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Hall Conductivity, Transition-State Theory and Their Effect on Entanglement and the Energy Density

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Abstract

Two isotropic harmonic oscillators in a magnetic field on the non-commutative phase space (*NCPS*) are studied in this paper. We derive the corresponding entanglement entropies of the vacuum state and we find analytically purity function, Hall conductivity and the energy density. Transition-state theory is also developed to examine the purity function and the energy density. By considering the action of the non-commutativity parameters, entanglement and Hall conductivity present a similar behavior. A large magnetic field acts as a disturbance of the accurate system information and the non-localization of the two harmonic oscillators in the atom by increasing its stored energy. The application of the transition-state theory increases entanglement and accelerates the non-localization phenomenon.

Subject Areas

Mathematical Analysis

Keywords

Two Isotropic Harmonic Oscillators in a Magnetic Field, *NCPS*, Entanglement, Hall Conductivity, Energy Density, Transition-State Theory

1. Introduction

A great deal of interest has been recently given to the formulation and possible experimental consequences of the extension of the standard quantum physical formalism to accommodate the non-commutativity of phase space operators [1]-[5]. The idea was inspired by quantum field theory and string theory [6]-[8]. The crucial difference with the standard quantum theory is to replace the usual product with the Moyal product [9] [10]. Therefore, they allow us to better understand various phenomena. So far, many examples have been studied intensively, such as the spectrum of the hydrogen atom [11] [12], the harmonic oscillator [13] [14], the Aharonov-Bohm effect under the action of a magnetic field [15]-[17], the Landau problem [18] [19], etc. We specify the particular system of two isotropic harmonic oscillators in a magnetic field, in the framework of non-commutative quantum mechanics, accordingly, because of the magnetic field application considering the Hall effect. This can be stated as follows: a semiconductor material through which an electric current flows perpendicular to the movement of charge carriers, a voltage is produced from the latter, so called Hall voltage, which has been attributed to the Hall effect. The Hall conductivity connected to the Hall effect has been taken into consideration in this work. Also, in the presence of the magnetic field B, we discuss the localized energy density, which is the amount of energy stored in a point of the material conductor. These two concepts are discussed in detail in this paper to clarify some properties of an entangled system. Entanglement concepts in non-commutative quantum mechanics are studied by many references, see for example [20]-[22] and we applied the transition-state theory to examine their effect on entanglement and the energy density.

2. Theoretical Framework (Definitions, Hamiltonian Diagonalization)

A harmonic oscillator can behave as an electron under a magnetic field of induction B, in this framework, consider two isotropic harmonic oscillators with unit masses and are exposed to a magnetic field [23]. The Hamiltonian is written:

$$H=\frac{1}{2}\left(-i\hbar\frac{\partial}{\partial y_1}+\frac{e}{2}By_2\right)^2+\frac{1}{2}\left(-i\hbar\frac{\partial}{\partial y_2}-\frac{e}{2}By_1\right)^2+\frac{1}{2}\omega^2\left(y_1^2+y_2^2\right). \quad (2.1)$$

where ω is the common angular frequency of the two harmonic oscillators, e is the electric charge. (2.1) can be reformulated as

$$H=-\frac{1}{2}\hbar^2\frac{\partial^2}{\partial y_1^2}-\frac{1}{2}\hbar^2\frac{\partial^2}{\partial y_2^2}+\frac{1}{2}\zeta y_1^2+\frac{1}{2}\zeta y_2^2-\frac{1}{2}\beta\left[y_1\left(-i\hbar\frac{\partial}{\partial y_2}\right)-\left(-i\hbar\frac{\partial}{\partial y_1}\right)y_2\right], \quad (2.2)$$

where $\zeta=\frac{e^2}{4}B^2+\omega^2$ and $\beta=eB$. To diagonalize (2.2), we define the unitary operator

$$\theta=\exp\left[-\gamma_1y_1y_2-\gamma_2\frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2}\right]. \quad (2.3)$$

where θ acts on Hamiltonian (2.2) as $\mathbf{H}=\theta H\theta^-$ to get the form

$$\mathbf{H} = \frac{1}{2}\left(\sigma_1 y_1^2 - \sigma_2 \frac{\partial^2}{\partial y_1^2}\right) + \frac{1}{2}\left(\sigma_1^* y_2^2 + \sigma_2 \frac{\partial^2}{y_2^2}\right), \tag{2.4}$$

where

$$\sigma_1 = \zeta - \hbar^2\gamma_1^2 + i\hbar\beta\gamma_1 \text{ and } \sigma_2 = i\hbar\beta\gamma_2. \tag{2.5}$$

The solution (2.4) gives

$$\gamma_1 = \frac{i\hbar\beta - 2\zeta\gamma_2}{2\hbar^2 + i\hbar\beta\gamma_2} \text{ and } \gamma_2 = \sqrt{\frac{\hbar^2}{\zeta}}. \tag{2.6}$$

3. Methods

The non-commutativity of positions y_k and momentums $-i\hbar\frac{\partial}{\partial y_l}$ operators in two-dimensional space is imposed by the relations [24] $[y_1, y_2] = i\theta_1$, $\left[-i\hbar\frac{\partial}{\partial y_1}, -i\hbar\frac{\partial}{\partial y_2}\right] = i\theta_2$ and $\left[y_k, -i\hbar\frac{\partial}{\partial y_l}\right] = i\delta_{kl}\hbar$, where $(k=1,2)$, $(l=1,2)$ and θ_1, θ_2 are the non-commutativity variables. As usual, we assume after, in the numerical section that $\hbar = 1$ and $e = 1$. Quantization deformation is the suitable method to study eignensolution of Hamiltonian (2.4) in non-commutative phase space [25]. $\star$ is the Moyal product introducing the non-commutativity, applied when we treat the classical quantities $\left(y, -i\hbar\frac{\partial}{\partial y}\right)$ to replacing the ordinary product, it is given as

$$\star = \exp\frac{i\hbar}{2}\left(\vec{\partial}_{y_1}\vec{\partial}_{\frac{\partial}{\partial y_2}} - \vec{\partial}_{\frac{\partial}{\partial y_1}}\vec{\partial}_{y_2}\right)\exp\frac{i\theta_1}{2}\xi_{kl}\left(\vec{\partial}_{y_1}\vec{\partial}_{y_2} - \vec{\partial}_{y_2}\vec{\partial}_{y_1}\right)\exp\frac{i\theta_2}{2}\xi_{kl}\left(\vec{\partial}_{\frac{\partial}{\partial y_1}}\vec{\partial}_{\frac{\partial}{\partial y_2}} - \vec{\partial}_{\frac{\partial}{\partial y_2}}\vec{\partial}_{\frac{\partial}{\partial y_1}}\right), \tag{3.1}$$

where ξ_{kl} is defined as a matrix of dimension 2 and we have

$$\xi_{kl} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hamiltonian (2.4) can be reformulated as the sum of two Hamiltonians $\mathbf{H}_1$ and $\mathbf{H}_2$ such as

$$\begin{aligned}\mathbf{H}_1 = &\left(i\frac{\partial}{\partial y_1}\sqrt{\sigma_2}\cos(\omega_1) - y_2\sqrt{\sigma_1^*}\sin(\omega_1)\right)^2 \\ &+ \left(\frac{\partial}{\partial y_2}\sqrt{\sigma_2}\cos(\omega_2) - y_1\sqrt{\sigma_1}\sin(\omega_2)\right)^2,\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\mathbf{H}_2 = &\left(i\frac{\partial}{\partial y_1}\sqrt{\sigma_2}\sin(\omega_1) + y_2\sqrt{\sigma_1^*}\cos(\omega_1)\right)^2 \\ &+ \left(\frac{\partial}{\partial y_2}\sqrt{\sigma_2}\sin(\omega_2) + y_1\sqrt{\sigma_1}\cos(\omega_2)\right)^2.\end{aligned} \tag{3.3}$$

In expressions (3.2) and (3.3):

$$\omega_1 = \frac{1}{2}\arctan\left(-\frac{2\hbar\left(\frac{\sqrt{\sigma_2}}{\sqrt{\sigma_1^*}}-1\right)\theta_2}{\vartheta}\right),$$

$$\omega_2 = \frac{1}{2}\arctan\left(\frac{2\hbar\frac{\sqrt{\sigma_2}}{\sqrt{\sigma_1^*}}\left(\frac{\sqrt{\sigma_2}}{\sqrt{\sigma_1^*}}-1\right)\theta_1}{\vartheta}\right), \tag{3.4}$$

where

$$\vartheta = \hbar^2\left(\frac{\sqrt{\sigma_2}}{\sqrt{\sigma_1^*}}-1\right)^2 - \left(\frac{\sigma_2}{\sigma_1^*}\theta_1^2 - \theta_2^2\right). \tag{3.5}$$

The Moyal product acts between $\mathbf{H}_1$ and $\mathbf{H}_2$ commutation relation as

$$[\mathbf{H}_1, \mathbf{H}_2]_\star = \mathbf{H}_1 \star \mathbf{H}_2 - \mathbf{H}_2 \star \mathbf{H}_1 = 0, \tag{3.6}$$

and verify the ordinary relation

$$\mathbf{H}_1 \star \mathbf{H}_2 = \mathbf{H}_2 \star \mathbf{H}_1 = \mathbf{H}_2\mathbf{H}_1. \tag{3.7}$$

The eigenvalue solution of Hamiltonian (2.4) is given by solving the eigenequation

$$\mathbf{H} \star \mathbf{W}_{n,m} = \mathbf{W}_{n,m} \star \mathbf{H} = E_{n,m}\mathbf{W}_{n,m}. \tag{3.8}$$

where $\mathbf{W}_{n,m}$ is the Wigner function of the (n,m) quantum states and $E_{n,m}$ is the corresponding eigenenergy. (3.8) corresponds to the standard two-dimensional Schrödinger equation, the solution is given by the following Wigner functions

$$\mathbf{W}_n = \frac{(-1)^n}{\pi\delta_1} \mathrm{e}^{-\frac{H_1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}} L_n\left(\frac{2\mathbf{H}_1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}\right), \tag{3.9}$$

and

$$\mathbf{W}_m = \frac{(-1)^m}{\pi\delta_2} \mathrm{e}^{-\frac{H_2}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}} L_m\left(\frac{2\mathbf{H}_2}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}\right). \tag{3.10}$$

δ_1 and δ_2 in expressions (3.9) and (3.10) have the forms:

$$\delta_1 = \hbar\sqrt{\sigma_2}\left(\sqrt{\sigma_1}\cos(\omega_1)\sin(\omega_2) - \sqrt{\sigma_1^*}\sin(\omega_1)\cos(\omega_2)\right) + \sqrt{\sigma_2}\left(\theta_1\sqrt{\sigma_1}\cos(\omega_1)\cos(\omega_2) - \theta_2\sqrt{\sigma_1^*}\sin(\omega_1)\sin(\omega_2)\right),$$

and

$$\delta_2 = \hbar\sqrt{\sigma_2}\left(\sqrt{\sigma_1}\sin(\omega_1)\cos(\omega_2) - \sqrt{\sigma_1^*}\cos(\omega_1)\sin(\omega_2)\right) + \sqrt{\sigma_2}\left(\theta_1\sqrt{\sigma_1}\sin(\omega_1)\sin(\omega_2) - \theta_2\sqrt{\sigma_1^*}\cos(\omega_1)\cos(\omega_2)\right). \tag{3.11}$$

We write the Wigner function and the eigenenergy of expression (3.8) as

$$\mathbf{W}_{nm} = \mathbf{W}_n \star \mathbf{W}_m = \mathbf{W}_n \mathbf{W}_m, \tag{3.12}$$

and

$$E_{n,m} = \mathbf{E}_n + \mathbf{E}_m. \tag{3.13}$$

Consequently, we get respectively

$$\mathbf{W}_{nm} = \frac{(-1)^{n+m}}{\pi^2 \delta_1 \delta_2} \mathrm{e}^{-\frac{H_1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}} \mathrm{e}^{-\frac{H_2}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}} L_n\left(\frac{2\mathbf{H}_1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}\right) L_m\left(\frac{2\mathbf{H}_2}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}\right), \tag{3.14}$$

$$E_{n,m} = \sqrt{\hbar\delta_1\sigma_1\sigma_2}\left(n+\frac{1}{2}\right) + \sqrt{\hbar\delta_2\sigma_1^*\sigma_2}\left(m+\frac{1}{2}\right). \tag{3.15}$$

One can easily show that

$$\int \mathbf{W}_{nm}\left(y_1, \frac{\partial}{\partial y_1} : y_2, \frac{\partial}{\partial y_2}\right) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}\left(\frac{\partial}{\partial y_1}\right) \mathrm{d}\left(\frac{\partial}{\partial y_2}\right) = 1. \tag{3.16}$$

Particularly the vacuum state, (3.14) becomes

$$\mathbf{W}_{00} = \frac{1}{\pi^2 \delta_1 \delta_2} \mathrm{e}^{-\frac{H_1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}} \mathrm{e}^{-\frac{H_2}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}}. \tag{3.17}$$

To calculate various entropies in non-commutative phase space, we need first to calculate the reduced Wigner functions of each of two harmonic oscillators. Using (3.17), one has

$$\begin{aligned} \mathbf{W}_{00}\left(y_1, \frac{\partial}{\partial y_1}\right) &= \int \mathbf{W}_{00}\left(y_1, \frac{\partial}{\partial y_1}; y_2, \frac{\partial}{\partial y_2}\right) \mathrm{d}y_2 \mathrm{d}\left(\frac{\partial}{\partial y_2}\right) \\ &= \frac{\gamma}{\pi\hbar} \mathrm{e}^{-\frac{1}{\hbar\delta_1\delta_2\sqrt{\sigma_1\sigma_2}\sqrt{\sigma_1^*\sigma_2}}\left(\frac{\sigma_1}{\vartheta_1} y_1^2 + \frac{\sigma_2}{\vartheta_2}\frac{\partial^2}{\partial y_1^2}\right)}, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \mathbf{W}_{00}\left(y_2, \frac{\partial}{\partial y_2}\right) &= \int \mathbf{W}_{00}\left(y_1, \frac{\partial}{\partial y_1}; y_2, \frac{\partial}{\partial y_2}\right) \mathrm{d}y_1 \mathrm{d}\left(\frac{\partial}{\partial y_1}\right) \\ &= \frac{\gamma'}{\pi\hbar} \mathrm{e}^{-\frac{1}{\hbar\delta_1\delta_2\sqrt{\sigma_1\sigma_2}\sqrt{\sigma_1^*\sigma_2}}\left(\frac{\sigma_1^*}{\vartheta_1'} y_2^2 + \frac{\sigma_2}{\vartheta_2'}\frac{\partial^2}{\partial y_2^2}\right)}. \end{aligned} \tag{3.19}$$

In expressions (3.18) and (3.19),

$$\gamma = \sqrt{\frac{1}{\delta_1^2\delta_2^2\sigma_1^*\sigma_2\vartheta_1\vartheta_2}}, \quad \gamma' = \sqrt{\frac{1}{\delta_1^2\delta_2^2\sigma_1\sigma_2\vartheta_1'\vartheta_2'}},$$

$$\vartheta_1 = \frac{1}{\delta_1\sqrt{\sigma_1^*\sigma_2}}\sin^2(\omega_1) + \frac{1}{\delta_2\sqrt{\sigma_1\sigma_2}}\cos^2(\omega_1),$$

$$\vartheta_2 = \frac{1}{\delta_1\sqrt{\sigma_1^*\sigma_2}}\cos^2(\omega_2) + \frac{1}{\delta_2\sqrt{\sigma_1\sigma_2}}\sin^2(\omega_2),$$

$$\vartheta_1' = \frac{1}{\delta_1\sqrt{\sigma_1^*\sigma_2}}\cos^2(\omega_1) + \frac{1}{\delta_2\sqrt{\sigma_1\sigma_2}}\sin^2(\omega_1),$$

and

$$\vartheta_2' = \frac{1}{\delta_1\sqrt{\sigma_1^*\sigma_2}}\sin^2(\omega_2) + \frac{1}{\delta_2\sqrt{\sigma_1\sigma_2}}\cos^2(\omega_2). \tag{3.20}$$

Since the system is entangled then expression (3.18) or (3.19) is sufficient to calculate entanglement entropies. The purity function is a derivation of the linear entropy and it also provides another form to examine entanglement, its expression is

$$P = \iint\left[(\mathbf{W}_{00})_\star^2\right]\mathrm{d}y_1\mathrm{d}\left(\frac{\partial}{\partial y_1}\right). \tag{3.21}$$

Following problem in ref. [26] and by applying the Moyal product $\star$ on expression (3.18), we obtain

$$\left(\mathbf{W}_{00}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_\star = \frac{1}{\pi\hbar}\frac{\gamma}{\sqrt{1-\gamma^2}}\exp_\star\left[-\frac{1}{2\hbar^2}\chi\left(\frac{\sigma_1}{\vartheta_1}y_1^2 + \frac{\sigma_2}{\vartheta_2}\frac{\partial^2}{\partial y_1^2}\right)\right], \tag{3.22}$$

where $\chi = \frac{1}{\gamma\delta_1\delta_2\sqrt{\sigma_1\sigma_2}\sqrt{\sigma_1^*\sigma_2}}\ln\left(\frac{1+\gamma}{1-\gamma}\right)$.

By substituting (3.22) in (3.21), we have

$$P = \frac{1}{\pi^2\hbar^2}\frac{\gamma^2}{1-\gamma^2}\left(\frac{\pi}{\frac{1}{\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}\iint\exp_\star\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}\left(y_1^2 + y_1'^2\right)\right]\mathrm{d}y_1\mathrm{d}y_1'. \tag{3.23}$$

Consequently, we can show that

$$P = \frac{1}{\sqrt{\pi}\hbar^2}\frac{\gamma^2}{1-\gamma^2}\left(\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\right)\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)\right)^{\frac{1}{2}}. \tag{3.24}$$

For all integer $n \geq 1$, the Rényi entropy writes

$$S_n(\mathbf{W}_{00}) = \frac{1}{1-n}\ln\left(\left(\frac{1}{2\pi\hbar}\right)^{1-n}\int(\mathbf{W}_{00})_\star^n\,\mathrm{d}y_1\mathrm{d}\left(\frac{\partial}{\partial y_1}\right)\right). \tag{3.25}$$

In order n,

$$\begin{aligned}\left(\mathbf{W}_{00}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_\star^n &= \frac{2\gamma^n}{(\pi\hbar)^n\left[(1+\gamma)^n + (1-\gamma)^n\right]}\\ &\times\exp\left[-\frac{1}{\hbar\delta_1\delta_2\sqrt{\sigma_1\sigma_2}\sqrt{\sigma_1^*\sigma_2}}\left(\frac{\sigma_1}{\vartheta_1}y_1^2 + \frac{\sigma_2}{\vartheta_2}\frac{\partial^2}{\partial y_1^2}\right)\frac{\left[(1+\gamma)^n - (1-\gamma)^n\right]}{\gamma\left[(1+\gamma)^n + (1-\gamma)^n\right]}\right].\end{aligned} \tag{3.26}$$

Insert (3.26) in (3.25), we have

$$S_n(\mathbf{W}_{00}) = \frac{n}{1-n}\ln(2\gamma) - \frac{1}{1-n}\ln\left[(1+\gamma)^n - (1-\gamma)^n\right]. \tag{3.27}$$

The particular case where $n \to 1$, expression (3.27) reduces to the von Neumann entropy as

$$S_1(\mathbf{W}_{00}) = \int \mathbf{W}_{00} \star \ln_\star (2\pi\hbar \mathbf{W}_{00}) \mathrm{d}y_1 \mathrm{d}\left(\frac{\partial}{\partial y_1}\right). \tag{3.28}$$

From (3.28), we can obtain $S_1(\mathbf{W}_{00})$ as

$$S_1(\mathbf{W}_{00}) = \frac{1}{2\gamma}\left[-\ln(2\gamma) - (1-\gamma)\ln(1-\gamma) + (1+\gamma)\ln(1+\gamma)\right]. \tag{3.29}$$

Such expressions (3.24), (3.27) and (3.29) are interesting because they show all the ingredients to investigate the system. In the ordinary case when $\theta_1 = \theta_2 = 0$, purity function, Rényi and the von Neumann entropies vanish. Recently, the quantization deformation method is extended to three dimensional to study entanglement of three isotropic harmonic oscillators by ref. [27].

4. Hall Conductivity and Energy Density

A natural generalization of two harmonic oscillators under a magnetic field in the framework of non-commutative quantum mechanics is devoted to deriving some properties of these two concepts: Hall conductivity and energy density, from the aspect of quantum information (entangled system). To start, we calculate the current density. Suppose that from (2.4),

$$H'\left(y_1, \frac{\partial}{\partial y_1}\right) = \frac{1}{2}\left(\sigma_1 y_1^2 - \sigma_2 \frac{\partial^2}{\partial y_1^2}\right), \tag{4.1}$$

and using expression (3.18), we have the current density as

$$\begin{aligned} J(y_1) &= \int \frac{\partial}{\partial\left(\frac{\partial}{\partial y_1}\right)} H'\left(y_1, \frac{\partial}{\partial y_1}\right) \mathbf{W}_{00}\left(y_1, \frac{\partial}{\partial y_1}\right) \mathrm{d}\left(\frac{\partial}{\partial y_1}\right) \\ &= \int \frac{\partial}{\partial\left(\frac{\partial}{\partial y_1}\right)} \left(\sigma_1 y_1^2 - \sigma_2 \frac{\partial^2}{\partial y_1^2}\right) \star \left(\mathbf{W}_{00}\left(y_1, \frac{\partial}{\partial y_1}\right)\right)_\star \mathrm{d}\left(\frac{\partial}{\partial y_1}\right). \end{aligned} \tag{4.2}$$

By applying the Moyal product (3.1), we can write

$$\begin{aligned} &\left(\sigma_1 y_1^2 - \sigma_2 \frac{\partial^2}{\partial y_1^2}\right) \star \left(\mathbf{W}_{00}\left(y_1, \frac{\partial}{\partial y_1}\right)\right)_\star \\ &= \frac{1}{2}\left(\sigma_1 y_1^2 - \sigma_2 \frac{\partial^2}{\partial y_1^2} - \frac{\hbar^2}{4}\left(\sigma_1 \partial^2_{\left(\frac{\partial}{\partial y_1}\right)} - \sigma_2 \partial^2_{y_1}\right)\right. \\ &\quad \left. + i\hbar\left(\sigma_1 y_1 \partial_{\left(\frac{\partial}{\partial y_1}\right)} + \sigma_2 \frac{\partial}{\partial y_1} \partial_{y_1}\right)\right)\left(\mathbf{W}_{00}\left(y_1, \frac{\partial}{\partial y_1}\right)\right)_\star \end{aligned} \tag{4.3}$$

Substituting (3.22) in (4.3), after some calculation we find expression (4.2) as:

$$J(y_1)=\frac{\gamma}{\pi\hbar\sqrt{1-\gamma^2}}\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}$$
$$\times\left[(-\sigma_2+D_2)\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}-i\sqrt{\pi}\sigma_1\sigma_2\chi\left(\frac{1}{\vartheta_2}-\frac{1}{\vartheta_1}\right)y_1\right] \tag{4.4}$$
$$\times\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_1^2\right].$$

The average of the current density operator $\langle J(y_1)\rangle$ is defined as

$$\langle J(y_1)\rangle=\int|\psi_0(y_1)|^2\star J(y_1)\mathrm{d}y_1. \tag{4.5}$$

where $|\psi_0(y_1)|^2$ is the probability distribution in the phase space. It is defined with the integral of the Wigner function on the momentum space as

$$|\psi_0(y_1)|^2=\frac{1}{2\hbar}\int\left(\mathbf{W}_{00}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_\star\mathrm{d}\left(\frac{\partial}{\partial y_1}\right). \tag{4.6}$$

We develop this expression using (3.22), we have

$$\frac{1}{2\hbar}\int\left(\mathbf{W}_{00}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_\star\mathrm{d}\left(\frac{\partial}{\partial y_1}\right)$$
$$=\frac{1}{2\hbar^2\sqrt{\pi}}\frac{\gamma}{\sqrt{1-\gamma^2}}\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}\times\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_1^2\right]. \tag{4.7}$$

Using expression (4.7), we can then express (4.5) which interests us as

$$\langle J(y_1)\rangle=\frac{1}{2\hbar^3\pi}\frac{\gamma^2}{1-\gamma^2}\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)\left(\frac{1}{\frac{1}{\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\right)^{\frac{1}{2}}\left[(-\sigma_2+D_2)\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}\right.$$
$$\left.-\frac{i}{2}\sigma_1\sigma_2\chi\left(\frac{1}{\vartheta_2}-\frac{1}{\vartheta_1}\right)\left(\frac{1}{\frac{1}{\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\right)^{\frac{1}{2}}\right], \tag{4.8}$$

where

$$D_2=-\frac{1}{4\hbar^2}\left(\frac{\sigma_2}{\vartheta_2}\right)^2\chi^2\zeta_1. \tag{4.9}$$

Hall conductivity is defined as the ratio of the current density $J(y_1)$ and the electric field $E(y_1)$:

$$\Gamma = \frac{\langle J(y_1)\rangle}{\langle E(y_1)\rangle}. \tag{4.10}$$

From (4.1), we have

$$V(y_1) = \frac{1}{2}\sigma_1 y_1^2, \tag{4.11}$$

consequently

$$\left(V(y_1)\right)_\star = \frac{1}{2}\left(\sigma_1 y_1^2 - \frac{\hbar^2}{4}\partial^2_{\left(\frac{\partial}{\partial y_1}\right)} + i\hbar y_1 \partial_{\left(\frac{\partial}{\partial y_1}\right)}\right). \tag{4.12}$$

So, it is easy to verify that

$$\left(E(y_1)\right)_\star = -\left(\frac{\mathrm{d}V(y_1)}{\mathrm{d}y_1}\right)_\star = -\sigma_1 y_1 - \frac{1}{2} i\hbar \partial_{\left(\frac{\partial}{\partial y_1}\right)}. \tag{4.13}$$

The average of (4.13)

$$\begin{aligned}\langle E(y_1)\rangle &= \int \left|\psi_0(y_1)\right|^2 \star \left(E(y_1)\right)_\star \mathrm{d}y_1 \\ &= \frac{2\sigma_1}{\sqrt{\pi}} \frac{\gamma}{\sqrt{1-\gamma^2}} \left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\right)\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right).\end{aligned} \tag{4.14}$$

Insert (4.8), (4.14) in (4.10), we obtain

$$\begin{aligned}\Gamma = \frac{1}{4\hbar^3}\frac{1}{\sqrt{\pi}\sigma_1}\frac{\gamma}{\sqrt{1-\gamma^2}}\frac{1}{\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\right)^{\frac{1}{2}}}&\left[\left(-\sigma_2 + D_2\right)\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}\right. \\ &\left. -\frac{i}{2}\sigma_1\sigma_2\chi\left(\frac{1}{\vartheta_2} - \frac{1}{\vartheta_1}\right)\left(\frac{1}{\frac{1}{\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\right)^{\frac{1}{2}}\right].\end{aligned} \tag{4.15}$$

Equation above summarizes some important specificity of an entangled system that can be quantified using Hall effect. Similarly, Hall conductivity is calculated in three dimensions using the Kubo formula from the bulk-edge [28]. Note that we have studied from two harmonic oscillators, entanglement entropies, Hall conductivity. It would be interesting to study their energy density. Their calculation in this context represents an important vision. Energy density reads

$$\varepsilon(y_1) = \int H'\left(y_1, \frac{\partial}{\partial y_1}\right) \star \left(\mathbf{W}_{00}\left(y_1, \frac{\partial}{\partial y_1}\right)\right)_\star \mathrm{d}\left(\frac{\partial}{\partial y_1}\right). \tag{4.16}$$

Go back expression (4.3), after integration, (4.16) becomes

$$\varepsilon(y_1)=\frac{1}{\pi\hbar}\frac{\gamma}{\sqrt{1-\gamma^2}}\left(\frac{1}{2\hbar^2\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}\left[\sqrt{\pi}(\sigma_1+D_1)y_1^2-\frac{i}{2}\sigma_1\sigma_2\chi\left(\frac{1}{\vartheta_2}-\frac{1}{\vartheta_1}\right)\left(\frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}y_1\right.$$
$$\left.+\frac{\sqrt{\pi}}{4}(-\sigma_2+D_2)\left(\frac{1}{2\hbar^2\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}\right]\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_1^2\right], \tag{4.17}$$

where

$$D_1=-\frac{1}{4\hbar^2}\left(\frac{\sigma_1}{\vartheta_1}\right)^2\chi^2\zeta_2. \tag{4.18}$$

The most famous examples that reflect the usefulness of both concepts are provided by the quantum Hall effect [29] [30], semiconductor quantum devices [31] [32], etc.

5. Application of Transition-State Theory

We will here briefly introduce some aspects of transition-state theory, focusing on the aspects that will be useful to us later. For more details, see references [33]-[35], from which this presentation is strongly inspired. We have performed a transition to the first excited state. Harmonic oscillator one moves into the first excited state, second harmonic oscillator in the ground state, so Wigner function (3.14) becomes

$$\mathbf{W}_{10}=-\mathbf{W}_{00}+\frac{2}{\pi^2\delta_1^2\delta_2\sqrt{\sigma_1\sigma_2}}\mathbf{H}_1\mathrm{e}^{-\frac{H_1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}}\mathrm{e}^{-\frac{H_2}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}}. \tag{5.1}$$

Consequently, of expression (5.1), the reduced Wigner function of the variables $\left(y_1,\frac{\partial}{\partial y_1}\right)$, is written

$$\mathbf{W}_{10}\left(y_1,\frac{\partial}{\partial y_1}\right)=\frac{2}{\hbar\sqrt{\sigma_1\sigma_2}}\left(\zeta_1y_1^2+\zeta_2\frac{\partial^2}{\partial y_1^2}+\zeta_3\right)\mathbf{W}_{00}\left(y_1,\frac{\partial}{\partial y_1}\right). \tag{5.2}$$

In expression (5.2),

$$\zeta_1=\frac{d^2}{4c^2}+\sqrt{\sigma_1\sigma_2}\frac{d}{c}\sin(\omega_2)\cos(\omega_2)+\sigma_1\sin^2(\omega_2),$$

$$\zeta_2=\frac{b^2}{4a^2}-\sqrt{\sigma_1\sigma_2}\frac{b}{a}\sin(\omega_1)\cos(\omega_1)-\sigma_1\sin^2(\omega_1),$$

and

$$\zeta_3=\frac{1}{2a}\sigma_1^*\sin^2(\omega_1)+\frac{1}{2c}\sigma_2\cos^2(\omega_2)-1, \tag{5.3}$$

where

$$a=\frac{1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}\sigma_1^*\sin^2(\omega_1)+\frac{1}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}\sigma_1^*\cos^2(\omega_1),$$

$$b=2i\sqrt{\sigma_1^*\sigma_2}\sin(\omega_1)\cos(\omega_1)\left(-\frac{1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}+\frac{1}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}\right),$$

$$c=\frac{1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}\sigma_2\cos^2(\omega_2)+\frac{1}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}\sigma_2\sin^2(\omega_2),$$

and

$$d=2\sqrt{\sigma_1\sigma_2}\sin(\omega_2)\cos(\omega_2)\left(-\frac{1}{\hbar\delta_1\sqrt{\sigma_1\sigma_2}}+\frac{1}{\hbar\delta_2\sqrt{\sigma_1^*\sigma_2}}\right). \tag{5.4}$$

Using ref. [36] and applying the Moyal product (3.1), we have

$$\begin{aligned}&\left(\mathbf{W}_{10}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_\star\\&=\left((\zeta_1+D_1)y_1^2+(\zeta_2+D_2)\frac{\partial^2}{\partial y_1^2}-\frac{1}{\hbar^2}\chi\left(\zeta_1\frac{\sigma_2}{\vartheta_2}+\zeta_2\frac{\sigma_1}{\vartheta_1}\right)y_1\frac{\partial}{\partial y_1}+\zeta_3\right)\\&\times\left(\mathbf{W}_{00}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_\star.\end{aligned} \tag{5.5}$$

At the barrier, the oscillator frequency is perturbed, defining thus the potential using (4.11) as

$$\sigma_1y_1^2\simeq\sigma_{1b}y_{1b}^2-\sigma_{1b}y_1^2, \tag{5.6}$$

where $\sigma_{1b}=-i\sigma_1$, (5.6) becomes

$$y_1^2\simeq i\left(y_1^2-y_{1b}^2\right), \tag{5.7}$$

Insert (5.7) in (5.5), we have

$$\begin{aligned}\left(\left(\mathbf{W}_{10}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_b\right)_\star&=\frac{\gamma}{\pi\hbar\sqrt{1-\gamma^2}}\left(i(\zeta_1+D_1)\left(y_1^2-y_{1b}^2\right)+(\zeta_2+D_2)\frac{\partial^2}{\partial y_1^2}\right.\\&\left.-\frac{1}{\hbar^2}\chi\left(\zeta_1\frac{\sigma_2}{\vartheta_2}+\zeta_2\frac{\sigma_1}{\vartheta_1}\right)\sqrt{i\left(y_1^2-y_{1b}^2\right)}\frac{\partial}{\partial y_1}+\zeta_3\right)\\&\times\exp_\star\left[-\frac{1}{2\hbar^2}\chi\left(i\frac{\sigma_1}{\vartheta_1}\left(y_1^2-y_{1b}^2\right)+\frac{\sigma_2}{\vartheta_2}\frac{\partial^2}{\partial y_1^2}\right)\right].\end{aligned} \tag{5.8}$$

At the barrier, we read the Wigner function as

$$\left(\mathbf{W}_{10}\left(y_{1b},\frac{\partial}{\partial y_1}\right)\right)_\star=\int\left(\left(\mathbf{W}_{10}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_b\right)_\star\mathrm{d}y_1 \tag{5.9}$$

To compute analytically integral of expression (5.9), we use ref. [37] and we ended up with

$$\left(\mathbf{W}_{10}\left(y_{1b},\frac{\partial}{\partial y_1}\right)\right)_{\star}$$

$$=\frac{\gamma}{\pi\hbar\sqrt{1-\gamma^2}}\left(\frac{\pi}{\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\right)^{\frac{1}{2}}\left(\lambda\frac{\partial}{\partial y_1}\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_{1b}^2\right]-i\left(\zeta_1+D_1\right)y_{1b}^2\right. \tag{5.10}$$

$$\left.-i\left(\zeta_2+D_2\right)\frac{\partial^2}{\partial y_1^2}+\lambda_1\right)f\left(y_{1b}\right)\exp_{\star}\left[-\frac{1}{2\hbar^2}\chi\left(\frac{\sigma_1}{\vartheta_1}y_{1b}^2+\frac{\sigma_2}{\vartheta_2}\frac{\partial^2}{\partial y_1^2}\right)\right].$$

$f\left(y_{1b}\right)$ is the Step function, it is defined as

$$f\left(y_{1b}\right)=\begin{cases}0 & y_{1b}<0\\ 1 & y_{1b}>0,\end{cases} \tag{5.11}$$

Now, we have just evaluated entanglement and the energy density via the transition-state theory, starting from expression

$$P_b=\iint\left[\left(\mathbf{W}_{10}\left(y_{1b},\frac{\partial}{\partial y_1}\right)\right)_{\star}^2\right]\mathrm{d}y_{1b}\mathrm{d}\left(\frac{\partial}{\partial y_1}\right) \tag{5.12}$$

This allows us to write

$$P_b=\iint\frac{1}{\pi^2\hbar^2}\frac{\gamma^2}{1-\gamma^2}\left(\frac{\pi}{\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\right)\left(\frac{\pi}{\frac{1}{\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}\exp_{\star}\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}\left(y_{1b}^2+y_{1b}'^2\right)\right]$$

$$\times\left[\lambda^2\left(\frac{1}{\frac{1}{\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}\left(y_{1b}^2+y_{1b}'^2\right)\right]+\frac{i}{\sqrt{\pi}}\left(\frac{1}{\frac{1}{\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right)^{\frac{1}{2}}\left(\zeta_1+D_1\right)\lambda\right.$$

$$\times\left(y_{1b}'^2\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_{1b}^2\right]-y_{1b}^2\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_{1b}'^2\right]\right)+i\lambda\lambda_2\left(\frac{1}{\frac{1}{\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\right) \tag{5.13}$$

$$\times\left(\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_{1b}^2\right]-\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_{1b}'^2\right]\right)+\frac{\lambda}{\sqrt{\pi}}$$

$$\times\left(\lambda_1^*\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_{1b}^2\right]+\lambda_1\exp\left[-\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}y_{1b}'^2\right]\right)+\left(\zeta_1+D_1\right)\left(\lambda_2\left(y_{1b}^2+y_{1b}'^2\right)\right.$$

$$\left.\left.+\left(\zeta_1+D_1\right)y_{1b}^2y_{1b}'^2+i\left(\lambda_1y_{1b}'^2-\lambda_1^*y_{1b}^2\right)\right)+\lambda_2\left(i\left(\lambda_1-\lambda_1^*\right)+\lambda_2\right)+\lambda_1\lambda_1^*\right]\mathrm{d}y_{1b}\mathrm{d}y_{1b}',$$

In expression (5.13),

$$\lambda = -\frac{\pi}{\Gamma\left(-\frac{1}{2}\right)}\frac{1}{\hbar^2}\chi\left(\zeta_1\frac{\sigma_2}{\vartheta_2}+\zeta_2\frac{\sigma_1}{\vartheta_1}\right)\left(\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}\right)^2,$$

$$\lambda_1 = \left(\frac{\pi}{\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\right)^{\frac{1}{2}}\left(\frac{i}{2}\lambda_3+\zeta_3\right),$$

$$\lambda_3 = \frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_1}{\vartheta_1}}\left(\zeta_1+D_1\right) \text{ and } \lambda_2 = \frac{1}{4}\frac{1}{\frac{1}{\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}\left(\zeta_2+D_2\right). \tag{5.14}$$

Expression (5.13) is a direct consequence of

$$P_b = \left(\frac{\pi}{\frac{1}{2\hbar^2}\frac{\sigma_1}{\vartheta_1}\chi}\right)\left[\left(\frac{1}{\frac{1}{\hbar^2}\frac{\sigma_2}{\vartheta_2}\chi}\right)\lambda^2 + i\left(\lambda_1-\lambda_1^*\right)\left(\lambda_2+\lambda_3\right)+\lambda_3\left(\lambda_2+\lambda_3\right)\right.$$
$$\left. + \frac{1}{\sqrt{2\pi}}\left(\lambda_1^*+\lambda_1\right)+\lambda_1\lambda_1^*\right]P. \tag{5.15}$$

In expression (5.15), P is the purity function of expression (3.24). Although with the transition performed to the first exited state and at the height of the barrier, the star product of (4.1) and (5.10) is written

$$\left(H'\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_b \star \left(\left(\mathbf{W}_{10}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_b\right)_\star$$
$$= \frac{1}{2}\left(i\sigma_1\left(y_1^2-y_{1b}^2\right)-\sigma_2\frac{\partial^2}{\partial y_1^2}\right)\star\left(\left(\mathbf{W}_{10}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_b\right)_\star$$
$$= \frac{1}{2}\left[i\sigma_1\left(y_1^2-y_{1b}^2\right)-\sigma_2\frac{\partial^2}{\partial y_1^2}+\frac{\hbar^2}{4}\sigma_2\partial_{y_1}^2\right.$$
$$\left.+i\hbar\left(i\sigma_1 y_1\partial_{\left(\frac{\partial}{\partial y_1}\right)}-i\sigma_1 y_{1b}\partial_{\left(\frac{\partial}{\partial y_1}\right)}+\sigma_2\frac{\partial}{\partial y_1}\partial_{y_1}\right)\right]\times\left(\left(\mathbf{W}_{10}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_b\right)_\star \tag{5.16}$$

Therefore, we generalize the energy density in the framework of transition-state theory as

$$\left(\varepsilon\left(y_1\right)_b\right) = \int\left(H'\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_b\star\left(\left(\mathbf{W}_{10}\left(y_1,\frac{\partial}{\partial y_1}\right)\right)_b\right)_\star \mathrm{d}\left(\frac{\partial}{\partial y_1}\right). \tag{5.17}$$

More explicitly, we have

$$\left(\varepsilon\left(y_1\right)_b\right) = \left\{y_1^4\alpha_1^2\sqrt{\pi\alpha_2}\left[\sigma_2 D_1+i\kappa\right]+y_1^3\alpha_2\left[-i\sigma_1\left(\sqrt{\zeta_1 D_2}+\sqrt{\zeta_2 D_1}\right)+\frac{\sigma_2}{\alpha_1}\left(\zeta_1-2iD_1\right)\right.\right.$$
$$\left.-\sigma_2\alpha_1^2\sqrt{\zeta_1 D_2}\right]+2i\sigma_1\zeta_1 y_{1b}^3+y_1^2\left[\sigma_2 D_1\sqrt{\pi\alpha_2}\left(\frac{1}{\alpha_1}+\alpha_2+\frac{9}{2}\hbar^2\right)\right.$$

$$
\begin{aligned}
&+\sigma_2 D_2\sqrt{\pi\alpha_2^5}\left(\frac{1}{\alpha_1^2}+\frac{1}{\alpha_1}-i\alpha_2\right)+\sigma_1\sqrt{\zeta_2 D_1}\left(\frac{1}{4}\sqrt{\pi\alpha_2^3}-\frac{1}{2}\sqrt{\pi\alpha_2}-\sqrt{\frac{1}{\alpha_2}}\right)\\
&+\sigma_1\sqrt{\zeta_1 D_2}\left(\sqrt{\pi\alpha_2^3}-\frac{1}{2\hbar^2}\right)+i\sigma_1\zeta_3\sqrt{\pi\alpha_2}\Bigg]+y_{1b}^2\left[\frac{i}{4}\kappa\sqrt{\pi\alpha_2^3}\right.\\
&\left.+\sigma_1\zeta_1\left(1+\frac{1}{2\alpha_1}-\frac{1}{2}\sqrt{\pi\alpha_2}\right)+2\hbar\sigma_1\sqrt{\zeta_1 D_2}+i\sigma_2\zeta_1\right]\\
&+y_1\left[\sigma_2\sqrt{\zeta_1 D_2}\left(\alpha_2^2-\frac{1}{2}\alpha_2+2\frac{\sigma_1}{\sigma_2}\right)+\frac{3}{2}\sigma_2\zeta_2\alpha_2+\frac{3}{2}i\sigma_2\zeta_1\frac{1}{\alpha_2}+i\frac{1}{\alpha_1}\sigma_2\zeta_3\alpha_2\right.\\
&\left.+2i\sigma_1\zeta_3\frac{1}{\alpha_1}\right]+y_{1b}\left[-i\frac{\sigma_1}{4}\sqrt{\zeta_2 D_1}+\sigma_2\zeta_1\alpha_2-\sigma_1\zeta_3\right]-i\sigma_1 y_1^2 y_{1b}^2\sqrt{\pi\alpha_2}D_1\\
&-i\sigma_1\zeta_1 y_1^2 y_{1b}+iy_{1b}^2 y_1\left[\sigma_1\sqrt{\zeta_2 D_1}\alpha_2+i\frac{1}{\alpha_1}\alpha_2\sigma_2\zeta_1-\hbar^2\sigma_1\zeta_1\right]\\
&+y_1 y_{1b}\left[2\sigma_1\sqrt{\zeta_2 D_1}\left(1-\sqrt{\pi\alpha_2}\right)\right]+\sigma_1\zeta_2\left[\alpha_2^2-2\alpha_2+\frac{1}{\alpha_1}\right]\\
&+\sigma_2\zeta_1\left[\frac{1}{\alpha_1}-\frac{i}{2}\sqrt{\pi\alpha_2^2}+\frac{i}{2}\right]+\frac{1}{4}\sigma_2\zeta_2\left[3\sqrt{\pi\alpha_2^5}+\frac{3}{\alpha_1}\sqrt{\pi\alpha_2^3}-\frac{1}{\alpha_1^2}\right]\\
&+\frac{1}{4}\sigma_2\zeta_3\left[\sqrt{\pi\alpha_2^2}-2\frac{1}{\alpha_1}\right]\Bigg\}\exp_\star\left[-\frac{1}{2\hbar^2}\chi i\frac{\sigma_1}{\vartheta_1}\left(y_1^2-y_{1b}^2\right)\right].
\end{aligned}
\tag{5.18}
$$

At the barrier, the energy density is written

$$\varepsilon\left(y_{1b}\right)=\int\left(\varepsilon\left(y_1\right)_b\right)\mathrm{d}y_1 \tag{5.19}$$

which gives according to (5.18)

$$
\begin{aligned}
\varepsilon\left(y_{1b}\right)=&\Bigg\{2i\sigma_1\zeta_1 y_{1b}^3+y_{1b}^2\left[\sigma_1\zeta_1\left(1+\frac{1}{2\alpha_1}-\frac{i}{2}\hbar^2\alpha_1-\frac{1}{2}\sqrt{\pi\alpha_2}\right)+\sigma_2\zeta_1\left(i-\frac{1}{2}\alpha_2\right)\right.\\
&\left.-\frac{i}{4}\pi^2\sigma_1 D_1\sqrt{\alpha_1^3\alpha_2}+2\hbar\sigma_1\sqrt{\zeta_1 D_2}+\frac{i}{2}\sigma_1\sqrt{\zeta_2 D_1}\alpha_1\alpha_2+\frac{i}{4}\kappa\sqrt{\pi\alpha_2^3}\right]\\
&+y_{1b}\left[\sigma_1\sqrt{\zeta_2 D_1}\left(\alpha_1-\sqrt{\pi\alpha_1^2\alpha_2}-\frac{i}{4}\right)+\sigma_2\zeta_1\alpha_2-\sigma_1\zeta_3-\frac{i}{4}\sigma_1\sqrt{\pi\alpha_1^3}\right]\\
&+\sigma_1\zeta_2\left[\frac{1}{\alpha_1}+\alpha_2^2-2\alpha_2\right]+i\sigma_1\zeta_3\left[1+\frac{1}{4}\pi\sqrt{\alpha_1^3\alpha_2}\right]+\sigma_2\zeta_1\left[\frac{1}{2}\alpha_1\alpha_2+\frac{3}{4}i\frac{\alpha_1}{\alpha_2}\right.\\
&\left.+\frac{1}{\alpha_1}-\frac{i}{2}\sqrt{\pi}\alpha_2+\frac{i}{2}\right]+\frac{1}{4}\sigma_2\zeta_2\left[-\frac{1}{\alpha_1^2}+3\sqrt{\pi}\frac{\sqrt{\alpha_2^3}}{\alpha_1}+3\sqrt{\pi\alpha_2^5}+3\alpha_1\alpha_2\right]\\
&+\frac{1}{2}\sigma_2\zeta_3\left[\frac{5}{2}i\alpha_2-\frac{1}{\alpha_1}\right]+\frac{1}{2}\sqrt{\zeta_1 D_2}\left[\sigma_1\left(-i\alpha_1^2\alpha_2+\frac{1}{2}\pi\sqrt{\alpha_1^3\alpha_2^3}\right.\right.\\
&\left.\left.+\frac{1}{4\hbar^2}\sqrt{\pi\alpha_1^3}+2\alpha_1\right)-\sigma_2\left(\alpha_1^4\alpha_2-\alpha_1\alpha_2^2+\frac{1}{2}\alpha_1\alpha_2\right)\right]\\
&+\frac{1}{2}\sqrt{\zeta_2 D_1}\left[\sigma_1\left(-i\alpha_1^2\alpha_2+\frac{1}{8}\pi\sqrt{\alpha_1^3\alpha_2^3}-\frac{1}{4}\pi\sqrt{\alpha_1^3\alpha_2}-\frac{1}{4}\sqrt{\pi\frac{\alpha_1^3}{\alpha_2}}\right)\right]\\
&-\frac{1}{4}\pi\sigma_2 D_2\left[\sqrt{\alpha_1^3\alpha_2^7}-\sqrt{\alpha_1\alpha_2^5}-\sqrt{\frac{\alpha_2^5}{\alpha_1}}\right]+\frac{3}{4}i\pi\alpha_1^4\sqrt{\alpha_1\alpha_2\kappa}\Bigg\}\exp_\star\left[\frac{1}{2\hbar^2}\chi i\frac{\sigma_1}{\vartheta_1}y_{1b}^2\right],
\end{aligned}
\tag{5.20}
$$

where

$$\kappa = \sigma_2\zeta_1 - \sigma_1\zeta_2, \alpha_1 = \frac{1}{\frac{1}{2\hbar^2}}\chi \text{ and } \alpha_2 = \frac{1}{\frac{1}{2\hbar^2}\chi\frac{\sigma_2}{\vartheta_2}}. \quad (5.21)$$

6. Some Particular Cases

Let us interpret the results obtained in (3.24), (3.27), (3.29), (4.15), (4.17), (5.15) and (5.20) in terms of the non-commutativity variables θ_1, θ_2 and the magnetic field B.

6.1. The First Case

If we assume from expression (3.4) that

$$\omega_1 = \omega_2 = \frac{\pi}{4}, \quad (6.1)$$

(3.11) become

$$\delta_1 = \delta_2 = \frac{\hbar}{2}\sigma_2\left(\sigma_1 - \sigma_1^*\right) + \sigma_2\left(\theta_1\sigma_1 - \theta_2\sigma_1^*\right) \quad (6.2)$$

As a consequence, (3.20) read

$$\gamma = \sqrt{\frac{1}{\delta_1^4\sigma_1^*\sigma_2\vartheta_1^2}}, \quad (6.3)$$

and

$$\vartheta_1 = \vartheta_2 = \frac{1}{2\delta_1}\left(\frac{1}{\sqrt{\sigma_1^*\sigma_2}} + \frac{1}{\sqrt{\sigma_1\sigma_2}}\right). \quad (6.4)$$

where θ_1 and θ_2 are fields and are functions of position and momentum, we consider from **Figure 1**, the evolution of the von Neumann entropy and the Hall conductivity with respect θ_1 and θ_2. These two concepts of quantum mechanics have a similar behavior and they increase with θ_1 and θ_2. This explains that non-commutativity introduces an intrinsic correlation into the system, affecting both the global quantum state (entanglement) and the electrical response (Hall conductivity). This behavior shows a duality between the quantities defined in the momentum space and those defined in the position space. The fields θ_1 and θ_2 that appear naturally in the expression of von Neumann entropy and Hall conductivity have a singularity at the origin of the coordinate position and momentum. **Figure 2** shows that the transition in the quantum state performs clearly increases entanglement.

6.2. The Second Case

The second case is considered as follows: $\theta_2 = 0$ which gives from (3.4), $\omega_1 = 0$. We define λ_2 in expression (3.4) such as $\omega_2 = \frac{1}{2}\arctan\left(\lambda_2\right)$ and

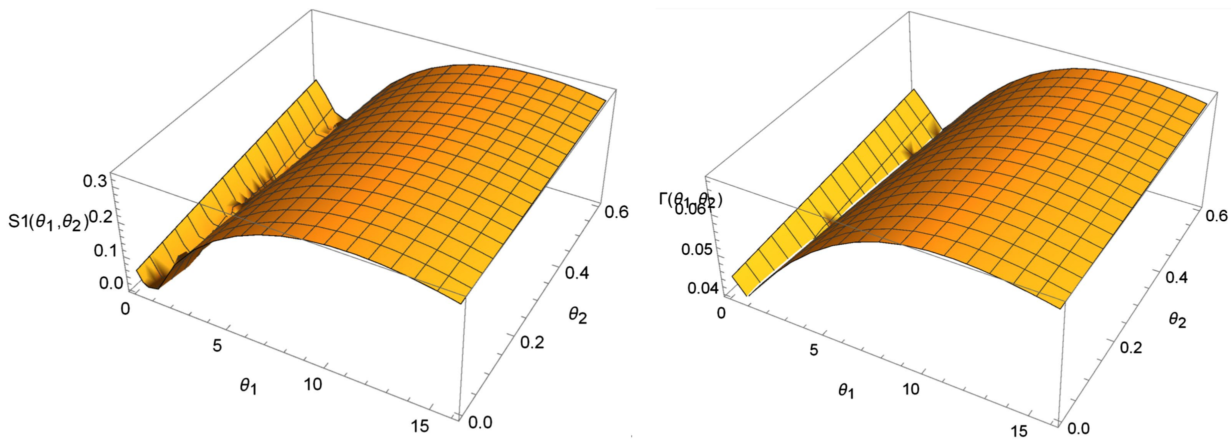

Figure 1. von Neumann entropy $S_1(\mathbf{W}_{00})$ (3.29) and Hall conductivity Γ (4.15) for $B=0.1$ and $\omega=1.2$.

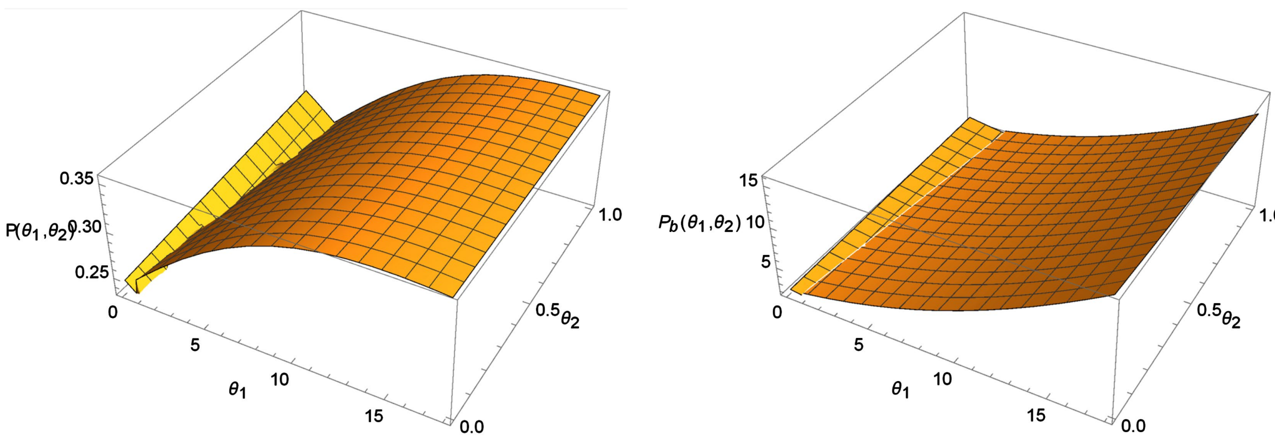

Figure 2. Purity functions P (3.24) and P_b (5.15) for $B=0.1$ and $\omega=0.7$.

$\lambda_2 = \dfrac{2\hbar\left(\sqrt{\frac{\sigma_2}{\sigma_1^*}}-1\right)\theta_1}{\hbar^2\left(\sqrt{\frac{\sigma_2}{\sigma_1^*}}-1\right)^2-\frac{\sigma_2}{\sigma_1^*}\theta_1^2}$. This formalism leads us to generalize the evolution of the Rényi entropy and the energy density by considering a field that depends on the position non-commutativity parameter θ_1 and the magnetic field B. (3.11) and (3.20) become

$$\delta_1 = \sqrt{\sigma_1\sigma_2}\left(\hbar\sin(\omega_2)+\theta_1\cos(\omega_2)\right), \tag{6.5}$$

$$\delta_2 = -\sqrt{\sigma_1^*\sigma_2}\,\hbar\sin(\omega_2), \tag{6.6}$$

and

$$\vartheta_1 = \frac{1}{2\delta_2}\frac{1}{\sqrt{\sigma_1\sigma_2}}. \tag{6.7}$$

Let us now evaluate the Rényi entropy on another basis according to λ_2 and n parameters. $n=2$ represents the lowest state, $n=8$ represents a more excited state.

We set $\theta_1 = 0.5$ and $\omega = 1.4$.

Observe **Figure 3** as the magnetic field increases, Rényi entropy values of the highest excited states appear together and merge into one curve to a single value close to zero. The transition performs clearly increases entanglement, acts as an information-relevant perturbation. Let's see how the energy density of an entangled system spreads according to the magnetic field B and the non-commutativity parameter θ_1.

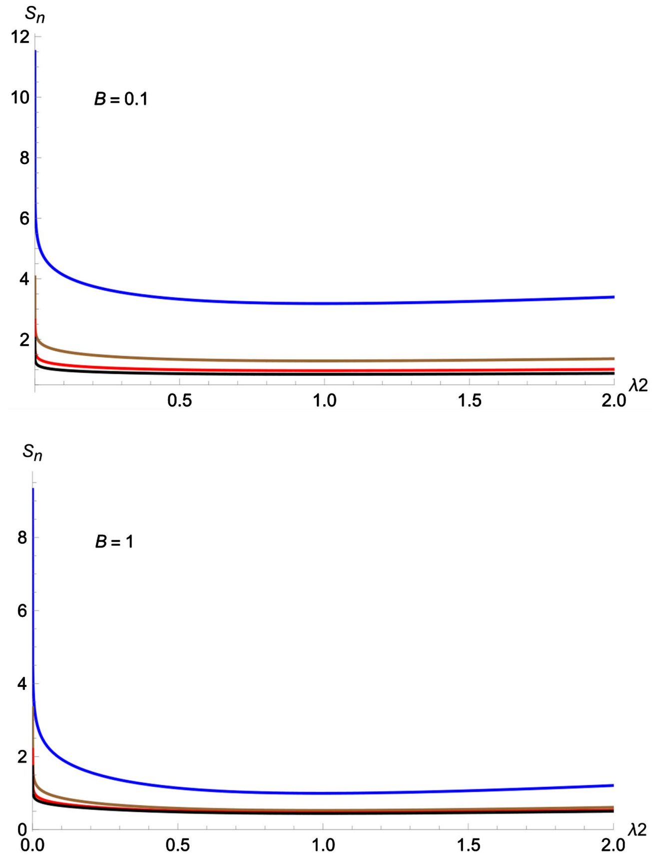

Figure 3. Rényi entropy (3.27) for different values of n; $n = 2$ (blue solid line), $n = 4$ (brown solid line), $n = 6$ (red solid line), $n = 8$ (black solid line).

We set $\omega = 1.4$.

The coupling θ_1 of the position operator expression to the non-commutative property implies the non-localization of the two harmonic oscillators to the atom. **Figure 4** and **Figure 5** show that, an increase of B increases the energy density, so it increases the energy stored in the masses of the system. Consequently, it accelerates the non-localization to the atom. By comparing the two (**Figure 4** and **Figure 5**), we prove that at the height of the barrier, a harmonic oscillator in a magnetic field θ_1, which is a priori a function of the position y_{1b}, has involuntarily and considerably increased its stored energy, so it increases its instabilization inside the atom. The effect of the θ_1-field then only manifests in the presence of a position-dependent potential y_1 and y_{1b}.

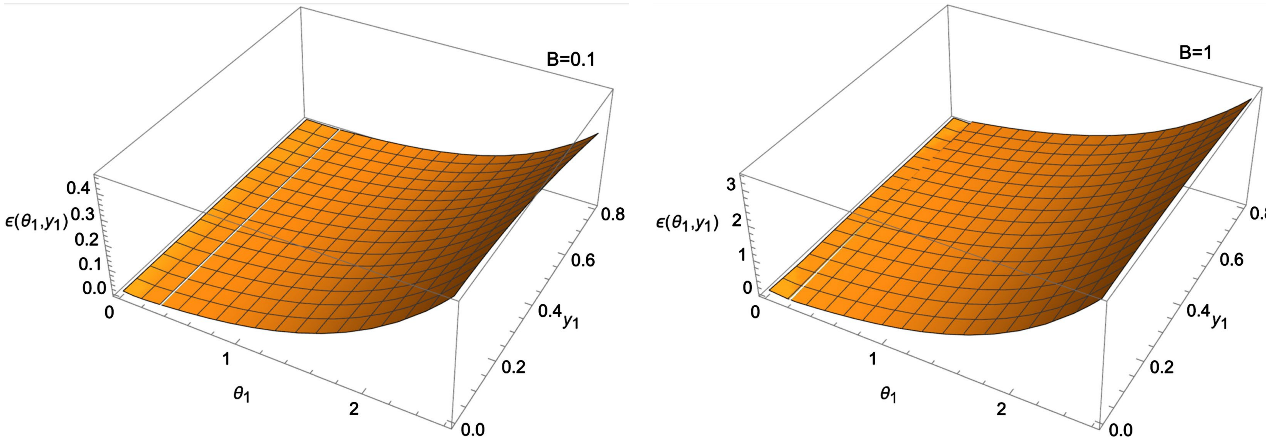

Figure 4. Energy density $\varepsilon(y_1)$ of (4.17).

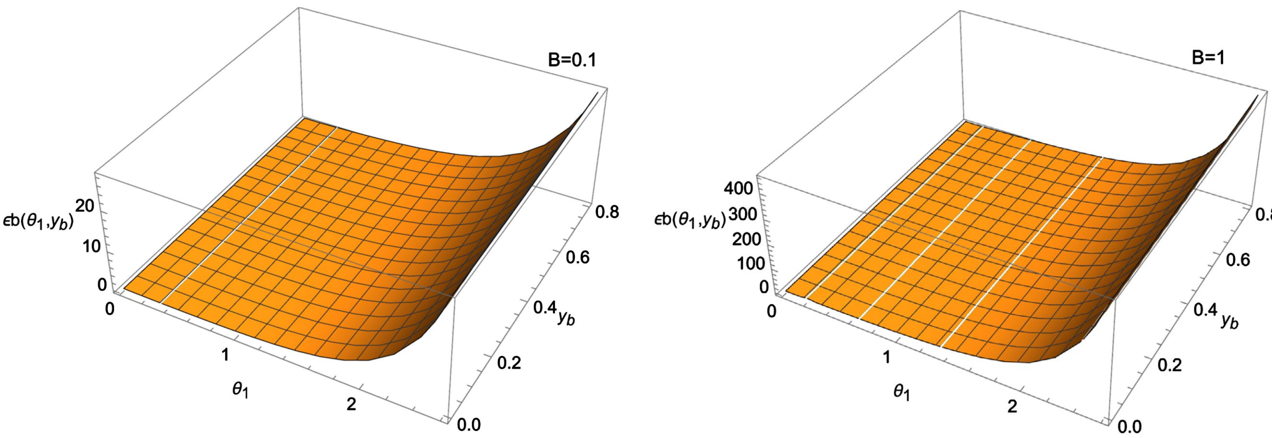

Figure 5. Energy density $\varepsilon(y_{1b})$ of (5.20).

7. Summary

In this work, we have examined quantum entanglement, as well as we have found the Hall conductivity and the energy density of two isotropic harmonic oscillators under a magnetic field in non-commutative phase space. The transition-state theory is applied to calculate the purity function and the energy density. Entanglement and Hall conductivity show a similar pattern because of the non-commutative parameters. A strong magnetic field disturbs the exact information of the highest excited states, and it accelerates the non-localization of the system in the atom by increasing its energy density. Transition-state theory makes the system more entangled and more instable.

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Conflicts of Interest

The author declares no conflicts of interest.

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