Liu, L., Chen, H. and Yang, J. (2021) Normalized Solutions to the Fractional Kirchhoff Equations with a Perturbation. *Applicable Analysis*, **102**, 1229-1249. https://doi.org/10.1080/00036811.2021.1979222

[21] Willem, M. (1996) Minimax Theorems. Birkhauser Verlag.

[22] Lieb, E.H. (1983) Sharp Constants in the Hardy-Littlewood-Sobolev and Related Inequalities. *The Annals of Mathematics*, **118**, 349-374. https://doi.org/10.2307/2007032

[23] Moroz, V. and Van Schaftingen, J. (2016) A Guide to the Choquard Equation. *Journal of Fixed Point Theory and Applications*, **19**, 773-813. https://doi.org/10.1007/s11784-016-0373-1

[24] Frank, R.L., Lenzmann, E. and Silvestre, L. (2015) Uniqueness of Radial Solutions for the Fractional Laplacian. *Communications on Pure and Applied Mathematics*, **69**, 1671-1726. https://doi.org/10.1002/cpa.21591

[25] Li, G. and Luo, X. (2019) Existence and Multiplicity of Normalized Solutions for a Class of Fractional Choquard Equations. *Science China Mathematics*, **63**, 539-558. https://doi.org/10.1007/s11425-017-9287-6