**Open Access Library Journal**
2025, Volume 12, e13882
ISSN Online: 2333-9721
ISSN Print: 2333-9705

# Extending Mohanta-Biswas Type Fixed Point Result Using Altering Distance Functions

**Meaza Bogale, Alemayehu Negash***

Department of Mathematics, Hampton University, Hampton, USA
Email: meaza.bogale@hamptonu.edu, *alemayehu.negash@hamptonu.edu

**How to cite this paper:** Bogale, M. and Negash, A. (2025) Extending Mohanta-Biswas Type Fixed Point Result Using Altering Distance Functions. *Open Access Library Journal*, **12**: e13882.
https://doi.org/10.4236/oalib.1113882

**Received:** June 30, 2025
**Accepted:** August 11, 2025
**Published:** August 14, 2025



## Abstract

In this paper, our purpose is to establish a common fixed point result for a pair of self mappings satisfying some generalized contraction type conditions involving altering distance and control function with two variables in partial metric spaces. Moreover, we provide an example in support of our main result.

## Subject Areas

Functional Analysis, Mathematics

## Keywords

Partial Metric, Altering Distance Function, 0-Completeness, Common Fixed Point

## 1. Introduction

The theory of fixed points plays a pivotal role in nonlinear analysis and has far-reaching applications in various areas of mathematics and computer science. In particular, *partial metric spaces*, introduced by Matthews [1], extend classical metric spaces by allowing the self-distance of a point to be nonzero. This framework has proven particularly useful in theoretical computer science, especially in the semantics of dataflow networks and the study of denotational semantics.

Over the past three decades, fixed point results have been extensively developed within partial metric spaces, with numerous generalizations of Banach-type contraction principles adapted to this setting [2]-[9]. Notably, researchers have introduced tools such as *altering distance functions* (due to Khan *et al.* [10]) and *control functions* involving two variables to capture more generalized contractive behavior [11]-[13]. These tools have provided new avenues for proving the existence and uniqueness of fixed points under broader assumptions, especially in *0-complete partial metric spaces*, where convergence is defined with respect to vanishing self-distances.

A significant advancement in this direction was made by *Mohanta and Biswas* [14], who introduced a generalized contraction condition using a control function $\psi \in \Psi$ of two variables. One of their main results (Theorem 3.9) established the existence and uniqueness of a common fixed point for a pair of self-mappings in a 0-complete partial metric space. This theorem not only unified several earlier results but also underscored the utility of combining classical contraction techniques with nonlinear control functions.

However, the approach in [14] still relied on the identity function $\gamma(t)=t$, thereby limiting the flexibility of the contraction condition in some nonlinear settings. To address this, we propose a further *generalization involving an altering distance function* $\gamma \in \Gamma$, in conjunction with the two-variable control function $\psi \in \Psi$. This new framework encompasses a broader class of mappings and allows for more refined contractive behavior, extending Theorem 3.9 of [14] as a special case.

In this paper, we establish a common fixed point theorem for a pair of self-mappings under this generalized contraction condition in 0-complete partial metric spaces. Our result not only subsumes earlier findings but also demonstrates, via a detailed example, the *necessity of altering distance functions* in achieving contractivity where the trivial choice $\gamma(t)=t$ fail. This contributes to the ongoing development of fixed point theory in generalized metric settings and expands its applicability to problems where traditional metric assumptions do not hold.

## 2. Some Basic Concepts

In this section, we begin with some basic facts and properties of partial metric spaces.

**Definition 2.1** [1] *A partial metric on a nonempty set* $X$ *is a function* $p: X \times X \to \mathbb{R}^+$ *such that for all* $x, y, z \in X$*:*

$(p_1)$ $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$,

$(p_2)$ $p(x,x) \le p(x,y)$,

$(p_3)$ $p(x,y) = p(y,x)$,

$(p_4)$ $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

The pair $(X,p)$ is called a partial metric space.

It is clear that if $p(x,y)=0$, then from $(p_1)$ and $(p_2)$, it follows that $x=y$. But if $x=y$, $p(x,y)$ may not be 0.

**Example 2.2** [1] *Let* $X=[0,\infty)$ *and* $p(x,y)=\max\{x,y\}$*, for all* $x,y \in X$. *Then* $(X,p)$ *is a partial metric space.*

**Example 2.3** [1] *Let* $X=\{[a,b]:a,b\in\mathbb{R},a\le b\}$ *and* $p([a,b],[c,d])=\max\{b,d\}-\min\{a,c\}$. *Then* $(X,p)$ *is a partial metric space.*

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has as a base the family of open $p$-balls $\{B_p(x,\epsilon):x\in X,\epsilon>0\}$, where $B_p(x,\epsilon)=\{y\in X:p(x,y)<p(x,x)+\epsilon\}$ for all $x\in X$ and $\epsilon>0$.

**Theorem 2.4.** [14] *If* $U\in\tau_p$ *and* $x\in U$, *then there exists* $r>0$ *such that* $B_p(x,r)\subseteq U$.

**Remark 2.5.** [14] *Let* $(X,p)$ *be a partial metric space,* $(x_n)$ *be a sequence in* $X$ *and* $x\in X$. *Then* $(x_n)$ *converges to* $x$ *with respect to (w.r.t.)* $\tau_p$ *if and only if* $\lim_{n\to\infty}p(x_n,x)=p(x,x)$.

Let $x_n\to x$ w.r.t. $\tau_p$ and $\epsilon>0$. Then there exists a natural number $n_0$ such that $x_n\in B_p(x,\epsilon)$ for all $n\ge n_0$. This gives that $p(x_n,x)-p(x,x)<\epsilon$ for all $n\ge n_0$. Since $p(x_n,x)-p(x,x)\ge 0$, it follows that $|p(x_n,x)-p(x,x)|<\epsilon$ for all $n\ge n_0$. This proves that $\lim_{n\to\infty}p(x_n,x)=p(x,x)$.

Conversely, suppose that $\lim_{n\to\infty}p(x_n,x)=p(x,x)$. We shall show that $x_n\to x$ w.r.t. $\tau_p$. Let $U\in\tau_p$ and $x\in U$. Then there exists $\epsilon>0$ such that $x\in B_p(x,\epsilon)\subseteq U$. By hypotheses, it follows that

$\lim_{n\to\infty}(p(x_n,x)-p(x,x))=0.$

So, there exists $n_0\in\mathbb{N}$ such that $p(x_n,x)-p(x,x)<\epsilon$ for all $n\ge n_0$. This ensures that $x_n\in B_p(x,\epsilon)$ for all $n\ge n_0$ and hence $x_n\in U$ for all $n\ge n_0$. Therefore, $(x_n)$ converges to $x$ w.r.t. $\tau_p$ on $X$.

**Definition 2.6** [1] *Let* $(X,p)$ *be a partial metric space and let* $(x_n)$ *be a sequence in* $X$. *Then*

(i) $(x_n)$ converges to a point $x\in X$ if $\lim_{n\to\infty}p(x_n,x)=p(x,x)$. This will be denoted as $\lim_{n\to\infty}x_n=x$ or $x_n\to x(n\to\infty)$.

(ii) $(x_n)$ is called a Cauchy sequence if $\lim_{n,m\to\infty}p(x_n,x_m)$ exists and is finite.

(iii) $(X,p)$ is said to be complete if every Cauchy sequence $(x_n)$ in $X$ converges to a point $x\in X$ such that $p(x,x)=\lim_{n,m\to\infty}p(x_n,x_m)$.

**Definition 2.7** [15] *A sequence* $(x_n)$ *in* $(X,p)$ *is called 0-Cauchy if*

$$\lim_{n,m\to\infty}p(x_n,x_m)=0.$$

The space $(X,p)$ is said to be 0-complete if every 0-Cauchy sequence in $X$ converges to a point $x\in X$ such that $p(x,x)=0$.

It is easy to verify that every closed subset of a 0-complete partial metric space is 0-complete.

**Lemma 2.8** *Let* $(X,p)$ *be a partial metric space.*

(a) (see [16]) If $p(x_n,z)\to p(z,z)=0$ as $n\to\infty$, then $p(x_n,y)\to p(z,y)$ as $n\to\infty$ for each $y\in X$.

(b) (see [15]) If $(X,p)$ is complete, then it is 0-complete.

The converse assertion of (b) may not hold, in general. The following example supports the above remark.

**Example 2.9** [15] *The space* $X=[0,\infty)\cap\mathbb{Q}$ *with the partial metric* $p(x,y)=\max\{x,y\}$ *is 0-complete, but it is not complete. Moreover, the* *sequence* $(x_n)$ *with* $x_n = 1$ *for each* $n \in \mathbb{N}$ *is a Cauchy sequence in* $(X, p)$, *but it is not a 0-Cauchy sequence.*

**Definition 2.10** [10] *A function* $\gamma : [0,\infty) \to [0,\infty)$ *is an* ***altering distance function*** *if:*

- $\gamma$ is continuous and nondecreasing,
- $\gamma(t) = 0 \Leftrightarrow t = 0$.

We denote the set of altering distance functions by $\Gamma$

**Example 2.11.** *Consider the function* $\gamma(t) = \ln(1+t)$. *We verify that* $\psi$ *is an altering distance function:*

(i) **Continuity & Monotonicity**:

- Since $\ln(1+t)$ is differentiable for $t \geq 0$ with derivative $\gamma'(t) = \frac{1}{1+t} > 0$, it is both continuous and strictly increasing.

(ii) **Zero Condition**:

- $\gamma(0) = \ln(1+0) = 0$, and if $\gamma(t) = 0$, then $\ln(1+t) = 0 \Rightarrow 1+t = 1 \Rightarrow t = 0$.

Thus, $\gamma(t) = \ln(1+t)$ satisfies all the conditions of an altering distance function.

**Definition 2.12.** [17] Let $(X, p)$ be a PMS, $C \subset X$ and $\varphi : C \to \mathbb{R}^+$ a function on *C*. Then, the function $\varphi$ is called *lower semi-continuous* (*l.s.c.*) on *C* whenever

$$\lim_{n\to\infty} p(x_n, x) = p(x,x) \Rightarrow \varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n) = \sup_{n\geq 1} \inf_{m\geq n} \varphi(x_m).$$

In 2013, Nashine *et al.* [18] introduced a class of generalized control functions as follows:

Let $\Psi$ denote the class of all functions $\psi : [0,\infty)^2 \to [0,\infty)$ satisfying the following conditions:

(a) $\psi$ is lower semicontinuous;

(b) $\psi(s,t) = 0$ if and only if $s = t = 0$.

In 2021, Mohanta and Biswas [14] established the following common fixed point result for a pair of self mappings satisfying some generalized contraction type conditions involving a control function with two variables in partial metric spaces.

**Theorem 2.13.** (Theorem 3.9 of [14]) *Let* $(X, p)$ *be a* 0*-complete partial metric space and let* $f, g : X \to X$ *be mappings. Suppose there exists* $\varphi \in \Phi$ *such that*

$$p(fx, gy) \leq N(x,y) - \varphi\left(p(x,y), \frac{p(x,fx) + p(y,gy)}{2}\right)$$

*for all* $x, y \in X$, *where*

$$N(x,y) = \max\left\{p(x,y), p(x,fx), p(y,gy), \frac{p(x,gy) + p(y,fx)}{2}\right\}.$$

*Then* $f$ *and* $g$ *have a unique common fixed point* $u$ *in* $X$ *with* $p(u,u) = 0$.

In the next section, we prove a common fixed point theorem for a pair of self-mappings on a 0-complete partial metric space, under a generalized contractive condition involving an altering distance function and a two-variable control function. This result generalizes Theorem 3.9 of [14].

## 3. Main Results

Next we present our second main theorem.

**Theorem 3.1.** *Let* $(X,p)$ *be a* 0*-complete partial metric space and let* $f,g:X\to X$ *be mappings. There exist functions* $\psi\in\Psi$ *and* $\gamma\in\Gamma$ *such that such that*

$$\gamma\left(p\left(fx,gy\right)\right)\le N\left(x,y\right)-\psi\left(\gamma\left(p\left(x,y\right)\right),\frac{\gamma\left(p\left(x,fx\right)\right)+\gamma\left(p\left(y,gy\right)\right)}{2}\right) \quad (3.1)$$

for all $x,y\in X$, where

$$N\left(x,y\right)=\max\left\{\gamma\left(p\left(x,y\right)\right),\gamma\left(p\left(x,fx\right)\right),\gamma\left(p\left(y,gy\right)\right),\frac{\gamma\left(p\left(x,gy\right)\right)+\gamma\left(p\left(y,fx\right)\right)}{2}\right\}.$$

Then $f$ and $g$ have a unique common fixed point $u$ in $X$ with $p\left(u,u\right)=0$.

*Proof.* We first prove that $u$ is a fixed point of $g$ if and only if $u$ is a fixed point of $f$ with $p\left(u,u\right)=0$.

Suppose that $u$ is a fixed point of $g$, *i.e.*, $gu=u$. Then, using condition (3.1), we obtain

$$\begin{aligned}\gamma\left(p\left(fu,u\right)\right)&=\gamma\left(p\left(fu,gu\right)\right)\\&\le N\left(u,u\right)-\psi\left(\gamma\left(p\left(u,u\right)\right),\frac{p\left(u,fu\right)+\gamma\left(p\left(u,gu\right)\right)}{2}\right),\end{aligned}$$

where

$$\begin{aligned}N\left(u,u\right)&=\max\left\{\gamma\left(p\left(u,u\right)\right),\gamma\left(p\left(u,fu\right)\right),\gamma\left(p\left(u,gu\right)\right),\frac{\gamma\left(p\left(u,gu\right)\right)+\gamma\left(p\left(u,fu\right)\right)}{2}\right\}\\&=\max\left\{\gamma\left(p\left(u,u\right)\right),\gamma\left(p\left(u,fu\right)\right),\frac{\gamma\left(p\left(u,u\right)\right)+\gamma\left(p\left(u,fu\right)\right)}{2}\right\}\\&=\max\left\{\gamma\left(p\left(u,u\right)\right),\gamma\left(p\left(u,fu\right)\right)\right\}=\gamma\left(p\left(u,fu\right)\right).\end{aligned}$$

Therefore,

$$\gamma\left(p\left(fu,u\right)\right)\le\gamma\left(p\left(u,fu\right)\right)-\psi\left(\gamma\left(p\left(u,u\right)\right),\frac{\gamma\left(p\left(u,fu\right)\right)+\gamma\left(p\left(u,u\right)\right)}{2}\right),$$

which implies that

$$\psi\left(\gamma\left(p(u,u)\right),\frac{\gamma\left(p(u,fu)\right)+\gamma\left(p(u,u)\right)}{2}\right)=0.$$

This gives $\gamma\left(p(u,u)\right)=0$ and $\gamma\left(p(u,fu)\right)=0$, *i.e.*, $p(u,fu)=p(u,u)=0$, and hence $fu=u$.

Conversely, by a similar argument, if $fu=u$ and $p(u,u)=0$, then $gu=u$ and $u$ is a fixed point of $g$ with $p(u,u)=0$.

Let $x_0\in X$ be arbitrary. We can construct a sequence $(x_n)$ in $X$ such that

$$x_n=\begin{cases} fx_{n-1}, & \text{if } n \text{ is odd},\\ gx_{n-1}, & \text{if } n \text{ is even}.\end{cases}$$

If $x_{2n}=x_{2n+1}$ for some $n\in\mathbb{N}\cup\{0\}$, then $x_{2n}=fx_{2n}$, and hence $x_{2n}$ is a fixed point of $f$. By our earlier reasoning, it follows that $x_{2n}$ is also a fixed point of $T$. Thus, $x_{2n}$ is a common fixed point of $f$ and $T$.

The case where $x_{2n+1}=x_{2n+2}$ for some $n\in\mathbb{N}\cup\{0\}$ can be treated analogously to reach the same conclusion.

Therefore, we may assume without loss of generality that $x_n\neq x_{n-1}$ for all $n\in\mathbb{N}$. Consequently, $p(x_n,x_{n-1})>0$ for every $n\in\mathbb{N}$, and thus

$$\psi\left(\gamma\left(p(x_n,x_{n-1})\right),\frac{\gamma\left(p(x_n,x_{n-1})\right)+\gamma\left(p(x_{m+1},x_m)\right)}{2}\right)>0,\ \forall n,m\in\mathbb{N}. \quad (3.2)$$

We now show that $\lim_{n\to\infty}p(x_n,x_{n+1})=0$.

Let $a_n:=\gamma\left(p(x_n,x_{n+1})\right)$. By using condition (3.1), we obtain

$$\begin{aligned} a_{2n+1}&=\gamma\left(p(x_{2n+1},x_{2n+2})\right)=\gamma\left(p(fx_{2n},gx_{2n+1})\right)\\ &\le N(x_{2n},x_{2n+1})\\ &\quad-\psi\left(\gamma\left(p(x_{2n},x_{2n+1})\right),\frac{\gamma\left(p(x_{2n},fx_{2n})\right)+\gamma\left(p(x_{2n+1},gx_{2n+1})\right)}{2}\right)\\ &\le N(x_{2n},x_{2n+1})-\psi\left(a_{2n},\frac{a_{2n}+a_{2n+1}}{2}\right),\end{aligned}$$

where

$$\begin{aligned} &N(x_{2n},x_{2n+1})\\ &=\max\left\{\gamma\left(p(x_{2n},x_{2n+1})\right),\gamma\left(p(x_{2n},fx_{2n})\right),\gamma\left(p(x_{2n+1},gx_{2n+1})\right),\right.\\ &\qquad\left.\frac{\gamma\left(p(x_{2n},gx_{2n+1})\right)+\gamma\left(p(x_{2n+1},fx_{2n})\right)}{2}\right\}\\ &=\max\left\{\gamma\left(p(x_{2n},x_{2n+1})\right),\gamma\left(p(x_{2n+1},x_{2n+2})\right),\right.\\ &\qquad\left.\frac{\gamma\left(p(x_{2n},x_{2n+2})\right)+\gamma\left(p(x_{2n+1},x_{2n+1})\right)}{2}\right\}\\ &=\max\left\{a_{2n},a_{2n+1},\frac{a_{2n}+a_{2n+1}}{2}\right\}\\ &=\max\left\{a_{2n},a_{2n+1}\right\}.\end{aligned}$$

Therefore,

$$a_{2n+1} \leq \max\{a_{2n}, a_{2n+1}\} - \psi\left(a_{2n}, \frac{a_{2n} + a_{2n+1}}{2}\right). \tag{3.3}$$

If $\max\{a_{2n}, a_{2n+1}\} = a_{2n+1}$, then by using (3.2), we obtain from condition (3.3) that

$$a_{2n+1} \leq a_{2n+1} - \psi\left(a_{2n}, \frac{a_{2n} + a_{2n+1}}{2}\right) < a_{2n+1},$$

which is a contradiction. Therefore,

$$\max\{a_{2n}, a_{2n+1}\} = a_{2n}.$$

Thus, condition (3.3) becomes

$$a_{2n+1} \leq a_{2n+1} - \psi\left(a_{2n}, \frac{a_{2n} + a_{2n+1}}{2}\right) < a_{2n}, \tag{3.4}$$

Similarly, we can show that

$$\begin{aligned} a_{2n} &= \gamma\left(p\left(x_{2n}, x_{2n+1}\right)\right) \\ &\leq \gamma\left(p\left(x_{2n-1}, x_{2n}\right)\right) \\ &\quad - \psi\left(\gamma\left(p\left(x_{2n-1}, x_{2n}\right)\right), \frac{\gamma\left(p\left(x_{2n+1}, x_{2n}\right)\right) + \gamma\left(p\left(x_{2n-1}, x_{2n}\right)\right)}{2}\right) \\ &< a_{2n-1}. \end{aligned} \tag{3.5}$$

Combining conditions (3.4) and (3.5), we get

$$\begin{aligned} a_n &= \gamma\left(p\left(x_n, x_{n+1}\right)\right) \\ &\leq \gamma\left(p\left(x_{n-1}, x_n\right)\right) - \psi\left(\gamma\left(p\left(x_{n-1}, x_n\right)\right), \frac{\gamma\left(p\left(x_{n-1}, x_n\right)\right) + \gamma\left(p\left(x_n, x_{n+1}\right)\right)}{2}\right) \\ &= a_n - \psi\left(a_{n-1}, \frac{a_{n-1} + a_n}{2}\right) \\ &< a_{n-1}, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.6}$$

Since $(a_n)$ is decreasing and bounded below by 0, it converges to some limit $L \geq 0$:

$$\lim_{n\to\infty} a_n = L. \tag{3.7}$$

Taking the upper limit as $n \to \infty$ in (3.6) and using (3.7) and lower semicontinuity of $\psi$ (0.12), we obtain

$$\begin{aligned} L &\leq L - \liminf_{n\to\infty} \psi\left(a_{n-1}, \frac{a_{n-1} + a_n}{2}\right) \\ &\leq L - \psi(L, L), \end{aligned}$$

which implies that $\psi(L, L) = 0$ and hence $L = 0$. Since $\gamma$ is continuous and $\gamma(t) = 0$ if and only if $t = 0$, it follows from (3.5) that

$$\lim_{n\to\infty} p\left(x_n, x_{n+1}\right) = 0. \tag{3.8}$$

We shall show that $(x_n)$ is a 0-Cauchy sequence in $X$. It is sufficient to show that $(x_{2n})$ is a 0-Cauchy sequence. If possible, suppose that $(x_{2n})$ is not a 0-Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences $(x_{2m_i})$ and $(x_{2n_i})$ of $(x_{2n})$ such that $n_i$ is the smallest positive integer for which

$$p\left(x_{2m_i}, x_{2n_i}\right) \geq \epsilon \text{ for } n_i > m_i > i. \tag{3.9}$$

This implies that

$$p\left(x_{2m_i}, x_{2n_i-2}\right) < \epsilon. \tag{3.10}$$

By repeated use of $(p_4)$ and by condition (3.10), we have

$$\begin{aligned} p\left(x_{2n_i+1}, x_{2m_i}\right) &\leq p\left(x_{2n_i+1}, x_{2n_i}\right) + p\left(x_{2n_i}, x_{2m_i}\right) - p\left(x_{2n_i}, x_{2n_i}\right) \\ &\leq p\left(x_{2n_i+1}, x_{2n_i}\right) + p\left(x_{2n_i}, x_{2n_i-1}\right) + p\left(x_{2n_i-1}, x_{2n_i-2}\right) + p\left(x_{2n_i-2}, x_{2m_i}\right) \\ &< p\left(x_{2n_i+1}, x_{2n_i}\right) + p\left(x_{2n_i}, x_{2n_i-1}\right) + p\left(x_{2n_i-1}, x_{2n_i-2}\right) + \epsilon. \end{aligned}$$

Passing to the upper limit as $i \to \infty$, we get

$$\limsup_{n\to\infty} p\left(x_{2n_i+1}, x_{2m_i}\right) \leq \epsilon.$$

From (3.9), we get

$$\epsilon \leq p\left(x_{2m_i}, x_{2n_i}\right) \leq p\left(x_{2m_i}, x_{2n_i+1}\right) + p\left(x_{2n_i+1}, x_{2n_i}\right).$$

Taking the upper limit as $i \to \infty$, we have

$$\epsilon \leq \limsup_{i\to\infty} p\left(x_{2m_i}, x_{2n_i+1}\right) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p\left(x_{2m_i}, x_{2n_i+1}\right) = \epsilon.$$

Similarly, $\liminf_{i\to\infty} p\left(x_{2m_i}, x_{2n_i+1}\right) = \epsilon$. Therefore,

$$\lim_{i\to\infty} p\left(x_{2m_i}, x_{2n_i+1}\right) = \epsilon. \tag{3.11}$$

Again,

$$\begin{aligned} p\left(x_{2n_i}, x_{2m_i-1}\right) &\leq p\left(x_{2n_i}, x_{2n_i-1}\right) + p\left(x_{2n_i-1}, x_{2n_i-2}\right) + p\left(x_{2n_i-2}, x_{2m_i}\right) + p\left(x_{2m_i}, x_{2m_i-1}\right) \\ &< \epsilon + p\left(x_{2n_i}, x_{2n_i-1}\right) + p\left(x_{2n_i-1}, x_{2n_i-2}\right) + p\left(x_{2m_i}, x_{2m_i-1}\right). \end{aligned}$$

Passing to the upper limit as $i \to \infty$, we obtain

$$\limsup_{i\to\infty} p\left(x_{2n_i}, x_{2m_i-1}\right) \leq \epsilon. \tag{3.12}$$

Also,

$$\epsilon \leq p\left(x_{2n_i}, x_{2m_i}\right) \leq p\left(x_{2n_i}, x_{2m_i-1}\right) + p\left(x_{2m_i-1}, x_{2m_i}\right).$$

Taking the upper limit as $i \to \infty$ and using conditions (3.8) and, we get

$$\epsilon \leq \limsup_{i\to\infty} p\left(x_{2n_i}, x_{2m_i-1}\right) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p\left(x_{2n_i}, x_{2m_i-1}\right) = \epsilon.$$

Similarly, we can obtain

$$\liminf_{i\to\infty} p\left(x_{2n_i}, x_{2m_i-1}\right) = \epsilon.$$

Therefore,

$$\lim_{i\to\infty} p\left(x_{2n_i}, x_{2m_i-1}\right) = \epsilon. \tag{3.13}$$

By an argument similar to that used above, we can obtain

$$\lim_{i\to\infty} p\left(x_{2n_i}, x_{2m_i}\right) = \epsilon \tag{3.14}$$

and

$$\lim_{i\to\infty} p\left(x_{2n_i+1}, x_{2m_i-1}\right) = \epsilon. \tag{3.15}$$

By using condition (3.1), we have

$$\begin{aligned}&\gamma\left(p\left(x_{2n_i+1}, x_{2m_i}\right)\right) = \gamma\left(p\left(fx_{2n_i}, Tx_{2m_i-1}\right)\right)\\ &\le N\left(x_{2n_i}, x_{2m_i-1}\right) - \psi\left(\gamma\left(p\left(x_{2n_i}, x_{2m_i-1}\right)\right), \frac{\gamma\left(p\left(x_{2n_i}, fx_{2n_i}\right)\right) + \gamma\left(p\left(x_{2m_i-1}, x_{2m_i}\right)\right)}{2}\right),\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}&N\left(x_{2n_i}, x_{2m_i-1}\right)\\ &= \max\left\{\gamma\left(p\left(x_{2n_i}, x_{2m_i-1}\right)\right), \gamma\left(p\left(x_{2n_i}, fx_{2n_i}\right)\right), \gamma\left(p\left(x_{2m_i-1}, Tx_{2m_i-1}\right)\right),\right.\\ &\qquad\left.\frac{\gamma\left(p\left(x_{2n_i}, Tx_{2m_i-1}\right)\right) + \gamma\left(p\left(x_{2m_i-1}, fx_{2n_i}\right)\right)}{2}\right\}\\ &= \max\left\{\gamma\left(p\left(x_{2n_i}, x_{2m_i-1}\right)\right), \gamma\left(p\left(x_{2n_i}, x_{2n_i+1}\right)\right), \gamma\left(p\left(x_{2m_i-1}, x_{2m_i}\right)\right),\right.\\ &\qquad\left.\frac{\gamma\left(p\left(x_{2n_i}, x_{2m_i}\right)\right) + \gamma\left(p\left(x_{2m_i-1}, x_{2n_i+1}\right)\right)}{2}\right\}.\end{aligned} \tag{3.17}$$

Taking the limit as $i \to \infty$ in (3.17) and using conditions (3.8), (3.13), (3.14), (3.15), we get

$$\lim_{i\to\infty} N\left(x_{2n_i}, x_{2m_i-1}\right) = \max\left\{\gamma(\epsilon), 0, 0, \frac{\gamma(\epsilon)+\gamma(\epsilon)}{2}\right\} = \gamma(\epsilon). \tag{3.18}$$

Passing to the upper limit as $i \to \infty$ in (3.16) and using conditions (3.8), (3.11), (3.13), (3.18) and apply the continuity of $\gamma$ and the lower semicontinuity of (Proposition 2.13), we get:

$$\begin{aligned}\gamma(\epsilon) &= \limsup_{i\to\infty}\gamma\left(p\left(x_{2n_i+1},x_{2m_i}\right)\right)\\ &\le \limsup_{i\to\infty} N\left(x_{2n_i},x_{2m_i-1}\right)\\ &\quad -\liminf_{i\to\infty}\psi\left(\gamma\left(p\left(x_{2n_i},x_{2m_i-1}\right)\right),\frac{\gamma\left(p\left(x_{2n_i},x_{2n_i+1}\right)\right)+\gamma\left(p\left(x_{2m_i-1},x_{2m_i}\right)\right)}{2}\right)\\ &\le \gamma(\epsilon)-\psi\left(\gamma(\epsilon),0\right),\end{aligned}$$

which implies:

$$\psi\left(\gamma(\epsilon),0\right)\le 0.$$

But by assumption, $\psi(s,t)>0$ for $s+t>0$, so:

$$\gamma(\epsilon)=0 \Rightarrow \epsilon=0,$$

a contradiction. Therefore, $(x_n)$ is a 0-Cauchy sequence in $X$. Since $(X,p)$ is 0-complete, there exists $u\in X$ such that $\lim_{n\to\infty}p(x_n,u)=p(u,u)=0$. This ensures that $\lim_{n\to\infty}p(x_{2n},u)=p(u,u)=0$ and $\lim_{n\to\infty}p(x_{2n+1},u)=p(u,u)=0$. Moreover, by Lemma 2.8, $\lim_{n\to\infty}p(x_{2n},gu)=p(u,gu)$ and $\lim_{n\to\infty}p(x_{2n+1},gu)=p(u,gu)$. By using condition (3.1), we obtain

$$\begin{aligned}\gamma\left(p\left(x_{2n+1},gu\right)\right) &= \gamma\left(p\left(fx_{2n},gu\right)\right)\\ &\le N\left(x_{2n},u\right)-\psi\left(\gamma\left(p\left(x_{2n},u\right)\right),\frac{\gamma\left(p\left(x_{2n},fx_{2n}\right)\right)+\gamma\left(p\left(u,gu\right)\right)}{2}\right),\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}N\left(x_{2n},u\right) &= \max\left\{\gamma\left(p\left(x_{2n},u\right)\right),\gamma\left(p\left(x_{2n},fx_{2n}\right)\right),\gamma\left(p\left(u,gu\right)\right),\right.\\ &\qquad \left.\frac{\gamma\left(p\left(x_{2n},gu\right)\right)+\gamma\left(p\left(u,fx_{2n}\right)\right)}{2}\right\}\\ &= \max\left\{\gamma\left(p\left(x_{2n},u\right)\right),\gamma\left(p\left(x_{2n},x_{2n+1}\right)\right),\gamma\left(p\left(u,gu\right)\right),\right.\\ &\qquad \left.\frac{\gamma\left(p\left(x_{2n},gu\right)\right)+\gamma\left(p\left(u,x_{2n+1}\right)\right)}{2}\right\}\\ &\to \gamma\left(p\left(u,gu\right)\right) \text{ as } n\to\infty.\end{aligned}$$

Taking the upper limit as $n\to\infty$ in (3.19), we have

$$\begin{aligned}\gamma\left(p\left(u,gu\right)\right) &\le \gamma\left(p\left(u,gu\right)\right)\\ &\quad -\liminf_{i\to\infty}\psi\left(\gamma\left(p\left(x_{2n},u\right)\right),\frac{\gamma\left(p\left(x_{2n},x_{2n+1}\right)\right)+\gamma\left(p\left(u,gu\right)\right)}{2}\right)\\ &\le \gamma\left(p\left(u,gu\right)\right)-\psi\left(0,\frac{1}{2}\gamma\left(p\left(u,gu\right)\right)\right),\end{aligned}$$

which gives that $\psi\left(0,\frac{1}{2}\gamma\left(p\left(u,gu\right)\right)\right)=0$. This assures that $\gamma\left(p\left(u,gu\right)\right)=0$

which implies $p(u,gu)=0$ and hence $gu=u$. By our previous discussion, $u$ is also a fixed point of $f$. Therefore, $u$ is a common fixed point of $f$ and $g$ with $p(u,u)=0$. To prove uniqueness, suppose $v$ is another such point with $p(v,v)=0$. Then: By applying condition (3.1), we get

$$\begin{aligned}\gamma(p(u,v)) &= \gamma(p(fu,gv)) \\ &\le N(u,v)-\psi\left(\gamma(p(u,v)),\frac{\gamma(p(u,fu))+\gamma(p(v,gv))}{2}\right),\end{aligned} \tag{3.20}$$

where

$$\begin{aligned} &N(u,v) \\ &= \max\left\{\gamma(p(u,v)),\gamma(p(u,fu)),\gamma(p(v,gv)),\frac{\gamma(p(u,gv))+\gamma(p(v,fu))}{2}\right\} \\ &= \max\{\gamma(p(u,v)),0,0,\gamma(p(u,v))\} \\ &= \gamma(p(u,v)). \end{aligned}$$

Thus, condition (3.20) becomes

$$\gamma(p(u,v)) \le \gamma(p(u,v))-\psi(\gamma(p(u,v)),0),$$

which implies that $\psi(\gamma(p(u,v)),0)=0$ and hence $\gamma(p(u,v))=0$, which implies $p(u,v)=0$ that is, $u=v$. Therefore, $f$ and $g$ have a unique common fixed point in $X$.

This completes the proof.

**Remark 3.2.** *If we take $\gamma(t)=t$ in Theorem 3.1, we recover Theorem 2.11 which is Theorem 3.9 of [14]. Moreover, Corollaries 3.10 through 3.13 of [14] follow directly as special cases of Theorem 3.1.*

**Example 3.3.** *Let $(X,p)$ be the partial metric space where $X=[0,1]$ and $p(x,y)=\max\{x,y\}$. Define the mappings $f,g:X\to X$ by:*

$$f(x)=\frac{x^2}{2}$$

$$g(x)=\frac{x^2}{3}$$

Consider the functions:

$$\psi(s,t)=\frac{s+t}{3} \quad (\text{control function})$$

$$\gamma(t)=t^5 \quad (\text{altering distance function})$$

We verify the contraction condition:

$$\gamma(p(fx,gy)) \le N(x,y)-\psi\left(\gamma(p(x,y)),\frac{\gamma(p(x,fx))+\gamma(p(y,gy))}{2}\right)$$

for all $x,y\in X$.

**Case Analysis Case I:** $y\le x$

$$p(fx, gy) = \max\left\{\frac{x^2}{2}, \frac{y^2}{3}\right\} = \frac{x^2}{2}$$

$$\gamma(p(fx, gy)) = \left(\frac{x^2}{2}\right)^5 = \frac{x^{10}}{32}$$

$$\begin{aligned}
&N(x, y) \\
&= \max\left\{\gamma(p(x,y)), \gamma(p(x,fx)), \gamma(p(y,gy)), \frac{\gamma(p(x,gy)) + \gamma(p(y,fx))}{2}\right\} \\
&= \max\left\{x^5, \left(\max\left\{x, \frac{x^2}{2}\right\}\right)^5, y^5, \frac{\left(\max\left\{x, \frac{y^2}{3}\right\}\right)^5 + \left(\max\left\{y, \frac{x^2}{2}\right\}\right)^5}{2}\right\} \\
&= \max\left\{x^5, x^5, y^5, \frac{x^5 + y^5}{2}\right\} = x^5
\end{aligned}$$

$$\psi(\cdot) = \frac{x^5 + \frac{x^5 + y^5}{2}}{3} = \frac{3x^5 + y^5}{6}$$

Verification:

$$\frac{x^{10}}{32} \le x^5 - \frac{3x^5 + y^5}{6} \Rightarrow 3x^{10} + 16y^5 \le 48x^5$$

This holds since for $x \in [0,1]$, $3x^{10} \le 3x^5$ and $16y^5 \le 16x^5 \le 45x^5$.

**Case II:** $x \le y$

$$p(fx, gy) = \max\left\{\frac{x^2}{2}, \frac{y^2}{3}\right\}$$

$$\gamma(p(fx, gy)) = \begin{cases} \left(\frac{y^2}{3}\right)^5 & \text{if } y \ge \sqrt{\frac{3}{2}}x \\ \left(\frac{x^2}{2}\right)^5 & \text{otherwise} \end{cases}$$

$$N(x, y) = \max\left\{y^5, x^5, y^5, \frac{y^5 + \left(\max\left\{x, \frac{y^2}{3}\right\}\right)^5}{2}\right\} = y^5$$

**Subcase II.1:** $y \ge \sqrt{\frac{3}{2}}x$

$$\gamma(p(fx, gy)) = \frac{y^{10}}{243}$$

$$\psi(\cdot) = \frac{3y^5 + x^5}{6}$$

$$\text{Condition}: \frac{y^{10}}{243} \le y^5 - \frac{3y^5 + x^5}{6} = \frac{3y^5 - x^5}{6}$$

This holds since for $y \in [0,1]$, $2y^{10} \le 243(3y^5 - x^5)/6$.

**Subcase II.2:** $y < \sqrt{\frac{3}{2}}x$

$$\gamma(p(fx, gy)) = \frac{x^{10}}{32}$$

$$\psi(\cdot) = \frac{3y^5 + x^5}{6}$$

$$\text{Condition}: \frac{x^{10}}{32} \le y^5 - \frac{3y^5 + x^5}{6} = \frac{3y^5 - x^5}{6}$$

Holds because $x \le y$ and $x \in [0,1]$.

**Special Case Demonstration**

At $x = 1, y = 1$ with $\gamma(t) = t$:

$$p(fx, gy) = \max\left\{\frac{1}{2}, \frac{1}{3}\right\} = \frac{1}{2}$$

$$N(x, y) = \max\left\{1, 1, 1, \frac{1+1}{2}\right\} = 1$$

$$\psi(\cdot) = \frac{1 + \frac{1+1}{2}}{3} = \frac{2}{3}$$

$$\text{Condition fails}: \frac{1}{2} \not\le 1 - \frac{2}{3} = \frac{1}{3}$$

This shows the necessity of the nonlinear altering distance function $\gamma(t) = t^5$.

## 4. Conclusions

We have established a new common fixed point theorem for pairs of self-mappings in 0-complete partial metric spaces, using a generalized contraction condition that combines: altering distance functions ( $\gamma \in \Gamma$ ) and two-variable control functions ( $\psi \in \Psi$ ). Our main result (Theorem 3.1) significantly extends previous work by providing a unified framework that recovers several known results as special cases. The concrete example (Example 3.3) demonstrates:

- The applicability of our results to explicit mappings
- The **necessity** of using a nonlinear altering distance function, specifically $\gamma(t) = t^5$, to satisfy the contraction condition, and
- The **insufficiency** of simpler linear choices such as $\gamma(t) = t$ in such settings.

Our results contribute to the growing body of fixed point theory in generalized metric spaces and provide tools for analyzing problems where standard metric space techniques are not directly applicable. The combination of altering distance functions with two-variable control functions appears particularly promising for future developments in this area.

## Conflicts of Interest

The authors declare no conflicts of interest.

## References

[1] Matthews, S.G. (1994) Partial Metric Topology. *Annals of the New York Academy of Sciences*, **728**, 183-197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x

[2] Abbas, M., Nazir, T. and Romaguera, S. (2012) Fixed Point Results for Generalized Cyclic Contraction Mappings in Partial Metric Spaces. *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, **106**, 287-297. https://doi.org/10.1007/s13398-011-0051-5

[3] Altun, I., Sola, F. and Simsek, H. (2010) Generalized Contractions on Partial Metric Spaces. *Topology and its Applications*, **157**, 2778-2785. https://doi.org/10.1016/j.topol.2010.08.017

[4] Ćirić, L., Samet, B., Aydi, H. and Vetro, C. (2011) Common Fixed Points of Generalized Contractions on Partial Metric Spaces and an Application. *Applied Mathematics and Computation*, **218**, 2398-2406. https://doi.org/10.1016/j.amc.2011.07.005

[5] Haghi, R.H., Rezapour, S. and Shahzad, N. (2011) Some Fixed Point Generalizations Are Not Real Generalizations. *Nonlinear Analysis: Theory, Methods & Applications*, **74**, 1799-1803. https://doi.org/10.1016/j.na.2010.10.052

[6] Haghi, R.H., Rezapour, S. and Shahzad, N. (2013) Be Careful on Partial Metric Fixed Point Results. *Topology and Its Applications*, **160**, 450-454. https://doi.org/10.1016/j.topol.2012.11.004

[7] Mohanta, S.K. (2011) A Fixed Point Theorem via Generalized *w*-Distance. *Bulletin of Mathematical Analysis and Applications*, **3**, 134-139.

[8] Mohanta, S.K. and Mohanta, S. (2012) A Common Fixed Point Theorem in G-Metric Spaces. *Cubo (Temuco)*, **14**, 85-101. https://doi.org/10.4067/s0719-06462012000300006

[9] Mohanta, S.K. and Biswas, S. (2017) Coincidence Points and Common Fixed Points for Hybrid Pair of Mappings in *b*-Metric Spaces Endowed with a Graph. *Journal of Linear and Topological Algebra*, **6**, 301-321.

[10] Khan, M.S., Swaleh, M. and Sessa, S. (1984) Fixed Point Theorems by Altering Distances between the Points. *Bulletin of the Australian Mathematical Society*, **30**, 1-9. https://doi.org/10.1017/s0004972700001659

[11] Shatanawi, W. and Postolache, M. (2010) Common Fixed Point Results for Mappings under Nonlinear Contraction of Cyclic Form in Ordered Metric Spaces. *Journal of Fixed Point Theory and Applications*, **2010**, Article 493298.

[12] Sastry, K.P.R., Naidu, S.V.R., Babu, G.V.R. and Naidu, G.A. (2000) Generalization of Common Fixed Point Theorems for Weakly Commuting Maps by Altering Distances. *Tamkang Journal of Mathematics*, **31**, 243-250. https://doi.org/10.5556/j.tkjm.31.2000.399

[13] Negash, A.G. and Bogale, M.F. (2025) A Common Fixed Point Result for Generalized Cyclic Contraction Pairs Involving Altering Distance and Control Functions in Partial Metric Spaces. *Advances in Pure Mathematics*, **15**, 491-504. https://doi.org/10.4236/apm.2025.157024

[14] Mohanta, S.K. (2021) Generalized Cyclic Contractions and Coincidence Points Involving a Control Function on Partial Metric Spaces. *Arab Journal of Mathematical Sciences*, **27**, 171-188. https://doi.org/10.1108/ajms-07-2020-0023

[15] Romaguera, S. (2009) A Kirk Type Characterization of Completeness for Partial

Metric Spaces. *Fixed Point Theory and Applications*, **2010**, Article No. 493298.

[16] Abdeljawad, T., Karapınar, E. and Taş, K. (2011) Existence and Uniqueness of a Common Fixed Point on Partial Metric Spaces. *Applied Mathematics Letters*, **24**, 1900-1904. https://doi.org/10.1016/j.aml.2011.05.014

[17] Karapinar, E. (2011) Generalizations of Caristi Kirk's Theorem on Partial Metric Spaces. *Fixed Point Theory and Applications*, **2011**, Article No. 4. https://doi.org/10.1186/1687-1812-2011-4

[18] Nashine, H.K. and Kadelburg, Z. (2013) Cyclic Contractions and Fixed Point Results via Control Functions on Partial Metric Spaces. *International Journal of Analysis*, **2013**, 1-9. https://doi.org/10.1155/2013/726387