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New Kinds of Transitivity Maps and Minimality Mappings

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Abstract

In this article, we have discussed the connection between two distinct types of map concepts, specifically topological α-transitive maps [1]-[3] and δ-transitive maps and explored certain characteristics within two constructed topological spaces from the original space (X,τ), denoted by (X,τ^{α}) and (X,τ^{δ}). Here, τ^{α} represents the α-topology and τ^{δ} represents the δ-topology of the specified topological space (X,τ). These two concepts are delineated by employing α-irresolute and δ-irresolute maps, respectively. Additionally, we have examined the correlation between two categories of minimal systems: α-minimal and δ-minimal systems. The principal findings are summarized in the ensuing propositions: 1) Every alpha-transitive map implies a delta-transitive map; however, the opposite may not always be the case. 2) Every alpha-minimal system implies a delta-minimal system; however, the opposite may not always be the case.

Subject Areas

Dynamical System

Keywords

Alpha-Transitive Maps, Delta-Minimal System, Alpha-Irresolute Maps

1. Introduction

Let $\boldsymbol{M}$ be the set of all positive integers, $f: X \to X$ be an α-irresolute map defined on an original space (X,τ), then the set $A \subseteq X$ is called topologically α-mixing subset of X [2], if, given any nonempty $U,V \in \tau^{\alpha}$ with A intersects U and V, then $\exists M > 0$ such that $f^{m}(U) \cap V \neq \varphi$ for all $m > M$, weakly α-

mixing set of (X,f) [3], if for any choice of nonempty α-open sets V_1,V_2 subsets of A and nonempty α-open sets U_1,U_2 subsets of X with A intersect U_1 *and* U_2 there exist some $m \in \boldsymbol{M}$ such that $f^m(V_1)\cap U_1 \neq \varphi$ *and* $f^m(V_1)\cap U_2 \neq \varphi$, *strongly* α *- mixing* if for every pair of open sets U and V with $U \cap A \neq \varphi$ and $V \cap A \neq \varphi$, there exists some $m \in \boldsymbol{M}$ such that $f^k(U)\cap V \neq \varphi$ for any $k \geq m$. A point p such that its orbit $O_f(p)$ is α-dense in X is called α-hyper-cyclic point.

A system is α-mixing [3], if, given α-open sets in X, there exists an integer M, such that, for all $m > M$, one has $f^m(U)\cap V \neq \varphi$, *topologically* α *-mixing if* for any *non-empty* α-open set U, there exists M in $\boldsymbol{M}$ such that $\bigcup_{m\geq M} f^m(U)$ is α-dense in X. For the definitions of delta-transitive and alpha-transitive, see [4] [5]. With the above concepts, some new propositions have been introduced. Additionally, we possess the following statements:

- Every α-transitive map implies δ-transitive map. However, the opposite may not always be the case.
- Every α-minimal map implies δ-minimal map, but the opposite may not always be the case.
- $(E_\alpha) \Rightarrow (ET_\alpha)$, but not conversely.
- $(TM_\alpha) \Rightarrow (WM_\alpha) \Rightarrow (TT_\alpha)$, but not conversely.

2. New Kinds of Chaos of Topological Spaces

In this section, we delve into α-transitive maps and α-minimal maps, initially introduced, and delineated by [6]. Subsequently, we will explore their characteristics, proving associated results and delving into various properties and characterizations of these newly defined mappings.

Definition 2.1 Recall that a map $f: X \to Y$ is called α-irresolute (resp. β-irresolute) if for every α-open (resp. β-open) set V of Y, $f^{-1}(V)$ is α-open (resp. β-open) in X.

Definition 2.2 Recall that a set $A \subseteq X$ is called β-open if $A \subseteq Cl(Int(Cl(A)))$, the compliment of β-open is β-closed and a function $f: X \to X$ is called βr-homeomorphism if f is β-irresolute bijective and $f^{-1}: X \to X$ is β-irresolute.

Definition 2.3 Two topological systems $f: X \to X$, $x_{n+1} = f(x_n)$ and $g: Y \to Y$, $y_{n+1} = g(y_n)$ are topologically βr-conjugate if there is βr-homeomorphism $h: X \to Y$ such that $h \circ f = g \circ h$ (*i.e.* $h(f(x)) = g(h(x))$). We call h a topological βr-Conjugacy.

Notation: (X,f) and (Y,g) constitute two mixing systems if and only if both maps are mixing.

Proposition 2.4 The product of two α-mixing systems must be α-mixing.

Proof: Suppose that (X,f) and (Y,g) are two α-mixing systems and consider any α-open sets W, W' in $X \times Y$. By the definition of the product topology, there exists α-open sets $U, U' \subset X$ and $V, V' \subset Y$ so that $U \times V \subset W$ and $U' \times V' \subset W'$. By the definition of topological α-mixing of (X,f), there exists M such that for any $m > M$, $f^m(U) \cap V \neq \varphi$. By the definition of topological α-mixing (M. N. M. Kaki, 2012) of (Y, g), there is a positive integer M' such that for any $m > M'$, $g^m(U') \cap V' \neq \varphi$. Then, for any $m > \max(M, M')$, both $f^m(U) \cap V$ and $g^m(U') \cap V'$ are nonempty, and therefore $(f \times g)^m (U \times U') \cap (V \times V')$ is nonempty as well. But this implies that $(f \times g)^m (W) \cap W' \neq \varphi$, since W and W' were arbitrary, this implies that $(X \times Y, f \times g)$ is topologically α-mixing.

Theorem 2.5 The product of two α-transitive maps is not necessarily α-transitive map [3].

Corollary 2.6 The product of two β-transitive systems is not necessarily β-transitive.

Definition 2.7 In a separable and second category topological space X lacking isolated points, a point $x \in X$ is labeled as a hyper-cyclic point if the set $\{f^n(x) : n \in \boldsymbol{M}\}$ is dense in *X*. If such an x exists in *X*, then f is termed a hyper-cyclic function, or it's said to possess a hyper-cyclic point. An important theorem follows f qualifies as a hyper-cyclic function if and only if it is transitive.

We will prove some of the following propositions:

1) The maps f and g have the same kind of dynamics.

2) If x is a periodic point of the map f with a stable set $W_f(x)$, then the stable set of $h(x)$ is $h(W_f(x))$.

3) The map f is β-exact $\Leftrightarrow$ g is β-exact.

4) The map f is β-mixing $\Leftrightarrow$ g is β-mixing.

5) The map f is β-chaotic $\Leftrightarrow$ g is β-chaotic.

6) The map f is *weakly* β *-mixing* $\Leftrightarrow$ g *is weakly* β *- mixing.*

Remark 2.8

If $\{x_0, x_1, x_2, \cdots\}$ represents an orbit of $x_{n+1} = f(x_n)$, then $\{y_1 = h(x_1), y_1 = h(x_1), y_2 = h(x_2), \cdots\}$ produces another orbit. In other words, h maps the periodic orbits of f onto periodic orbits of g. This is because $y_{n+1} = h(x_{n+1}) = h(f(x_n)) = g(h(x_n)) = g(y_n)$, indicating that f and g exhibit the same type of dynamics.

A new form of transitivity has been introduced and defined in a manner that ensures its preservation under topological βr- conjugation.

Proposition 2.9 Let X and Y are β-separable and β-second category spaces. If $f : X \to X$ and $g : Y \to Y$ are βr-conjugated by the βr-homeomorphism $h : Y \to X$ then, *for each* β-hyper-cyclic point *y in Y if and only if h*(*y*) *is* β *-hyper-cyclic point in X*.

Proof: Suppose that $f : X \to X$ and $g : Y \to Y$ are two maps and βr-conjugated via $h : Y \to X$ such that $h \circ g = f \circ h$, then if $y \in Y$ is β-hyper-cyclic in *Y i.e. the orbit* $O_g(y) = \{y, g(y), g^2(y), \cdots\}$ *is* β-dense in *Y*, let $V \subset X$ be a nonempty β-open set. Then since h is a βr-homeomorphism, $h^{-1}(V)$ is β-open in *Y*, so there exists $n \in \boldsymbol{N}$ with $g^n(y) \in h^{-1}(V)$. From $h \circ g^n = f^n \circ h$ it follows that $h(g^n(y)) = f^n(h(y)) \in V$, so that $O_f(h(y)) = \{h(y), f(h(y)), f^2(h(y)), \cdots\}$ is β-dense in X *so* $h(y)$ *is* *β-hyper-cyclic in* X. Similarly, if $h(y)$ is β-hyper-cyclic in X, then y is β-hyper-cyclic in Y.

Proposition 2.10 if $f: X \to X$ and $g: Y \to Y$ are βr-conjugate via $h: X \to Y$. Then

1) T is β-transitive subset of $X \Leftrightarrow h(T)$ is β- transitive subset of Y;

2) $T \subset X$ is β-mixing set $\Leftrightarrow h(T)$ is β-mixing subset of Y.

Proof (1)

Assume that $f: X \to X$ and $g: Y \to Y$ are topological systems which are topologically βr-conjugated by $h: X \to Y$. Thus, h is βr-homeomorphism (that is, h is bijective and thus invertible and both maps h and h^{-1} are β-irresolute) and $h \circ f = g \circ h$

Suppose T is β -transitive subset of X. Let A, B be β-open subsets of Y with $B \cap h(T) \neq \varphi$ and $A \cap h(T) \neq \varphi$. (To show $g^n(A) \cap B \neq \phi$ for some $n > 0$). $U = h^{-1}(A)$ and $V = h^{-1}(B)$ are β-open subsets of X since h is an β-irresolute. Then there exists some $n > 0$ such *that* $f^n(U) \cap V \neq \phi$ *since the set* T is β-transitive subset of X, with $U \cap T \neq \varphi$ and $V \cap T \neq \varphi$. Thus (as $f \circ h^{-1} = h^{-1} \circ g$ implies $f^n \circ h^{-1} = h^{-1} \circ g^n$).

$\varphi \neq f^n(h^{-1}(A)) \cap h^{-1}(B) = h^{-1}(g^n(A)) \cap h^{-1}(B)$. Therefore, $h^{-1}(g^n(A) \cap B) \neq \varphi$ implies $g^n(A) \cap B \neq \varphi$ since h^{-1} is invertible. So, $h(T)$ is a β-transitive subset of Y.

Proof (2)

We have only to prove that if B is β-mixing subset of Y then $h^{-1}(B)$ is also β-mixing subset of X. Let D and V be two β-open subsets of X with $D \cap h^{-1}(B) \neq \varphi$ and $V \cap h^{-1}(B) \neq \varphi$. We need to demonstrate that there is a positive N such that for any n greater than N, the intersection of $f^n(D)$ and V is not empty. $h^{-1}(D)$ and $h^{-1}(V)$ are two β-open sets since h is β-irresolute with $h^{-1}(V) \cap B \neq \varphi$ and $h^{-1}(D) \cap B \neq \varphi$. If the set B is β-mixing then $\exists$ $M > 0$ such that $\forall$ $n > M$, $g^n(h^{-1}(D)) \cap h^{-1}(V) \neq \varphi$. So $\exists x \in g^n(h^{-1}(D)) \cap h^{-1}(V)$. That is $x \in g^n(h^{-1}(D))$ and $x \in h^{-1}(V)$ $\Leftrightarrow x = g^n(y)$ for $y \in h^{-1}(D)$. $h(x) \in V$. Thus, since $h \circ g^n = f^n \circ h$, so that $h(x) = h(g^n(y)) = f^n(h(y)) \in f^n(D)$ and we have $h(x) \in V$ that is $f^n(D) \cap V \neq \varphi$. So, $h^{-1}(B)$ is β-mixing set.

Proposition 2.11 Let B be a β-closed in a system (X, f). Then the following conditions must be equivalent.

1) B is a β-transitive set of (X, f).

2) Let B_1 be a nonempty β-open subset of A and B_2 be a nonempty β-open subset of X with $B_2 \cap B \neq \varphi$. Then there exists $n \in \mathbf{N}$ such that $B_1 \cap f^{-n}(B_2) \neq \varphi$.

3) Let B be a nonempty β-open a set of X with $B \cap A \neq \varphi$. Then $\bigcup_{n \in \mathbf{N}} f^{-n}(B)$ is β-dense in A.

Theorem 2.12 Let (X, f) be a topological system and D be a nonempty β-closed invariant set of X. Then D is a β- transitive subset of (X, f) if and only if (D, f) is β - transitive system.

Proof:

$\Rightarrow$) Let V_1 and B_1 be two nonempty β-open subsets of D. For a nonempty β-open subset B_1 of D, there exists a β- open set B of X such that $B_1 = B \cap D$. Since D is a β-transitive set of (X, f), there exists $n \in N$ such that $f(V_1) \cap B \neq \varphi$. Moreover, D is invariant, *i.e.*, $f(D) \subset D$. Therefore, $f(V_1) \cap D \cap B \neq \varphi$, *i.e.* $f(V_1) \cap B_1 \neq \varphi$. This shows that (D, f) is β-transitive.

$\Leftarrow$) Let V_1 be a nonempty β-open set of D and B be a nonempty β-open set of X with $D \cap B \neq \varphi$. Since B is a β-open set in X and $D \cap B \neq \varphi$, it follows that $B \cap D$ is a nonempty β-open set of D. Since (D, f) is a β-transitive, there exists $n \in N$ such that $f(V_1) \cap (D \cap B) \neq \varphi$, which implies that $f(V_1) \cap B \neq \varphi$. This shows that D is a β-transitive set of (X, f).

3. Development of Generalized Transitivity in Topological Dynamics

The first paper, "Topologically α-Transitive Maps and Minimal Systems" (2012), introduces the concept of α-transitive maps and investigates their connection with minimal dynamical systems, establishing foundational relationships between generalized transitivity and orbit density in topological spaces. This work serves as the theoretical basis for studying how modified openness conditions affect dynamical behavior.

The fourth paper, "New Types of δ-Transitive Maps" (2012), further advances the theory by presenting δ-transitive maps, another refinement of generalized transitivity using δ-open sets and δ-topological operations. This study deepens the hierarchy of transitivity concepts and demonstrates how alternative generalized open-set structures influence dynamical properties.

Building upon this foundation, the fifth paper, "Introduction to θ-Type Transitive Maps on Topological Spaces" (2012), extends the previous framework by defining θ-type transitive maps, introducing another generalized form of transitivity based on θ-open sets and θ-topological structures. This paper broadens the applicability of transitivity theory to spaces with weaker separation or generalized topological conditions.

Overall, these papers are strongly interconnected and constitute a progressive research program aimed at constructing a hierarchy of generalized transitivity notions—namely, α-transitivity, θ-transitivity, and δ-transitivity—to extend classical topological dynamics to broader generalized topological settings. Together, they contribute to the theoretical advancement of dynamical systems by providing new frameworks for analyzing continuity, orbit behavior, and minimality under generalized openness conditions. For more information on generalized transitivity and chaotic notions [6]-[12].

4. Conclusion

In summary, Propositions 2.4, 2.9, 2.10, and 2.11, along with Theorem 2.12, collectively contribute to our understanding of the dynamics of topological systems. Proposition 2.4 establishes that the product of two β-mixing systems remain β-mixing, which is a fundamental property in studying the behavior of topological systems. Propositions 2.9 and 2.10 delve into the concept of βr-conjugacy between spaces, offering insights into the preservation of β-hyper-cyclic points, β-transitive and β-mixing subsets under βr-homeomorphisms. Proposition 2.11 provides equivalent conditions for β-transitive sets within β-closed invariant sets, emphasizing the importance of β-openness and β-density in defining β-transitivity. Finally, Theorem 2.12 establishes a crucial link between β-closed invariant sets and β-transitive systems, providing a deeper understanding of β-transitivity within topological systems. Together, these propositions and theorems contribute to a more comprehensive understanding of the dynamics and behavior of topological systems, enriching our mathematical toolkit for analyzing complex systems.

Conflicts of Interest

The authors declare no conflicts of interest.

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