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Public Debt Dynamics When $r < g$: Stability Is not Guaranteed

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Abstract

This paper reexamines the stability of public debt dynamics in economies where real interest rate is lower than real economic growth rate ($r < g$). While this condition is often interpreted as ensuring debt sustainability, we develop a simple dynamic model in which the real interest rate depends endogenously on public-debt-to-GDP ratio. Using difference equations, we show that a low-debt steady state can become unstable if the debt ratio temporarily exceeds a critical threshold. Once this threshold is crossed, the endogenous rise in interest rates destabilizes debt dynamics and leads to divergence, even without expectation-driven crises or default risk. The results suggest that the $r < g$ condition should be viewed as a local, state-dependent criterion rather than a guarantee of long-run fiscal safety.

Keywords

Public Debt Dynamics, Debt-to-GDP Ratio, Interest-Growth Differential ($r < g$), Endogenous Interest Rates, Fiscal Sustainability

1. Introduction

In recent years, relationship between real interest rate and economic growth rate—namely, the condition $r < g$ —has played a central role in discussions of public debt sustainability. When the real growth rate exceeds the real interest rate, public-debt-to-GDP ratio decreases over time, implying that government debt may evolve in a stable manner even in the presence of persistent fiscal deficits. This view is rooted in the classical argument raised by Domar (1944), who clearly demonstrated that burden of public debt depends not on level of fiscal deficits per se, but rather on relationship between the interest rate and the growth rate.

A prominent study that reassessed this $r < g$ perspective in a modern context is Blanchard (2019). He documented that, historically, real interest rates have of-

ten remained below real growth rates in many advanced economies, and argued that under such conditions the fiscal cost of public debt may be substantially smaller than traditionally assumed.

By contrast, much of theoretical literature on public debt has focused on emergence of crises as situations in which the condition $r < g$ breaks down. For example, Cole and Kehoe (2000) showed that in economies where public debt reaches sufficiently high levels, changes in investors' expectations can give rise to self-fulfilling rollover crises. In their model, both a normal equilibrium and a crisis equilibrium may coexist under identical fundamentals, with crises triggered by shifts in expectations.

Extending this line of research, Lorenzoni and Werning (2019) demonstrated that debt crises need not occur abruptly, but may instead evolve gradually through interaction between rising interest rates and increasing debt—a phenomenon they describe as "slow-moving" debt crises. While their analysis highlights that higher initial debt levels raise the probability of a crisis, its primary focus is on discontinuous events such as crises and defaults.

Another important strand of literature emphasizes the role of fiscal policy responses. Bohn (1998) claimed that public debt is sustainable as long as primary fiscal balance responds positively to the public-debt-to-GDP ratio. This result has since served as a standard benchmark in empirical studies of fiscal sustainability. Based on this framework, Ghosh, Kim, Mendoza, Ostry and Qureshi (2013) introduced the concept of "fiscal fatigue", whereby political and economic constraints weaken fiscal adjustment at high debt levels, leading to endogenous debt limits and sharp increases in interest rate spreads.

Despite these important contributions, a common feature of existing literature is that it primarily focuses on whether a crisis occurs or whether public debt is ultimately sustainable. Much less attention has been paid to the mechanisms through which the steady state of public debt itself may become unstable. In particular, little is known about how a temporary increase in the public-debt-to-GDP ratio affects the stability of debt dynamics in economies where the condition $r < g$ holds initially.

This paper aims to fill this gap by developing a simple model that explicitly incorporates financial assets and endogenizes the real interest rate as a function of the public-debt-to-GDP ratio. Within this framework, we show, by using difference equations, that a temporary increase in the public-debt-to-GDP ratio can undermine the stability of the steady state. A key feature of our analysis is that instability arises from the deterministic structure of debt dynamics itself, without relying on shifts in investors' expectations or sunspot equilibria.

Main conclusion of this paper is that the condition $r < g$ does not guarantee long-run debt safety. In this sense, our analysis provides a theoretical reexamination of recent fiscal optimism based on the $r < g$ condition and highlights the importance of endogenous interaction between debt levels and interest rates when assessing public debt sustainability.

Remainder of this paper is organized as follows. Section 2 presents the analytical framework. Section 3 derives the steady state of the public-debt-to-GDP ratio and examines how a temporary increase in the public-debt-to-GDP ratio affects the steady state. Section 4 concludes.

2. Basic Model

Let GDP, government debt, fiscal deficit and interest rate in period t be denoted by $Y(t)$, $S(t)$, $D(t)$, and $r(t)$, respectively. Assume that economic growth rate is constant over time and equal to g. We adopt the following timing convention. Government debt $D(t)$ is measured at the end of period t, while the fiscal deficit $D(t-1)$ is incurred during period $t-1$ and adds to debt at the beginning of period t. This convention allows us to express the law of motion for the debt-to-GDP ratio in a simple recursive form.

Then, GDP and government debt in period t evolve according to

$$Y(t) = (1+g)Y(t-1),$$

$$S(t) = (1+r(t-1))S(t-1) + D(t-1).$$

Hence, the government-debt-to-GDP ratio in period t is given by

$$\frac{S(t)}{Y(t)} = \frac{(1+r(t-1))S(t-1) + D(t-1)}{(1+g)Y(t-1)}. \tag{1}$$

Let $s(t) \equiv S(t)/Y(t)$ and $d(t) \equiv D(t)/Y(t)$. Equation (1) can then be rewritten as

$$s(t) = \frac{1+r(t-1)}{1+g}s(t-1) + \frac{1}{1+g}d(t-1). \tag{2}$$

$d(t)$ is the primary fiscal deficit as a ratio to GDP, excluding interest payments. Interest payments are captured separately through the endogenous interest rate $r(s(t))$, which applies to the outstanding stock of government debt.

Subtracting $s(t-1)$ from both sides yields

$$s(t) - s(t-1) = \frac{r(t-1)-g}{1+g}s(t-1) + \frac{1}{1+g}d(t-1) \tag{3}$$

Next, assume that the interest rate $r(t)$ remains constant at a value $\bar{r}$ as long as $s(t)$ does not exceed a threshold $\bar{s}$, but increases once this threshold is crossed, according to $r(s(t)) = \bar{r} + \alpha(s(t) - \bar{s})$. Equivalently,

$$r(s(t)) = \begin{cases} \bar{r}, & \text{if } s(t) \le \bar{s}, \\ \bar{r} + \alpha(s(t) - \bar{s}), & \text{if } s(t) > \bar{s}. \end{cases} \tag{4}$$

The piecewise linear specification of the interest rate serves two purposes. First, it captures in a parsimonious manner the idea that borrowing costs remain insensitive to debt at low levels but increase sharply once fiscal capacity is strained. Second, this functional form allows for transparent analytical characterization of stability and instability using difference equations.

Importantly, the qualitative results of the model do not rely on the kinked structure per se. Any smooth and sufficiently convex interest rate function—such as those commonly employed in the fiscal fatigue literature—would generate a similar threshold beyond which debt dynamics become unstable.

The parameter α captures the sensitivity of borrowing costs to increases in public debt. In practice, its magnitude reflects institutional and macroeconomic factors such as fiscal credibility, monetary policy frameworks, financial market depth, and the investor base. Economies with strong fiscal institutions and credible central banks are likely to exhibit a lower α, whereas countries facing rollover risk or weak policy credibility may experience a sharply rising interest rate response.

3. Dynamics and Steady States of the Government-Debt-to-GDP Ratio

Based on the above discussion, this section derives the steady states and shows that the condition $r < g$ does not guarantee long-run debt safety.

Under the above assumptions, equation (3) becomes

$$s(t)-s(t-1)=\begin{cases}\dfrac{\bar{r}-g}{1+g}s(t-1)+\dfrac{1}{1+g}d(t-1), & \text{if } s(t-1)\le \bar{s},\\ \dfrac{\bar{r}+\alpha\left(s(t-1)-\bar{s}\right)-g}{1+g}s(t-1)+\dfrac{1}{1+g}d(t-1), & \text{if } s(t-1)>\bar{s}.\end{cases} \tag{5}$$

If we let s_L^* the steady state in the region $s(t)\le \bar{s}$, we have

$$s_L^*=\frac{d}{g-\bar{r}}$$

by solving

$$\frac{r-g}{1+g}s^*+\frac{1}{1+g}d=0.$$

In what follows, we assume $0<s_L^*<\bar{s}$, which requires

$$g>\bar{r} \text{ and } \frac{d}{g-\bar{r}}<\bar{s}.$$

On the other hand, if we let s_H^* denote the steady state in the region $s(t)>\bar{s}$, we have

$$s_H^*=\frac{-\left[(\bar{r}-g)-\alpha\bar{s}\right]+\sqrt{\left[(\bar{r}-g)-\alpha\bar{s}\right]^2-4\alpha d}}{2\alpha}$$

by solving

$$\frac{\bar{r}+\alpha\left(s^*-\bar{s}\right)-g}{1+g}s^*+\frac{1}{1+g}d=0.$$

In what follows, we assume that s_H^* is real, which requires

$$\left[(\bar{r}-g)-\alpha\bar{s}\right]^2-4\alpha d\ge 0.$$

Now we are ready to show that the condition $r < g$ does not guarantee long-run debt safety.

The phase diagram corresponding to this dynamic system is typically depicted as in Figure 1. Downward sloping straight line to the left of vertical line at $\bar{s}$ represents

$$s(t)-s(t-1)=\frac{\bar{r}-g}{1+g}s(t-1)+\frac{1}{1+g}d(t-1),$$

while U shaped curve to the right of the vertical line at $\bar{s}$ represents

$$s(t)-s(t-1)=\frac{\bar{r}-g}{1+g}s(t-1)+\frac{1}{1+g}d(t-1).$$

Moreover, s_L^* is a stable steady state, whereas s_H^* is an unstable steady state. That is, when $s(t-1)\le s_L^*$ or $s_L^*\le s(t-1)\le s_H^*$, the government-debt-to-GDP ratio, $s(t)$, converges to s_L^*. In contrast, when $s(t-1)>s_H^*$, the government-debt-to-GDP ratio, $s(t)$, diverges.

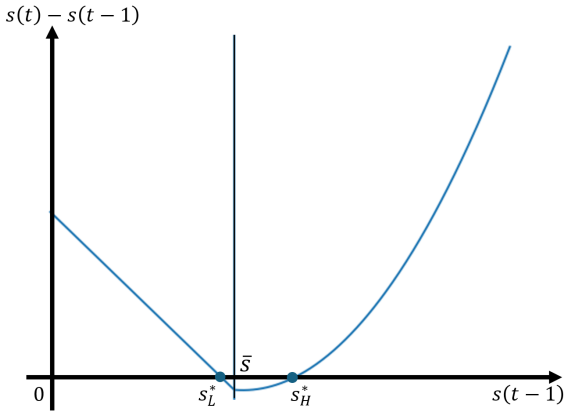

Figure 1. Dynamics of government-debt-to-GDP ratio.

It is important to distinguish between two conceptually different objects. The parameter $\bar{s}$ represents a debt threshold at which interest rates begin to increase endogenously. By contrast, s_H denotes an unstable steady state that governs divergence once the debt ratio exceeds a critical level.

From this phase diagram, the following proposition is derived.

Proposition.

Even if the condition $r < g$ holds, the debt-to-GDP ratio diverges once $s(t-1)$ exceeds s_H^*, even temporarily.

This proposition is illustrated by the numerical example presented below.

Numerical Example

Suppose that

$$s(0)=0.6, g=0.02, d=0.01,$$

$$r(s(t)) = \begin{cases} 0.005, & \text{if } s(t) \le 0.8, \\ 0.005 + 0.35(s(t) - 0.8), & \text{if } s(t) > 0.8. \end{cases}$$

If nothing happens to d, since $r < g$ holds over time, the debt-to-GDP ratio converges to a steady-state value s^*, given by

$$s^* = \frac{d}{g - r},$$

which is determined as

$$s^* = \frac{2}{3},$$

by substituting $g = 0.02$, $r = 0.005$, and $d = 0.01$.

Next, we consider the case of a temporary increase in the deficit. In period 1, the government runs a small fiscal deficit, $d = 0.01$, followed by a one-time large deficit of $d = 0.25$ in period 2. From period 3 onward, the deficit ratio returns to its original level, $d = 0.01$. Under this fiscal path, numerical simulations show that the debt-to-GDP ratio remains almost unchanged in period 1, staying at a stable level of approximately $s(1) \approx 0.60$. However, the large deficit in period 2 raises the debt-to-GDP ratio sharply to about $s(2) \approx 0.84$, thereby exceeding the threshold level $s = 0.8$ embedded in the interest-rate rule.

Once this threshold is crossed, the interest rate becomes an increasing function of the debt-to-GDP ratio, rendering the debt dynamics effectively nonlinear, with a quadratic term in s. As a consequence, even if the deficit ratio is reduced back to 0.01 from period 3 onward, the debt-to-GDP ratio does not decline. Instead, as illustrated in Table 1, it continues to rise monotonically and eventually diverges. This example therefore demonstrates that even a temporary fiscal expansion can irreversibly push the economy into an unstable debt regime when the interest rate responds nonlinearly to the level of public debt.

These results strongly suggest that long-run debt sustainability depends not only on the level of fiscal deficits but also on whether the debt-to-GDP ratio crosses a critical threshold even once.

Table 1. Sequence of the debt-to-GDP ratio under a temporary increase in the deficit.

	0	1	2	3	4	5	10	15	18	19	20	21
d	–	0.01	0.25	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
debt/GDP	0.600	0.601	0.837	0.845	0.856	0.869	1.041	2.219	16.76	108.3	4107.50	5,792,181.0015

4. Conclusion

This paper has examined the stability of public debt dynamics in an economy where the real interest rate depends endogenously on the public-debt-to-GDP ratio. Motivated by recent fiscal optimism based on the condition $r < g$, we developed a simple model with financial assets and showed that this condition does not guarantee the long-run debt safety.

Our analysis demonstrates that even when $r < g$ holds initially and the econ-

omy is supposed to converge to a low-debt steady state, a temporary increase in the public-debt-to-GDP ratio can fundamentally alter debt dynamics. Once the debt ratio exceeds a critical threshold, the interest rate rises endogenously, destabilizing the steady state and leading to self-reinforcing debt accumulation. Importantly, this divergence occurs without relying on changes in investors' expectations, sunspot equilibria or explicit default risk. Instead, instability arises purely from the deterministic structure of debt dynamics itself.

The results imply that the condition $r < g$ should be interpreted as a local, state-dependent condition rather than as a global criterion for fiscal safety. As a consequence, assessments of debt sustainability that rely solely on observed interest-growth differentials may be misleading if they ignore the endogenous response of interest rates to rising debt levels.

From a policy perspective, the results emphasize the importance of preventing temporary surges in public debt from pushing the economy beyond critical thresholds. Even when borrowing appears inexpensive in the short run, fiscal authorities must consider the possibility that higher debt levels can endogenously raise interest rates and undermine long-run stability.

Future research could extend the present framework by incorporating endogenous growth, stochastic shocks or alternative fiscal reaction functions, as well as by exploring quantitative implications for specific economies. Nonetheless, the simple structure of the model presented here already suffices to show that fiscal sustainability cannot be assessed independently of the dynamic interaction between debt levels and interest rates.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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